Static Diffusive Couplings in Heterogeneous Linear Networks*

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Abstract: Recently, necessary and sufficient conditions for output synchronization of linear systems via diffusive couplings have been reported. In this paper, we study the case when such conditions are not satisfied and exact synchronization is impossible. In particular, we study two kinds of heterogeneous linear networks: (i) non-identical harmonic oscillators and (ii) double-integrators. We show that static diffusive couplings render heterogeneous networks of harmonic oscillators asymptotically stable. Networks of non-identical double-integrators, in contrast, are not asymptotically stable but synchronize with bounded synchronization error depending on the network topology and the heterogeneity in the agent dynamics. Numerical examples illustrate the results.

1. INTRODUCTION

Over the last decade, large-scale and distributed dynamical systems have attracted great attention in the field of control theory. Of particular interest are so-called multi-agent systems consisting of individual subsystems which interact with neighboring subsystems, or agents, according to some distributed control law. Such models are suitable to describe and analyze consensus and synchronization phenomena.

A common approach to consensus and synchronization problems in networks of dynamic agents is static diffusive couplings, i.e., distributed controllers without dynamics that take into account the output differences of neighboring agents. Famous examples are the classical consensus protocol, cf., Olfati-Saber and Murray [2004], Moreau [2004], Ren and Beard [2005], and its extensions to double-integrators, Ren and Atkins [2007], harmonic oscillators, Ren [2008], Su et al. [2009], and general linear agents, Wieland et al. [2011a]. It has been shown by Scardovi and Sepulchre [2009], Wieland [2010], Wieland et al. [2011b] that dynamic diffusive couplings provide more flexibility and allow to solve synchronization problems in a larger class of networks of linear systems. A major challenge is synchronization in heterogeneous linear networks, i.e., multi-agent systems consisting of non-identical linear agents. Wieland [2009] presents a necessary condition for synchronization in heterogeneous linear networks. The result resembles the internal model principle of classical output regulation and states that the agents have to embed a common internal model in order to synchronize.

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In this paper, we study heterogeneous linear networks with static diffusive couplings and focus on the case when such necessary conditions are not fulfilled. We analyze the dynamics of heterogeneous linear networks in which exact synchronization is not possible. The goal is to see whether we can achieve "practical" synchronization with a small synchronization error in such situations and assess the robustness of synchronization with respect to heterogeneities in the agent dynamics.

The contributions of this paper are the following: First, the internal model principle for synchronization stated by Wieland [2009] is formulated for the special case of static diffusive couplings. Then, heterogeneous networks of harmonic oscillators are studied. We show that the internal model condition is not satisfied and that static diffusive couplings have a stabilizing effect in such networks. In particular, the network of oscillators is rendered asymptotically stable if and only if there are oscillators with non-identical frequency in the network. Last, heterogeneous networks of double-integrators are analyzed. In this case, the trajectories stay "close" to synchronization, depending on the graph topology and the heterogeneity in the network.

The remainder of this paper is organized as follows. Section 2 contains mathematical preliminaries and the graph theoretic background. In Section 3 the internal model principle for synchronization is reviewed. Our main results on heterogeneous networks of harmonic oscillators and double-integrators are presented in Section 4 and Section 5, respectively. Section 6 concludes the paper.

2. PRELIMINARIES

2.1 Mathematical Preliminaries

The following notation is used in this paper. For a vector $v \in \mathbb{R}^n$, $\mathrm{diag}(v)$ is the diagonal matrix with the elements v_i, $i = 1, \dots, n$, of v on the diagonal. The all ones and all zeros vectors are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively, and $I = \mathrm{diag}(\mathbf{1})$ is the identity matrix. The null space and image of a linear map defined by a matrix M are denoted by $\ker(M)$ and $\mathrm{im}(M)$, respectively. The norm $\|\cdot\|$ is understood as 2-norm for vectors and induced 2-norm for matrices. The spectrum of a square matrix M is denoted by $\sigma(M)$. With a slight abuse of notation, $\sigma(M)$ is to be understood as the set of roots of the characteristic polynomial of M, i.e., it respects the multiplicity of the eigenvalues. For a complex number $z \in \mathbb{C}$, $\Re(z)$ is the real part and $\Im(z)$ the imaginary part of z. The closed right-half complex plane is denoted by $\bar{\mathbb{C}}^+$.

2.2 Graph Theory

The network topology is modeled by a time-invariant directed graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, A_{\mathscr{G}}\}$. In the following, we review selected definitions and results on directed graphs, which are needed in the remainder of this paper. For a comprehensive discussion, the reader is referred to Godsil and Royle [2001], Wu [2005], Wieland [2010], Wieland et al. [2011a]. Each vertex v_k in the set $\mathscr{V} = \{v_1, \ldots, v_N\}$ corresponds to a dynamical subsystem (agent) k in the network. There is a directed edge from vertex v_j to v_k, i.e., $(v_j, v_k) \in \mathscr{E}$, if and only if v_k is influenced by (receives information from) v_j. The vertexes v_k, v_j are called head and tail of edge (v_j, v_k), respectively. A consecutive sequence of directed edges is called a directed path. The adjacency matrix $A_{\mathscr{G}} \in \mathbb{R}^{N\times N}$ describes the graph structure and edge weights, i.e., $a_{kj} > 0 \Leftrightarrow (v_j, v_k) \in \mathscr{E}$. The Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as $L = \mathrm{diag}(A_{\mathscr{G}}\mathbf{1}) - A_{\mathscr{G}}$. By construction, L is a Metzler matrix and has zero row sums, i.e., $L\mathbf{1} = \mathbf{0}$. The vector of ones $\mathbf{1}$ is the eigenvector corresponding to the zero eigenvalue $\lambda_1(L) = 0$.

