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A graph-theoretic approach on optimizing informed-node selection in multi-agent tracking control☆

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HIGHLIGHTS

- Structure optimization for leader–follower multi-agent systems is studied.
- Both upper bound and lower bounds of the convergence rate are obtained.
- It is shown that the convergence rate depends on a leader-induced distance.
- An approximate solution is presented by solving a metric k-center problem.

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ABSTRACT

A graph optimization problem for a multi-agent leader–follower problem is considered. In a multi-agent system with n followers and one leader, each agent's goal is to track the leader using the information obtained from its neighbors. The neighborhood relationship is defined by a directed communication graph where k agents, designated as informed agents, can become neighbors of the leader. This paper establishes that, for any given strongly connected communication graph with k informed agents, all agents will converge to the leader. In addition, an upper bound and a lower bound of the convergence rate are obtained. These bounds are shown to explicitly depend on the maximal distance from the leader to the followers. The dependence between this distance and the exact convergence rate is verified by empirical studies. Then we show that minimizing the maximal distance problem is a metric k-center problem in classical combinatorial optimization studies, which can be approximately solved. Numerical examples are given to illustrate the properties of the approximate solutions.

1. Introduction

In the past decade numerous research efforts have been devoted to the study of the coordination of a group of autonomous agents, in order to obtain a better understanding of the mechanism of distributed decision-making of multi-agent systems from various scientific areas including biology, engineering, and social science [1–8].

In most cases individuals in a multi-agent system are considered as equal members [4,9,10]. On the other hand, motivated from biological systems such as animal groups [1] or robotics networks [11,6], leader–follower models are introduced where the agents in the system are categorized according to the different functioning abilities. A "leader", or informed agent, is usually a special agent which carries more information in order to guide the whole group, while a "follower" is usually equipped with simple functions based on the information received from the leaders and other connected agents. Leader–follower models have been extensively studied in the literature in terms of controllability, formation, and target tracking [12–20].

However, few works have discussed how to choose the leaders in an optimal way such that the overall multi-agent system can reach a better performance. The selection of effective leadership was discussed in [1] for biological systems where it was shown that the larger the group the smaller the proportion of informed individuals needed to guide the group. The optimal choice of node dynamics for multi-agent networks to reach a fast consensus was studied in [21–23], where the structure of the system was fixed and optimization was carried out on the weights of the arcs, i.e., strength of the information flow. Distributed leader-selection was studied for a formation of autonomous systems where agents do not communicate with each other directly [24].

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In this paper, we consider an informed-agents selection problem for tracking control of first-order multi-agent systems. There are n follower nodes trying to track a static leader, where only k of them can be connected to the leader. These followers that can communicate to the leader are called *informed* agents. The weights of the arcs are normalized, so the optimal choice of the selected informed agents leads to a structure optimization problem. Both upper and lower bounds are established for the convergence rates, which is determined by the maximal distance in the communication graph from the leader to the followers. The optimal choice for the k informed follower nodes that are connected to the leader can be approximately solved by minimizing this maximal distance. We show that it is in fact a metric k-center problem in classical combinatorial optimization studies [25,26].

The rest of the paper is organized as follows. Section 2 introduces the system model and the precise definition of the considered problem. In Section 3, we establish the convergence rate of the network dynamics and present upper and lower bounds for the convergence rate given by the maximal distance in the communication graph from the leader to the followers, which is a graphical metric of the communication graph. A numerical example is provided to explore the relation between this graphical metric and the actual convergence rate. Section 4 discusses an approximate solution for the optimal informed agents selection by solving a k-center problem, and numerical simulations are presented to illustrate the properties of the solutions. Finally some concluding remarks are given in Section 5.

2. Problem definition

Consider a multi-agent system consisting of n follower agents and one leader agent. The set of the follower agents is denoted as $\mathcal{V}_F \triangleq \{v_1, \dots, v_n\}$, and the leader agent is denoted as v_0. Then $\mathcal{V} = \{v_0, v_1, \dots, v_n\}$ is the overall agent set (including the leader and followers). The underlying communication of the follower agents is described by a directed graph $\mathcal{G}_F = (\mathcal{V}_F, \mathcal{E}_F)$, where elements in $\mathcal{E}_F$ are arcs as ordered pair of nodes. We call $j \in \mathcal{V}_F$ a neighbor of node $i \in \mathcal{V}_F$ if there is an arc $(j, i) \in \mathcal{E}_F$.

When there is no possible confusion, we will identify node v_i with its index i. For the communication graph of the follower nodes, we use the following assumption.

Assumption (*Connectivity*). $\mathcal{G}_F$ is strongly connected.

The leader node v_0 keeps a static state, denoted by $\theta_0 \in \mathbb{R}$. The state of follower node $v_i \in \mathcal{V}_F$ is denoted as $x_i(t) \in \mathbb{R}$, $i = 1, \dots, n$. The goal of the followers is to track the leader, i.e., to reach a state consensus at θ_0. The evolution of the followers' states is given by

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} \left(x_j(t) - x_i(t)\right) + b_i\left(\theta_0 - x_i(t)\right), \quad i = 1, \dots, n \qquad (1)$$

where $\mathcal{N}_i$ represents the neighbor set of i, defined by $\mathcal{N}_i = \{j \in \mathcal{V}_F : (j, i) \in \mathcal{E}_F\}$, and

$$b_i = \begin{cases} 1, & \text{if } v_i \text{ is connected to } v_0, \\ 0, & \text{otherwise} \end{cases}$$

marks whether node $i \in \mathcal{V}_F$ is connected to the leader or not. If a node v_i is connected to the leader, we assume there is an arc from v_0 to v_i, and it is called an *informed* follower. The overall communication graph for both the leader and the followers is then denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Remark 1. The node dynamics (1) is obtained through distributed controller with each node taking feedback of the state difference from its neighbors. Such setups have been widely used in the literature on continuous-time multi-agent systems [4,27,9,28,19,20].

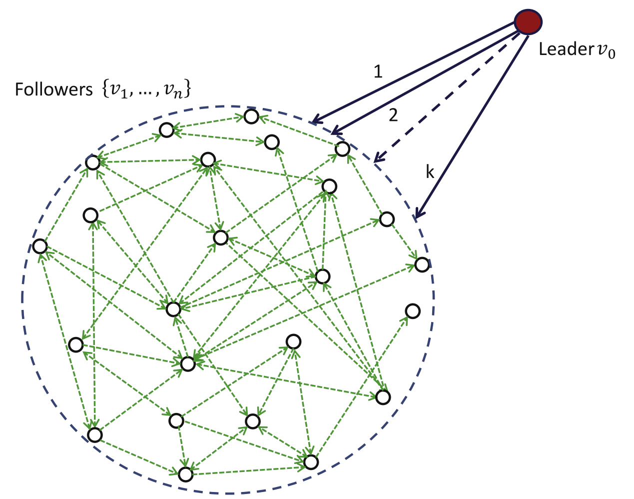

Fig. 1. What nodes should be connected to the leader for a faster convergence?

In this paper, we consider the following structural optimization problem regarding the convergence rate of the multi-agent system (see Fig. 1).

Problem. How should $0 \le k < n$ informed followers be selected so that the fastest consensus is reached?

3. Leader-induced diameter: convergence rate estimations

Denote $\xi_i(t) = x_i(t) - \theta_0$. Then System (1) is transformed into the following form:

$$\dot{\xi}_i = \sum_{j \in \mathcal{N}_i} \left(\xi_j - \xi_i\right) - b_i \xi_i. \qquad (2)$$

Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be the adjacency matrix of graph $\mathcal{G}_F$, where a_{ij}'s take values only from $\{0, 1\}$ and $a_{ij} = 1$ only if $(j, i) \in \mathcal{E}_F$. Denote $D = \mathrm{diag}(d_1 \dots d_n)$, where $d_i = \sum_{j=1, j \neq i}^{n} a_{ij}$. Then $L_F = D - A$ is the Laplacian matrix of $\mathcal{G}_F$. Let $B = \mathrm{diag}(b_1, \dots, b_n)$ mark the connections from the leader to the followers. Denoting $\xi = (\xi_1 \dots \xi_n)^T$, we can now rewrite (2) in a compact form:

$$\dot{\xi} = -\left(L_F + B\right)\xi. \qquad (3)$$

Now we see that consensus convergence of (1) is equivalent to the asymptotic stability of (3). Thus, the convergence rate of (1) to a consensus is determined by the minimal real part of all eigenvalues of $L_F + B$, i.e., the stability margin of (3). Therefore, in order for reaching a fastest consensus given only k informed follower nodes, we need to solve the following problem:

$$\begin{aligned} \text{maximize} \quad & \min_{\lambda_i \in \sigma(L_F + B)} \mathrm{Re}(\lambda_i) \\ \text{subject to} \quad & b_i \in \{0, 1\} \\ & \sum_{i=1}^{n} b_i = k. \end{aligned} \qquad (4)$$

Here $\sigma(L_F + B)$ is the spectrum of $L_F + B$.

However, (4) is a non-convex problem, and there are overall $\frac{n!}{k!(n-k)!} = \binom{n}{k} \approx O(n^k)$ possible options for choosing the followers connected to the leader when $k \ll n$. In other words, for a network with a large number of nodes, even if the complexity of computing the spectrum of $L_F + B$ is ignored, finding the exact solution of (4) is practically impossible. Then it is natural to ask, can we find some graph-theoretical measures that play an important role in influencing the convergence rate? If we do, classical graph

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optimization methods can be adopted to provide approximate solutions.

The rest of the section focuses on the estimations of the convergence rate. Upper and lower bounds are established based on a generalized diameter of the underlying communication graph. Then an empirical example will be presented to verify the dependence of the stability margin of $L_F + B$ on this generalized diameter.

Remark 2. In [23,22,21], the optimization of arc weights was considered for reaching fastest consensus on leaderless multi-agent systems, where the underlying communication graph is fixed. Here we consider the fastest consensus problem for a continuous-time leader–follower model from another perspective: the weights of arcs are fixed to one, then how can we choose the best communication graph?

3.1. Convergence rate upper bound

Introduce

$$M(t) = \max\{\theta_0, x_i(t), i = 1, \dots, n\},$$
$$m(t) = \min\{\theta_0, x_i(t), i = 1, \dots, n\}.$$

Denote $V(t) = M(t) - m(t)$ as the convergence measure. Let

$$d_* = \max_{i \in \mathcal{V}_F} \mathrm{d}(v_0, v_i)$$

be the maximum distance from the leader to the followers. Here $\mathrm{d}(v_0, v_i)$ represents the length of the shortest path from v_0 to v_i in graph $\mathcal{G}$. We call d_* the *leader-induced diameter* for the considered multi-agent system.