Definition 1. (connected graph). The graph $\mathscr{G}$ is called connected if it contains a directed spanning tree, i.e., if there exists a vertex v_k such that there is a path from v_k to every other vertex $v_j \in \mathscr{V}$. In this case, v_k is called centroid.

Definition 2. (strongly connected graph). The graph $\mathscr{G}$ is called stongly connected if there exists a directed path from any vertex to any other vertex in $\mathscr{V}$. In this case, every vertex is a centroid.

Lemma 3. (Ren and Beard [2005]). All eigenvalues of L are contained in the closed right-half plane, i.e., $\lambda_k(L) \in \bar{\mathbb{C}}^+$ for $k = 1, \ldots, N$. The zero eigenvalue $\lambda_1(L) = 0$ is simple and all other eigenvalues have positive real parts $\Re(\lambda_k(L)) > 0$ for $k = 2, \ldots, N$, if and only if $\mathscr{G}$ is connected.

Lemma 4. (Li and Duan [2009], Brualdi and Ryser [1991]). If the graph $\mathscr{G}$ is connected, then there exists a vertex permutation such that L reduces to the Frobenius normal form

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{m1} & L_{m2} & \cdots & L_{mm} \end{bmatrix}, \qquad (1)$$

where L_{ii}, $i = 1, \ldots, m-1$, are irreducible square matrices, each L_{ii} has at least one row with positive row sum, and L_{mm} is irreducible or a scalar zero.

Definition 5. (induced subgraph). An induced subgraph of $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$ is a graph $\tilde{\mathscr{G}} = \{\tilde{\mathscr{V}}, \tilde{\mathscr{E}}\}$ with $\tilde{\mathscr{V}} \subseteq \mathscr{V}$ and $\tilde{\mathscr{E}} = \{(v, w) \in \mathscr{E} : v, w \in \tilde{\mathscr{V}}\}$.

Definition 6. (iSCC, Wieland [2010]). An independent strongly connected component (iSCC) of a directed graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$ is an induced subgraph $\tilde{\mathscr{G}} = \{\tilde{\mathscr{V}}, \tilde{\mathscr{E}}\}$ which is maximal, subject to being strongly connected, and satisfies $(v, \tilde{v}) \notin \mathscr{E}$ for any $v \in \mathscr{V} \backslash \tilde{\mathscr{V}}$ and $\tilde{v} \in \tilde{\mathscr{V}}$.

In other words, an iSCC $\tilde{\mathscr{G}} = \{\tilde{\mathscr{V}}, \tilde{\mathscr{E}}\}$ is strongly connected and the directed graph induced by any set $\hat{\mathscr{V}}$ with $\tilde{\mathscr{V}} \subseteq \hat{\mathscr{V}} \subseteq \mathscr{V}$ is strongly connected if and only if $\tilde{\mathscr{V}} = \hat{\mathscr{V}}$. Furthermore, there is no edge in $\mathscr{E}$ with tail outside $\tilde{\mathscr{V}}$ and head inside $\tilde{\mathscr{V}}$.

Figure 1 shows a directed graph which is connected. It is not strongly connected since there is, e.g., no path from v_6 to any other node. The iSCC contains vertexes $\mathscr{V}_{iSCC} = \{v_1, v_2, v_3, v_4\}$, any vertex in $\mathscr{V}_{iSCC}$ is a centroid in this example.

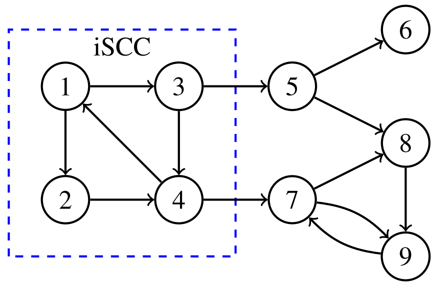

Fig. 1. A connected directed graph $\mathscr{G}$. The dashed box indicates the iSCC of the graph.

Theorem 7. (Wieland et al. [2011a]). If $\mathscr{G}$ has r distinct iSCCs $\mathscr{V}_{iSCC,j} \subseteq \mathscr{V}$, $j = 1, \ldots, r$, then

1. $\mathrm{rank}(L) = N - r$,
2. the null space of $L^\top$ admits a non-negative orthogonal basis $p_j \in \mathbb{R}^N$, $j = 1, \ldots, r$,
3. the basis vectors p_j, $j = 1, \ldots, r$, can be ordered such that for any vertex $k \in \mathscr{V}$, $k \in \mathscr{V}_{iSCC,j} \Leftrightarrow p_j^\top e_k \neq 0$, $j = 1, \ldots, r$, where e_k is the k-th canonical basis vector.