The following result gives an upper bound for the convergence rate of System (1).

Proposition 1. *Denote $\mu_0 = 1 - e^{-nT}$. For all node states $x_1(t), \dots, x_n(t)$ and $T > 0$, we have*

$$V(t + d_*T) \le \left(1 - \frac{1}{2}\left(\frac{\mu_0}{n}\right)^{d_*}\right) V(t).$$

Proof. It is straightforward to see that $m(t)$ is nondecreasing, and $M(t)$ is non-increasing. Pick a t_* and assume that

$$\theta_0 \le \frac{1}{2} m(t_*) + \frac{1}{2} M(t_*). \tag{5}$$

The case when (5) does not hold is considered at the end of the proof.

Denote $\mathcal{V}_1 = \{j \in \mathcal{V}_F : \mathrm{d}(v_0, v_j) = 1\}$. Then $\mathcal{V}_1$ contains the nodes which are directly connected to the leader. Take $i_1 \in \mathcal{V}_1$. We have

$$\begin{aligned}
\frac{d}{dt} x_{i_1}(t) &= \sum_{j \in \mathcal{N}_{i_1}} \left(x_j(t) - x_{i_1}(t)\right) + b_i\left(\theta_0 - x_{i_1}(t)\right) \\
&\le (n-1)\left(M(t) - x_{i_1}(t)\right) + \left(\theta_0 - x_{i_1}(t)\right) \\
&\le (n-1)\left(M(t_*) - x_{i_1}(t)\right) \\
&\quad + \left(\frac{1}{2} m(t_*) + \frac{1}{2} M(t_*) - x_{i_1}(t)\right),
\end{aligned} \tag{6}$$

for all $t \ge t_*$, which implies

$$\begin{aligned}
x_{i_1}(t) &\le e^{-n(t-t_*)} x_{i_1}(t_*) + \left(1 - e^{-n(t-t_*)}\right) \\
&\quad \times \frac{\frac{1}{2} m(t_*) + \left(n - \frac{1}{2}\right) M(t_*)}{n} \\
&\le e^{-n(t-t_*)} M(t_*) + \left(1 - e^{-n(t-t_*)}\right) \\
&\quad \times \frac{\frac{1}{2} m(t_*) + \left(n - \frac{1}{2}\right) M(t_*)}{n} \\
&= \left(1 - e^{-n(t-t_*)}\right) \frac{1}{2n} m(t_*) \\
&\quad + \left(1 - \left(1 - e^{-n(t-t_*)}\right) \frac{1}{2n}\right) M(t_*)
\end{aligned} \tag{7}$$

by Grönwall's Inequality [29].

This leads to

$$x_{i_1}(t) \le \frac{\mu_0}{2n} m(t_*) + \left(1 - \frac{\mu_0}{2n}\right) M(t_*), \quad t \in [t_* + T, \infty) \tag{8}$$

for all $i_1 \in \mathcal{V}_1$, where $\mu_0 = 1 - e^{-nT}$.

We continue to define $\mathcal{V}_2 = \{j \in \mathcal{V}_F : \mathrm{d}(v_0, v_j) = 2\}$. Then for any node $i_2 \in \mathcal{V}_2$, there exists a node $i_1 \in \mathcal{V}_1$ such that i_1 is a neighbor of i_2. Thus, we obtain

$$\begin{aligned}
\frac{d}{dt} x_{i_2}(t) &= \sum_{j \in \mathcal{N}_{i_2}} \left(x_j(t) - x_{i_2}(t)\right) + b_i\left(\theta_0 - x_{i_2}(t)\right) \\
&\le (n-2)\left(M(t_*) - x_{i_2}(t)\right) \\
&\quad + \frac{\mu_0}{2n} m(t_*) + \left(1 - \frac{\mu_0}{2n}\right) M(t_*) - x_{i_2}(t),
\end{aligned} \tag{9}$$

for $t \in [t_* + T, \infty)$, which yields

$$\begin{aligned}
x_{i_2}(t) &\le e^{-(n-1)(t-t_*-T)} x_{i_2}(t_* + T) \\
&\quad + \left(1 - e^{-n(t-t_*-T)}\right) \frac{\frac{\mu_0}{2n} m(t_*) + \left(n - 1 - \frac{\mu_0}{2n}\right) M(t_*)}{n-1} \\
&\le e^{-(n-1)(t-t_*-T)} M(t_*) \\
&\quad + \left(1 - e^{-n(t-t_*-T)}\right) \frac{\frac{\mu_0}{2n} m(t_*) + \left(n - 1 - \frac{\mu_0}{2n}\right) M(t_*)}{n-1} \\
&\le \frac{\mu_0^2}{2n^2} m(t_*) + \left(1 - \frac{\mu_0^2}{2n^2}\right) M(t_*)
\end{aligned} \tag{10}$$

for all $t \in [t_* + 2T, \infty)$ and $i_2 \in \mathcal{V}_2$.