Lemma 8. (Wieland et al. [2011a]). If $\mathscr{G}$ is connected, then $\mathscr{G}$ has exactly one iSCC.

Corollary 9. If $\mathscr{G}$ is connected, then $\ker(L) = \mathrm{im}(\mathbf{1})$ and the left eigenvector p of L corresponding to eigenvalue zero with $p^\top\mathbf{1} = 1$ is non-negative, i.e., $p^\top L = \mathbf{0}^\top$ and $p \geq 0$ element-wise. If $\mathscr{G}$ is strongly connected, then p is positive, i.e., $p > 0$ element-wise.

Lemma 10. (Zhang et al. [2012]). Suppose $\mathscr{G}$ is strongly connected and $P = \mathrm{diag}(p)$, where $p^\top L = \mathbf{0}^\top$ and $p^\top\mathbf{1} = 1$. Then,

1. $P > 0$,
2. $(PL + L^\top P) \geq 0$,
3. $\ker(PL + L^\top P) = \mathrm{im}(\mathbf{1})$.

3. THE INTERNAL MODEL PRINCIPLE

The internal model principle for synchronization of heterogeneous linear systems has originally been presented by Wieland [2009]. It provides a necessary and sufficient condition for synchronization, Wieland et al. [2011b]. In Wieland [2009], Wieland et al. [2011b], dynamic diffusive couplings are considered. Here the internal model principle is reviewed and formulated for the special case of static diffusive couplings. Consider a heterogeneous group of N linear agents, given by

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u_k \\ y_k &= C_k x_k, \end{aligned} \qquad (2)$$

with state $x_k \in \mathbb{R}^{n_k}$, input $u_k \in \mathbb{R}^{q_k}$, and output $y_k \in \mathbb{R}^p$, for $k = 1, \ldots, N$. Suppose the agents are interconnected by static diffusive couplings according to

$$u_k = K_k \sum_{j=1}^{N} a_{kj}(y_j - y_k), \qquad (3)$$

where $K_k \in \mathbb{R}^{q_k \times p}$ is a coupling gain matrix and a_{kj} are the elements of the adjacency matrix $A_{\mathscr{G}}$ of the underlying communication graph $\mathscr{G}$. The network of N agents (2) with couplings (3) is said to reach non-trivial output synchronization, if

$$(y_j(t) - y_k(t)) \to 0 \quad \text{as} \quad t \to \infty$$

for all pairs $k, j \in \{1, ..., N\}$ and the closed-loop system has no asymptotically stable equilibrium set on which $y_k(t) = 0$, $k = 1, ..., N$. We impose the following standing assumption.

Assumption 11. The pair (A_k, C_k) is detectable for $k = 1, ..., N$.

In this setup, the internal model principle of Wieland [2009], Wieland et al. [2011b] can be stated as follows.

Theorem 12. A necessary condition for non-trivial output synchronization of a heterogeneous linear network of N agents (2) with static diffusive couplings (3) is that there exist an integer $m > 0$ and matrices $\Pi_k \in \mathbb{R}^{n_k \times m}$ with full column rank, $S \in \mathbb{R}^{m \times m}$ and $R \in \mathbb{R}^{p \times m}$, where $\sigma(S) \subset \bar{\mathbb{C}}^+$ and (S, R) is observable, such that

$$A_k \Pi_k = \Pi_k S, \qquad (4)$$
$$C_k \Pi_k = R, \qquad (5)$$

for $k = 1, ..., N$. Furthermore, in this case there exists a $w_0 \in \mathbb{R}^m$ such that $\lim_{t \to \infty} \|y_k(t) - Re^{St} w_0\| = 0$.

Remark 13. Equation (4) is equivalent to A_k-invariance of $\text{im}(\Pi_k)$. Furthermore, since Π_k has full column rank, every eigenvalue of S is an eigenvalue of A_k, i.e.,

$$\sigma(S) \subseteq \sigma(A_k), \quad k = 1, ..., N. \qquad (6)$$

Consequently, the eigenvalues of S are a subset of the largest common subset $\bigcap_{k=1}^N \sigma(A_k)$ of all agent's spectra.

Remark 14. It is possible to check in a systematic way whether the necessary condition in Theorem 12 is fulfilled. The possible spectra of S can be listed according to the condition (6). The candidates for matrix S can be chosen in Jordan normal form of dimension $1 \le m \le \hat{m}$, where $\hat{m}$ is the cardinality of $\bigcap_{k=1}^N \sigma(A_k)$, with $\sigma(S) \subseteq \bigcap_{k=1}^N \sigma(A_k)$. Assume $\tilde{S}$ is not in Jordan normal form and we want to find a solution $\tilde{\Pi}_k$, $\tilde{R}$ to

$$A_k \tilde{\Pi}_k = \tilde{\Pi}_k \tilde{S},$$
$$C_k \tilde{\Pi}_k = \tilde{R}.$$

Then, we can find a matrix T such that $S = T^{-1} \tilde{S} T$ has Jordan normal form, and we obtain

$$A_k \tilde{\Pi}_k T = \tilde{\Pi}_k T S,$$
$$C_k \tilde{\Pi}_k T = \tilde{R} T,$$

which is equivalent to (4), (5), with new variables $\Pi_k = \tilde{\Pi}_k T$ and $R = \tilde{R} T$. Therefore, it suffices to check the candidates of S in Jordan normal form.

In words, the internal model principle for synchronization in heterogeneous networks of linear systems states that the agents can only synchronize to a trajectory generated by a dynamical system $\dot{w} = Sw$, which is contained in the dynamics of each agent. Furthermore, if the agents in the network have no eigenvalues in common, then (non-trivial) synchronization is impossible.

4. HETEROGENEOUS HARMONIC OSCILLATORS

In this section, networks of harmonic oscillators with non-identical frequencies are analyzed. The agents (2) are given with matrices

$$A_k = \begin{bmatrix} 0 & (\omega + \delta_k) \\ -(\omega + \delta_k) & 0 \end{bmatrix},$$

and $B_k = C_k = I$, and the coupling gains in (3) are $K_k = I$ for $k = 1, ..., N$. The individual oscillators deviate by $\delta_k \in \mathbb{R}$ from the nominal frequency $\omega \in \mathbb{R}$. Suppose there exist two agents $k, j \in \{1, ..., N\}$ such that $\delta_k \neq \delta_j$. Then the intersection of the agents' spectra $\sigma(A_k)$ is empty and exact non-trivial synchronization is impossible as discussed before.

However, we are interested in the behavior of the dynamic network in this case. In particular, we would like to see whether small perturbations in the frequencies lead to small synchronization errors. The following result characterizes the dynamic behavior of the network.

Lemma 15. Consider a network of N harmonic oscillators interconnected by static diffusive couplings, i.e.,

$$\dot{x}_k = (\omega + \delta_k) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \sum_{j=1}^N a_{kj}(x_j - x_k) \qquad (7)$$

for $k = 1, ..., N$. Suppose that the directed graph $\mathscr{G}$ is strongly connected. Furthermore, suppose that there exists a pair k, j of oscillators such that $\delta_k \neq \delta_j$, i.e., not all oscillators have identical frequencies. Then, the network of oscillators is asymptotically stable.

Proof. With $x_k = [r_k, v_k]^\mathsf{T}$, $r_k, v_k \in \mathbb{R}$, (7) can be written as

$$\dot{r}_k = (\omega + \delta_k) v_k + \sum_{j=1}^N a_{kj}(r_j - r_k),$$
$$\dot{v}_k = -(\omega + \delta_k) r_k + \sum_{j=1}^N a_{kj}(v_j - v_k).$$

With stack vectors $r = [r_1, ..., r_N]^\mathsf{T}$ and $v = [v_1, ..., v_N]^\mathsf{T}$, the dynamics of the network can compactly be written as

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -L & \omega I + \Delta \\ -\omega I - \Delta & -L \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix}$$

where $\Delta = \text{diag}(\delta)$ and $\delta = [\delta_1, ..., \delta_N]^\mathsf{T}$.

Suppose the directed graph $\mathscr{G}$ is strongly connected. Then, there exists a unique vector $p \in \mathbb{R}^N$ such that $p^T L = 0$, $p^T \mathbf{1} = 1$ and $p_k > 0$ for $k = 1, ..., N$, i.e., the left eigenvector of L corresponding to the eigenvalue zero has positive elements, cf., Corollary 9. In order to assess stability of the system above, we use the Lyapunov function

$$V = r^\mathsf{T} P r + v^\mathsf{T} P v,$$

where $P = \text{diag}(p)$. V is positive definite by Lemma 10. The Lie-derivative of V is

$$\begin{aligned} \dot{V} &= \begin{bmatrix} r \\ v \end{bmatrix}^\mathsf{T} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix}^\mathsf{T} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} \\ &= -\begin{bmatrix} r \\ v \end{bmatrix}^\mathsf{T} \begin{bmatrix} PL + L^\mathsf{T} P & \Delta P - P\Delta \\ P\Delta - \Delta P & PL + L^\mathsf{T} P \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} \\ &= -\begin{bmatrix} r \\ v \end{bmatrix}^\mathsf{T} \begin{bmatrix} PL + L^\mathsf{T} P & 0 \\ 0 & PL + L^\mathsf{T} P \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} \\ &= -r^\mathsf{T}(PL + L^\mathsf{T} P) r - v^\mathsf{T}(PL + L^\mathsf{T} P) v. \end{aligned}$$

The cross terms between r and v cancel since the diagonal matrices P and Δ commute. The resulting Lie-derivative $\dot{V}$ is negative semi-definite, i.e.,

$$\dot{V} \le 0$$

by Lemma 10. The set on which $\dot{V} = 0$ is given by

$$\mathscr{I} = \{x \in \mathbb{R}^{2N} : r, v \in \text{im}(\mathbf{1})\},$$

where $x = [x_1^\top, \ldots, x_N^\top]^\top$. Since $L\mathbf{1} = \mathbf{0}$, the dynamics on $\mathscr{I}$ are given by

$$\dot r = (\omega I + \Delta)v,$$
$$\dot v = -(\omega I + \Delta)r.$$

Hence, $\mathscr{I}$ is invariant, if and only if $\delta \in \mathrm{im}(\mathbf{1})$. From there we can see that the oscillators synchronize in case they have all identical frequencies. However, by assumption there exist two agents k, j in the network, for which $\delta_k \neq \delta_j$. Therefore $\dot r, \dot v \notin \mathrm{im}(\mathbf{1})$ and $\mathscr{I}$ is not invariant. Thus the only trajectory contained in $\mathscr{I}$ is $r \equiv 0$, $v \equiv 0$. By LaSalle's invariance principle, it follows that the equilibrium $r = 0$, $v = 0$, i.e., $x = 0$, is asymptotically stable. ∎

In case of strongly connected graphs, the static diffusive couplings between the oscillators have a stabilizing effect in case of non-identical frequencies. The following theorem characterizes general connected directed graphs and is the main result of this section.