Proceeding the analysis, $\mathcal{V}_3, \dots, \mathcal{V}_{d_*}$ can be defined, and upper bounds for $x_{i_j}(t) \in \mathcal{V}_j$ can be established, respectively. Eventually we have

$$x_i(t_* + d_*T) \le \frac{\mu_0^{d_*}}{2n} m(t_*) + \left(1 - \frac{\mu_0^{d_*}}{2n}\right) M(t_*), \quad i \in \mathcal{V}_F, \tag{11}$$

which yields

$$M(t_* + d_*T) \le \frac{1}{2}\left(\frac{\mu_0}{n}\right)^{d_*} m(t_*) + \left(1 - \frac{1}{2}\left(\frac{\mu_0}{n}\right)^{d_*}\right) M(t_*). \tag{12}$$

Based on the definition of $V(t)$ and the nondecreasing property of $m(t)$, (12) leads to

$$\begin{aligned}
V(t_* + d_*T) &= M(t_* + d_*T) - m(t_* + d_*T) \\
&\le \frac{1}{2}\left(\frac{\mu_0}{n}\right)^{d_*} m(t_*) \\
&\quad + \left(1 - \frac{1}{2}\left(\frac{\mu_0}{n}\right)^{d_*}\right) M(t_*) - m(t_*) \\
&= \left(1 - \frac{1}{2}\left(\frac{\mu_0}{n}\right)^{d_*}\right) V(t_*).
\end{aligned} \tag{13}$$

For the other case with

$$\theta_0 > \frac{1}{2} m(t_*) + \frac{1}{2} M(t_*), \tag{14}$$

(13) can be established by a symmetric argument estimating the lower bound for $m(t_* + d_*T)$. Moreover, since t_* is chosen arbitrarily, the desired conclusion holds. □

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3.2. Convergence rate lower bound

We present another conclusion on the lower bound of the convergence rate.

Proposition 2. *For any $T > 0$, there exists initial values $x_1(t_0), \ldots, x_n(t_0)$, such that*

$$V(t_0+T) \geq \left(1-\left(1-e^{-(n-1)T}\right)^{d_*}\left(1-e^{-T}\right)\right)V(t_0). \tag{15}$$

Proof. Without loss of generality we assume $\theta_0 = 0$. Take $x_i(t_0) = 1$ for $i = 1, \ldots, n$. Let $\mathcal{V}_1, \ldots, \mathcal{V}_{d_*}$ be defined as the proof of Proposition 1.

For any $i_1 \in \mathcal{V}_1$, it is not hard to see that

$$\frac{d}{dt}x_{i_1}(t) \geq -x_{i_1}(t), \quad t \geq t_0, \tag{16}$$

which implies

$$x_{i_1}(t_0+T) \geq e^{-T}, \quad t \in [t_0, t_0+T]. \tag{17}$$

With (17), for $i_2 \in \mathcal{V}_2$, we have

$$\frac{d}{dt}x_{i_2}(t) = \sum_{j\in\mathcal{N}_{i_2}}\left(x_j(t)-x_{i_2}(t)\right) \geq (n-1)\left(e^{-T}-x_{i_2}(t)\right), \quad t \in [t_0, t_0+T]. \tag{18}$$

This leads to

$$\begin{aligned} x_{i_2}(t) &\geq e^{-(n-1)T} + \left(1-e^{-(n-1)T}\right)e^{-T} \\ &= \left(1-\left(1-e^{-(n-1)T}\right)\right) + \left(1-e^{-(n-1)T}\right)e^{-T}, \\ &\quad t \in [t_0, t_0+T]. \end{aligned} \tag{19}$$

Similarly, for $i_3 \in \mathcal{V}_3$, we conclude that

$$\begin{aligned} x_{i_3}(t) &\geq e^{-(n-1)T} + \left(1-e^{-(n-1)T}\right) \\ &\quad \times \left(e^{-(n-1)T} + \left(1-e^{-(n-1)T}\right)e^{-T}\right) \\ &= \left(1-\left(1-e^{-(n-1)T}\right)^2\right) + \left(1-e^{-(n-1)T}\right)^2 e^{-T}, \\ &\quad t \in [t_0, t_0+T]. \end{aligned} \tag{20}$$

Proceeding the estimation we obtain

$$\begin{aligned} x_i(t_0+T) &\geq \left(1-\left(1-e^{-(n-1)T}\right)^{d_*}\right) + \left(1-e^{-(n-1)T}\right)^{d_*}e^{-T} \\ &= 1-\left(1-e^{-(n-1)T}\right)^{d_*}\left(1-e^{-T}\right) \end{aligned} \tag{21}$$

for all $i \in \mathcal{V}_F$. Since $V(t_0) = 1$, it follows from (21) that

$$V(t_0+T) \geq \left(1-\left(1-e^{-(n-1)T}\right)^{d_*}\left(1-e^{-T}\right)\right)V(t_0). \tag{22}$$

The proof is completed. □

3.3. Empirical verification

In this subsection, we present a numerical example investigating the relation between the stability margin of (3) and the leader-induced diameter, i.e., maximum distance from the leader to the followers $d_* = \max_{i\in\mathcal{V}_F} \mathrm{d}(v_0, v_i)$.

We take 1000 strongly connected samples from directed random Erdös–Rényi (ER) graphs. The graph has 100 nodes, and independently with probability 0.04 there is a directed arc for any ordered pair of nodes (i.e., about 400 arcs on average). For each sample, a leader is randomly chosen, and the corresponding leader-induced diameter and the stability margin of $-(L_F+B)$ are computed.