Theorem 16. Consider a network of N harmonic oscillators interconnected by static diffusive couplings, i.e.,

$$\dot x_k = (\omega + \delta_k)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \sum_{j=1}^{N} a_{kj}(x_j - x_k)$$

for $k = 1, \ldots, N$. Suppose that the directed graph $\mathscr{G}$ is connected. Then, the network of oscillators is asymptotically stable if and only if there exists a pair k, j of oscillators in the iSCC of $\mathscr{G}$ such that $\delta_k \neq \delta_j$.

Proof. *Only if:* Within the iSCC, the network is strongly connected. The oscillators contained in the iSCC are not affected by the oscillators which are not contained in the iSCC. Assume that all oscillators in the iSCC have identical frequencies. Then, they synchronize to a (generally) non-trivial trajectory, cf., proof of Lemma 15, and the network is not asymptotically stable.

If: Since $\mathscr{G}$ is connected, there exists a vertex permutation such that the Laplacian L reduces to the Frobenius normal form (1), cf., Lemma 4, where the first block L_{11} corresponds to the iSCC, i.e., vertexes $\mathscr{V}_{iSCC}$. By assumption there exists a pair k, j of oscillators in the iSCC of $\mathscr{G}$ such that $\delta_k \neq \delta_j$. It follows from Lemma 15 that $x_k \to 0$ as $t \to \infty$ for all $k \in \mathscr{V}_{\mathrm{iSCC}}$.

It remains to show that this implies $x_j \to 0$ as $t \to \infty$ for $j \in \mathscr{V} \backslash \mathscr{V}_{\mathrm{iSCC}}$. We partition the vectors r, v according to the size of the blocks on the diagonal of L, i.e., $r = [r_1^\top, \ldots, r_m^\top]^\top$, $v = [v_1^\top, \ldots, v_m^\top]^\top$, and $\Delta = \mathrm{diag}(\Delta_{11}, \ldots, \Delta_{mm})$. This yields

$$\dot r_i = -L_{ii} r_i - \sum_{l=1}^{i-1} L_{il} r_l + (\omega I_{ii} + \Delta_{ii}) v_i,$$
$$\dot v_i = -L_{ii} v_i - \sum_{l=1}^{i-1} L_{il} v_l - (\omega I_{ii} + \Delta_{ii}) r_i,$$

for $i = 2, \ldots, m$. By Lemma 4, each L_{ii} has at least one row with positive row sum. Therefore it is possible to decompose $L_{ii} = \tilde L_{ii} + D_{ii}$, such that $\tilde L_{ii}$ is the Laplacian matrix corresponding to a strongly connected graph $\mathscr{G}_{ii}$ and D_{ii} is a non-negative diagonal matrix with at least one positive element, cf., Brualdi and Ryser [1991]. Now we prove asymptotic stability blockwise by induction. For block $i = 1$, exponential stability follows from Lemma 15. For any block $i > 1$, it can be shown that $r_i, v_i \to 0$ as $t \to \infty$ if $r_l, v_l \to 0$ as $t \to \infty$ for $l = 1, \ldots, i-1$ by the following argumentation. If $r_l, v_l \to 0$ as $t \to \infty$ for $l = 1, \ldots, i-1$, then the dynamics of r_i, v_i are asymptotically described by

$$\dot r_i = -(\tilde L_{ii} + D_{ii}) r_i + (\omega I_{ii} + \Delta_{ii}) v_i,$$
$$\dot v_i = -(\tilde L_{ii} + D_{ii}) v_i - (\omega I_{ii} + \Delta_{ii}) r_i.$$

Consider the Lyapunov function $V_i = r_i^\top P_{ii} r_i + v_i^\top P_{ii} v_i$, where $P_{ii} = \mathrm{diag}(\tilde p_i)$ is the diagonal matrix consisting of the elements of the left eigenvector $\tilde p_i$ of $\tilde L_{ii}$ corresponding to zero. Since $\mathscr{G}_{ii}$ is strongly connected, $P_{ii} > 0$ and hence V_i is positive definite. Furthermore, we obtain

$$\begin{aligned}\dot V_i &= -r_i^\top (P_{ii}(\tilde L_{ii} + D_{ii}) + (\tilde L_{ii} + D_{ii})^\top P_{ii}) r_i \\ &\quad - v_i^\top (P_{ii}(\tilde L_{ii} + D_{ii}) + (\tilde L_{ii} + D_{ii})^\top P_{ii}) v_i \\ &= -r_i^\top (P_{ii}\tilde L_{ii} + \tilde L_{ii}^\top P_{ii}) r_i - v_i^\top (P_{ii}\tilde L_{ii} + \tilde L_{ii}^\top P_{ii}) v_i \\ &\quad - r_i^\top (2P_{ii}D_{ii}) r_i - v_i^\top (2P_{ii}D_{ii}) v_i.\end{aligned}$$