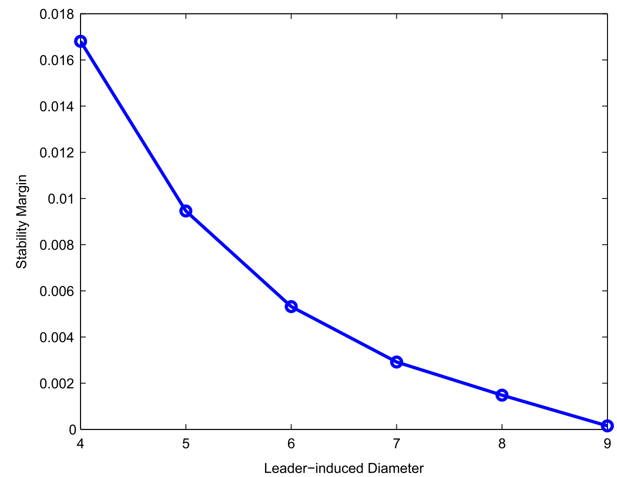


Fig. 2. The vertical coordinate is obtained by taking the average of the stability margin of $-(L_F+B)$ out of the 1000 samples, fixing the leader-induced diameter. Numerical simulation shows the stability margin, i.e., the convergence rate, decreases rapidly as the leader-induced diameter increases.

Fig. 2 shows the mean correlation between the leader-induced diameter and the stability margin of $-(L_F+B)$. The simulation results show that the average stability margin (the exact convergence rate) decreases almost exponentially as the leader-induced diameter increases. This is consistent with the convergence rate bounds established in Propositions 1 and 2.

4. Diameter minimization: an approximate solution

As pointed out in previous discussions, (4) is a non-convex problem, while enumerating all possible conditions takes $O(n^k)$ times of computing the stability margin of $-(L_F+B)$. Therefore, finding the exact best selections of the informed followers in order for reaching the fastest consensus convergence is not scalable.

Now note that the established convergence rate upper bound in Proposition 1, the lower bound in Proposition 2, and the presented empirical example all point out that a faster convergence can be expected if the leader-induced diameter, $d_* = \max_{i\in\mathcal{V}_F} \mathrm{d}(v_0, v_i)$, can be reduced. Therefore, instead of finding the exact solution to (4), a reasonably good approximate solution can be obtained by solving the following optimization problem based purely on the communication graph:

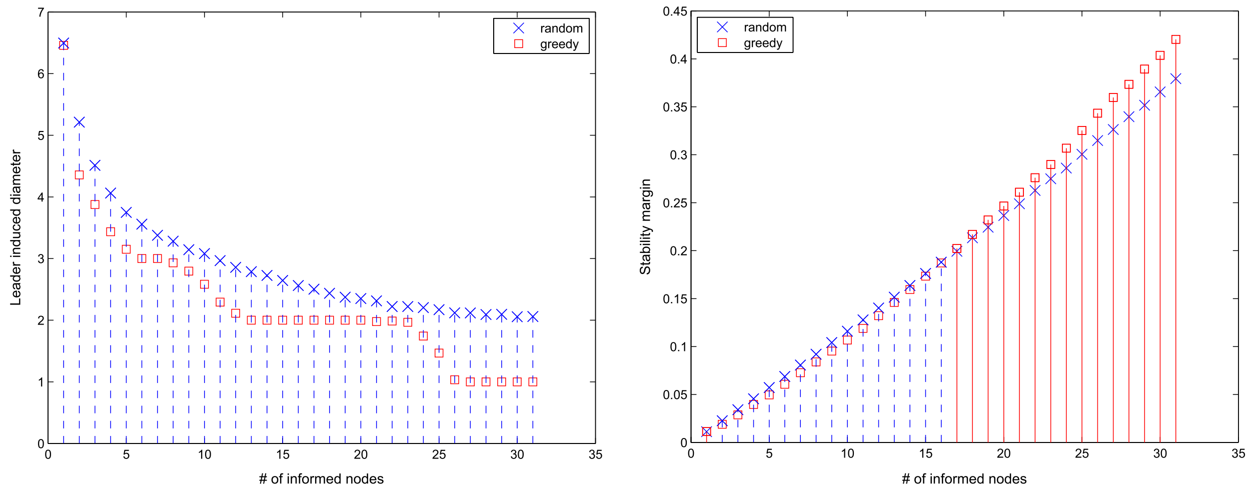
$$\begin{aligned} \text{minimize} \quad & \max_{v_i\in\mathcal{V}_F} \mathrm{d}(v_0, v_i) \\ \text{subject to} \quad & v_{i_1}, \ldots, v_{i_k} \in \mathcal{V}_F \\ & (v_0, v_{i_m}) \in \mathcal{E}, \quad m = 1, \ldots, k. \end{aligned} \tag{23}$$

Based on the definition of the node distance on graphs, it is straightforward to see that problem (23) can be rewritten into the following equivalent form:

$$\begin{aligned} \text{minimize}_S \quad & \max_{v_i\in\mathcal{V}_F}\min_{v_m\in S} \mathrm{d}(v_m, v_i) \\ \text{subject to} \quad & S \subseteq \mathcal{V}_F \\ & |S| = k, \end{aligned} \tag{24}$$

where $\mathrm{d}(v_m, v_i)$ denotes the shortest path from v_m to v_i in graph $\mathcal{G}_F$, S represents the set of selected followers that are connected to the leader and $|S|$ represents the number of elements in S.

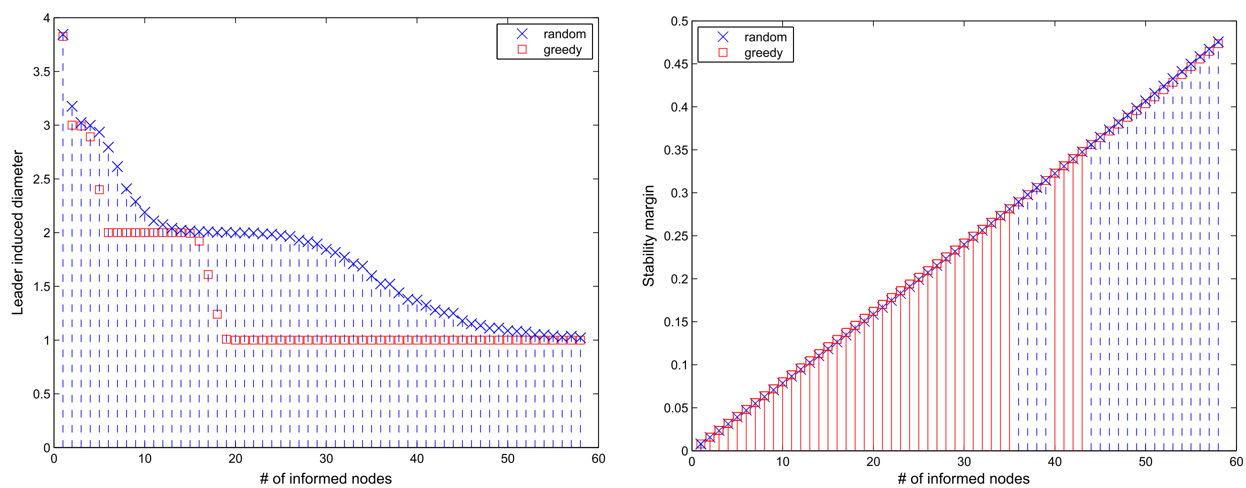
Now we see that problem (24) is a standard k-center problem in combinatorial optimization studies [25,26], which can be solved within an approximation factor of two using $O(nk)$ running time by simple greedy algorithms [30]. This means the approximate solution has twice the optimal diameter in the worst case. Thus, the approximate minimization of the leader-induced diameter for the



(a) Ensemble average of leader-induced diameter.

(b) Ensemble average of stability margin.

Fig. 3. Ensemble averages of leader-induced diameter and stability margin due to random (labeled as random) and approximate k-center (labeled as greedy) strategies, as functions of the number of informed nodes. The underlying network is the 62-node dolphin social network.
Source: From [31].



(a) Ensemble average of leader-induced diameter.

(b) Ensemble average of stability margin.

Fig. 4. Ensemble averages of leader-induced diameter and stability margin due to random (labeled as random) and approximate k-center (labeled as greedy) strategies, as functions of the number of informed nodes. The underlying network is the 115-node American college football network.
Source: From [32].

informed follower selection is a scalable problem. It can be solved even for networks with large number of nodes.

4.1. Numerical examples

We present some numerical examples to demonstrate how the leader-induced diameter and the stability margin are affected by the number of informed nodes and the methods with which these informed nodes are selected. Two selection processes are considered. One process is to randomly choose the informed nodes according to uniform distribution of the nodes, and this selection method is referred to as "random". The other process is to choose the informed nodes based on the greedy approximate k-center solution, and this method is referred to as "greedy".

In the first example the 62-node dolphin social network from [31] is considered. For different numbers of allowable informed nodes up to half the total number of nodes, both the random and greedy methods for informed nodes selection are employed. The greedy method is also randomized because the first informed node is chosen randomly, uniform over possible nodes. For each selection of the informed nodes, the resulted leader-induced diameter and stability margin can be computed. For this example, 1000 instances of informed nodes selections are obtained for each number of allowable informed nodes. The ensemble averages of the leader-induced diameter and stability margin are shown in Fig. 3(a) and (b), respectively. Fig. 3(a) demonstrates that the leader-induced diameters are smaller due to the greedy method, as expected. Fig. 3(b) shows that in this example the stability margin increase due to the greedy method is better than that of the random method, when the number of informed nodes is large.

While the trend in Fig. 3(a) is expected, the trend in Fig. 3(b) needs not hold true in general. For instance, consider the second example where the experiment from the Dolphin network example is repeated with the only exception that the network considered in this case is the 115-node American college football network from [32]. The corresponding results are shown in Fig. 4(a) and (b), respectively. In particular, Fig. 4(a) shows that the greedy