It holds that $(P_{ii}\tilde L_{ii} + \tilde L_{ii}^\top P_{ii}) \geq 0$ and $\ker(P_{ii}\tilde L_{ii} + \tilde L_{ii}^\top P_{ii}) = \mathrm{im}(\mathbf{1})$. Since $(2P_{ii}D_{ii})$ is a non-negative diagonal matrix with at least one positive element, $\mathbf{1}^\top (2P_{ii}D_{ii})\mathbf{1} > 0$ and therefore $\dot V_i < 0$, i.e., the Lie-derivative of the Lyapunov function is negative definite. This proves that $r_i, v_i \to 0$ as $t \to \infty$. By induction, we conclude that $r, v \to 0$ as $t \to \infty$ and hence $x \to 0$ as $t \to \infty$. ∎

Theorem 16 shows that the iSCC of a connected graph plays an important role. In particular, the network of oscillators is asymptotically stable if and only if there is a pair of oscillators inside the iSCC which do not have identical frequencies. Furthermore, this shows that (non-trivial) synchronization of harmonic oscillators via static diffusive couplings is not at all robust with respect to variations of the frequencies. It suffices to change the frequency of one single oscillator in the iSCC by an arbitrarily small $\varepsilon > 0$ in order to render the entire network asymptotically stable.

Example 17. Consider a network of 9 oscillators according to Figure 1 with nominal frequency $\omega = 10$ and random offset $\delta_k \in [0, 3]$, $k = 1, \ldots, N$. Consequently the oscillators in $\mathscr{V}_{iSCC}$ have non-identical frequencies. The simulation in Figure 2 shows that the network is asymptotically stable. Next, the frequencies of all oscillators within the iSCC are set to the nominal value $\omega = 10$. The oscillators outside the iSCC have random frequency offsets. The corresponding simulation result is shown in Figure 3. The oscillators within the iSCC synchronize and excite the oscillators outside the iSCC, as expected from Theorem 16.

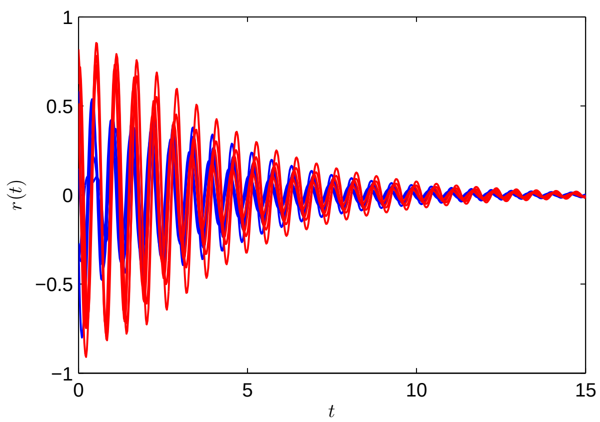

Fig. 2. Simulation of a network of harmonic oscillators with heterogeneous iSCC. The oscillators in the iSCC (red line) and outside the iSCC (blue line) are stabilized.

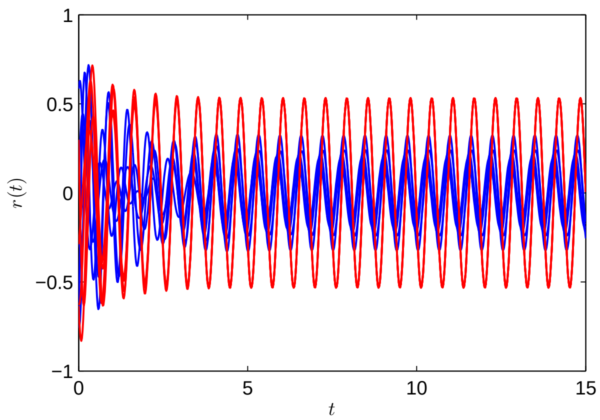

Fig. 3. Simulation of a network of harmonic oscillators with homogeneous iSCC. The identical oscillators in the iSCC (——) synchronize asymptotically and excite the oscillators outside the iSCC (——).

5. HETEROGENEOUS DOUBLE-INTEGRATORS

In this section, heterogeneous networks of double-integrators are analyzed. The agents (2) are given with matrices

$$A_k = \begin{bmatrix} 0 & 1+\delta_k \\ 0 & 0 \end{bmatrix},$$

where $\delta_k \in \mathbb{R}$ and $B_k = C_k = I$, and the coupling gains in (3) are $K_k = I$ for $k = 1,\dots,N$.