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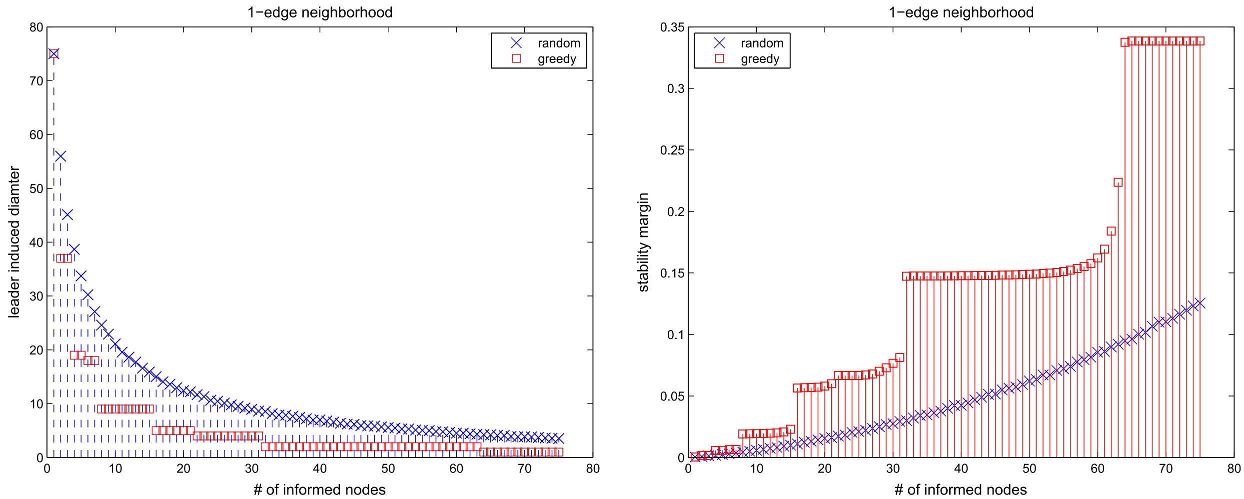
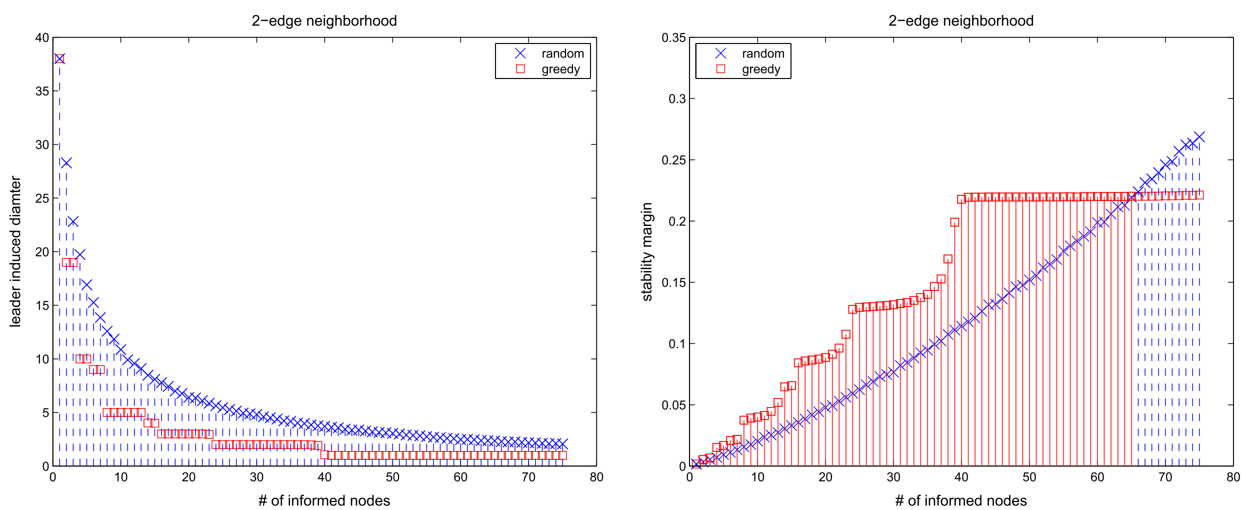


(a) Ensemble average of leader-induced diameter. (b) Ensemble average of stability margin.

Fig. 5. Ensemble averages of leader-induced diameter and stability margin due to random (labeled as random) and approximate k-center (labeled as greedy) strategies, as functions of the number of informed nodes. The underlying network is the 150-node modified ring.
Source: From [33] with $k = 1$.



(a) Ensemble average of leader-induced diameter. (b) Ensemble average of stability margin.

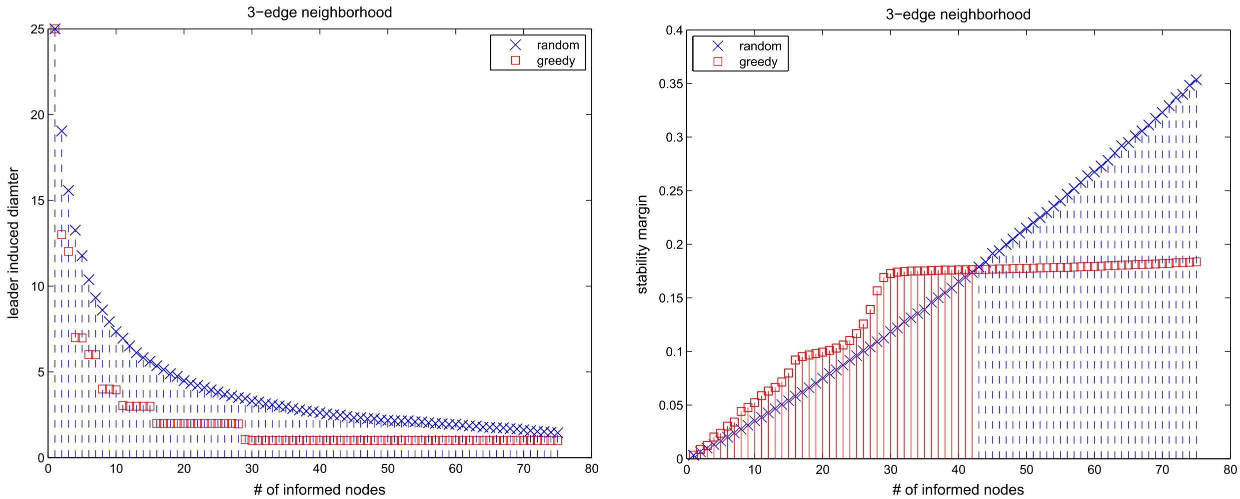
Fig. 6. Ensemble averages of leader-induced diameter and stability margin due to random (labeled as random) and approximate k-center (labeled as greedy) strategies, as functions of the number of informed nodes. The underlying network is the 150-node modified ring.
Source: From [33] with $k = 2$.