A candidate matrix S for the matrix A_k has to fulfill $\sigma(S) \subseteq \sigma(A_k) = \{0,0\}$ for $k = 1,\dots,N$. Thus, there are three candidates

$$S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad S_3 = 0.$$

Since $C_k = I$, condition (5) yields $\Pi_k = \Pi_j$ for all $k, j \in \{1,\dots,N\}$. Therefore, a necessary condition for non-trivial synchronization in the double-integrator network is that there exists a matrix Π with full column rank such that $A_k\Pi = \Pi S_l$ for some $l \in \{1,2,3\}$ and all $k = 1,\dots,N$. In general, $\delta_k \neq -1$ and thus there is no solution for $l = 1$. There is also no solution for $l = 2$. For $l = 3$, the condition is fulfilled for $\Pi = [1,\ 0]^\top$. This is not surprising since the internal model S_3 is contained in A_k as the lower right element, exact synchronization to a trajectory generated by a single-integrator may be possible.

In contrast, exact synchronization to a trajectory generated by a double-integrator model is not possible. The following theorem characterizes the dynamic behavior of the network.

Theorem 18. Consider a network of N double-integrator agents interconnected by static diffusive couplings, i.e.,

$$\dot{x}_k = (1+\delta_k)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + \sum_{j=1}^{N} a_{kj}(x_j - x_k), \tag{8}$$

for $k = 1,\dots,N$. Suppose that the directed graph $\mathscr{G}$ is connected. Furthermore, suppose that there exists a pair k, j of agents such that $\delta_k \neq \delta_j$. Let $x_k = [r_k, v_k]^\top$, $p^\top L = \mathbf{0}^\top$, and $p^\top \mathbf{1} = 1$. Then, $v(t) \to \mathbf{1}p^\top v_0$ as $t \to \infty$ and the states $r(t)$ do not synchronize but grow asymptotically with constant and identical speed. In particular, $(r(t) - r_\perp) \to \operatorname{im}(\mathbf{1})$ and $\dot{r}(t) \to \mathbf{1}(p^\top v_0 + c)$ as $t \to \infty$, where $c \in \mathbb{R}$ and the asymptotic disagreement $r_\perp \in \mathbb{R}^N$ with $\mathbf{1}^\top r_\perp = 0$ are given by

$$\begin{bmatrix} r_\perp \\ c \end{bmatrix} = \begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} \delta p^\top v_0 \\ 0 \end{bmatrix}. \tag{9}$$

Proof. The dynamics of the network (8) can be written as

$$\dot{r}_k = (1+\delta_k)v_k + \sum_{j=1}^{N} a_{kj}(r_j - r_k),$$

$$\dot{v}_k = \sum_{j=1}^{N} a_{kj}(v_j - v_k),$$

for $k = 1,\dots,N$. The states v_k of the agents form a classical single-integrator network. With stack vectors r, v, and diagonal matrix $\Delta = \operatorname{diag}(\delta)$, we obtain

$$\dot{r} = -Lr + (I+\Delta)v, \tag{10}$$

$$\dot{v} = -Lv. \tag{11}$$

The network (11) converges to consensus exponentially, in particular

$$v(t) \to \mathbf{1}p^\top v_0 \quad \text{as} \quad t \to \infty, \tag{12}$$

where $v(0) = v_0$, cf., Wieland [2010]. Suppose that $r, v \in \operatorname{im}(\mathbf{1})$. Then, $\dot{r} = (I+\Delta)v \notin \operatorname{im}(\mathbf{1})$ since $\delta \notin \operatorname{im}(\mathbf{1})$ by assumption. Thus, $\operatorname{im}(\mathbf{1})$ is not invariant for (10) and the states $r(t)$ do not synchronize, i.e., $r(t) \nrightarrow \operatorname{im}(\mathbf{1})$ as $t \to \infty$.

Let $\xi = \dot{r}$. Then, with (10) and (11),

$$\dot{\xi} = -L\xi - (I+\Delta)Lv.$$

Due to (12), $Lv(t) \to \mathbf{0}$ as $t \to \infty$, and $\xi(t)$ converges exponentially to a solution of the unforced system $\dot{\xi} = -L\xi$. Hence,

$$\dot{r}(t) = \xi(t) \to \operatorname{im}(\mathbf{1}) \quad \text{as} \quad t \to \infty. \tag{13}$$

Asymptotically, the states $r(t)$ grow with constant and identical velocity. From (10), (12), and (13), it follows that for $t \to \infty$,

$$-Lr(t) + \delta p^\top v_0 \in \operatorname{im}(\mathbf{1}). \tag{14}$$

The state r can be decomposed into a sum of two components, one component in the subspace $\operatorname{im}(\mathbf{1})$ and the other, denoted by $r_\perp$, in the orthogonal complement $\operatorname{im}(\mathbf{1})^\perp$ of $\operatorname{im}(\mathbf{1})$, i.e., $\mathbf{1}^\top r_\perp = 0$. We are interested in the component $r_\perp$ since it determines the distance of r from $\operatorname{im}(\mathbf{1})$. The quantity $\|r_\perp\|$ can be seen as asymptotic synchronization error.