method has much better leader-induced diameter reduction than the random method. On the other hand, Fig. 4(b) indicates that the influence of the stability margin is not as clear as in the previous example with the dolphin social network.

The previous two examples consider networks whose nodes are relatively well-connected. The next example considers instead a relatively sparse network, which is a 150-node ring with possible additional edges so that the $k = 1, 2, \ldots$ neighbors of the original ring are connected (see Fig. 2a in [33]). The same experiment as in the previous two examples is repeated. Figs. 5–8 show the leader-induced diameter and stability margin results for the cases with $k = 1, 2, 3, 10$, respectively. In all these figures, for each pair of blue cross and red square associated with a particular number of informed nodes, the blue vertical dash-line means that the random method related quantity (i.e., diameter or stability margin) is greater than the corresponding one for the greedy method. The red vertical solid-line means vice-versa. These results indicate that the greedy method (the one that attempts to minimize leader-induced diameter) is advantageous in terms of both diameter reduction and stability margin increase when the networks are relatively sparsely connected (i.e., $k = 1, 2$). However, for denser networks (e.g., $k = 10$) the greedy method is inferior to the random method.

5. Conclusions

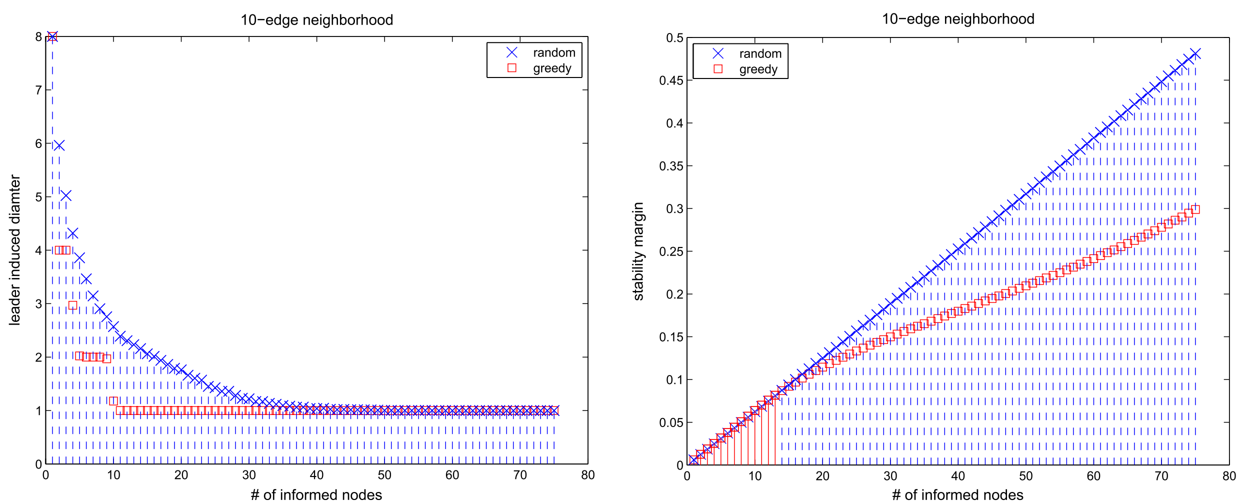
Optimizing the structure of a multi-agent systems for tracking a static leader was considered. There were n follower nodes targeting to track a static leader, where only k of them, i.e., informed followers, can be connected to this leader. Both upper and lower bounds were established for the convergence rates, which are explicitly determined by the maximal distance from the leader among the followers. It was shown that the optimal selection of the k informed followers can be approximately obtained by minimizing this maximal distance, which turns out to be a metric k-center problem in combinatorial optimization, which can be approximately solved within $O(nk)$ running time. Numerical examples were presented



(a) Ensemble average of leader-induced diameter.

(b) Ensemble average of stability margin.

Fig. 7. Ensemble averages of leader-induced diameter and stability margin due to random (labeled as random) and approximate k-center (labeled as greedy) strategies, as functions of the number of informed nodes. The underlying network is the 150-node modified ring.
Source: From [33] with $k = 3$.



(a) Ensemble average of leader-induced diameter.

(b) Ensemble average of stability margin.

Fig. 8. Ensemble averages of leader-induced diameter and stability margin due to random (labeled as random) and approximate k-center (labeled as greedy) strategies, as functions of the number of informed nodes. The underlying network is the 150-node modified ring.
Source: From [33] with $k = 10$.

to verify the proposed convergence bounds and to illustrate the properties of the approximate solutions.

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