Since $L\mathbf{1} = \mathbf{0}$, it holds that $Lr = Lr_\perp$ and therefore with (14), $-Lr_\perp + \delta p^\top v_0 \in \operatorname{im}(\mathbf{1})$. This can be rewritten as $Lr_\perp + c\mathbf{1} = \delta p^\top v_0$ for some $c \in \mathbb{R}$, or equivalently,

$$\begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} r_\perp \\ c \end{bmatrix} = \begin{bmatrix} \delta p^\top v_0 \\ 0 \end{bmatrix} \tag{15}$$

It holds that $\operatorname{im}(L)^\perp = \ker(L^\top) = \operatorname{im}(p)$, where $p^\top L = \mathbf{0}^\top$, $p^\top \mathbf{1} = 1$. Since $p^\top \mathbf{1} \neq 0$, it follows that $\operatorname{im}([L\,\mathbf{1}]) = \mathbb{R}^N$, i.e., the rank of the $(N \times (N+1))$-matrix $[L\,\mathbf{1}]$ is N. It holds that $[L\,\mathbf{1}][\mathbf{1}^\top\ 0]^\top = \mathbf{0}$ and $[\mathbf{1}^\top\ 0][\mathbf{1}^\top\ 0]^\top \neq 0$. Therefore the matrix

$$\begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}$$

has full rank $(N+1)$, i.e., is invertible, and the linear system of equations (15) has the unique solution (9). With (10), we can finally conclude that $\dot{r}(t) \to \mathbf{1}(p^\top v_0 + c)$ as $t \to \infty$, i.e., the constant c is the deviation of the agents' velocity from the nominal case, where $\dot{r}(t) \to \mathbf{1}p^\top v_0$ as $t \to \infty$. ■

Theorem 18 shows that networks of double-integrators with static diffusive couplings have a certain robustness with respect to heterogeneity in the dynamics, in the sense that they synchronize practically for small parameters δ_k, $k = 1,\dots,N$. Moreover, the velocities of the agents synchronize for arbitrary parameters δ_k. Both the final velocity and the asymptotic offsets between the agents can be computed explicitly according to (9), depending on the graph topology, parameters δ, and the initial state.

Example 19. Consider a network of 9 double-integrators according to Figure 1, with random non-identical parameters $\delta_k \in [-0.5, 0.5]$ with $\mathbf{1}^\mathsf{T}\delta = 0$. Simulation results for random initial conditions are shown in Figure 4. As expected from Theorem 18, the states $v(t)$ synchronize whereas the states $r(t)$ grow asymptotically with constant and identical speed, while maintaining constant offsets $r_\perp$ according to (9), i.e.,

$$r_\perp \approx [-0.29, -0.03, 0, -0.1, -0.13, -0.12, 0.26, 0.13, 0.28]^\mathsf{T}.$$

In Figure 4, it can also be seen that the final velocity of the states $r(t)$ is $p^\mathsf{T} v_0 + c \approx 0.46$. It differs by $c \approx -0.08$ from the velocity in the nominal case.

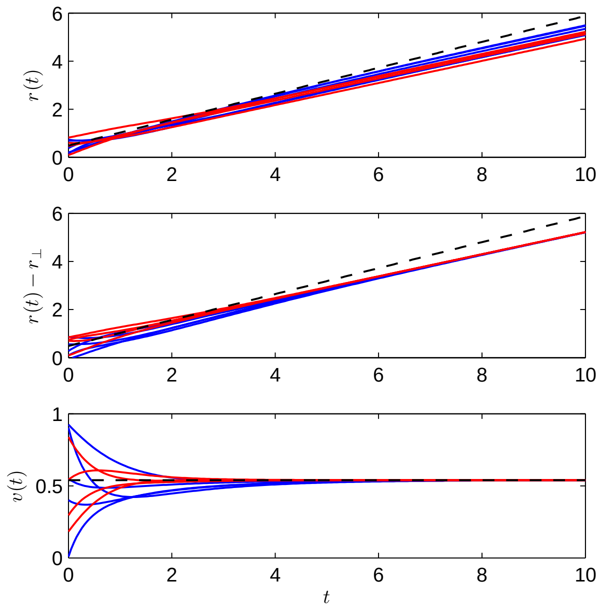

Fig. 4. Simulation of a network of non-identical double-integrators. The dashed lines (**– – –**) indicate the asymptotic solution in the nominal case ($\delta = \mathbf{0}$). The second states $v(t)$ of all agents reach consensus (bottom). The first states $r(t)$ grow with constant and identical speed but with constant offsets $r_\perp$ (top), i.e., $(r(t) - r_\perp) \to \mathrm{im}(\mathbf{1})$ (middle).

6. CONCLUSIONS

We have investigated heterogeneous linear networks with static diffusive couplings among neighboring agents. We have discussed the internal model principle as a necessary condition for synchronization in this setup. This condition is not fulfilled in networks of harmonic oscillators with non-identical frequencies. The analysis of the dynamic behavior of such networks revealed that heterogeneous networks of harmonic oscillators are rendered asymptotically stable, if and only if there are non-identical oscillators within the iSCC of the connected directed graph. Heterogeneous networks of double-integrator agents with static diffusive couplings show a different behavior. The second states of all agents reach consensus as in the nominal case, whereas the first states of all agents maintain constant offsets depending on the graph topology and the heterogeneity in the network. Nevertheless, the first states grow with constant and identical velocity, i.e., the network "practically" synchronizes for small heterogeneity.

Future work will further investigate the robustness of synchronization methods in networks of linear systems with respect to uncertainties and heterogeneities in the dynamics of individual agents.

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