Consensus Control for Multi-agent Systems with a Faulty Node

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Abstract: This paper studies consensus control for a multi-agent system with a faulty node. The node dynamics follow a continuous-time consensus protocol with negative feedback from the relative state of the neighbors, where the faulty node is instead using positive feedback from the state. Conditions for reaching consensus are established, and a fault threshold is introduced. Numerical examples investigate how the fault threshold determines the system behavior.

Keywords: Multi-agent systems, Consensus, Faulty agents

1. INTRODUCTION

In recent years, cooperative control of multi-agent systems has been extensively investigated in the literature for the consensus, formation, flocking, aggregation and coverage of a group of autonomous agents, see Jadbabaie et al. (2003); Lin et al. (2005, 2007); Tsitsiklis et al. (1986); Olfati-Saber and Murray (2004); Olfati-Saber and Shamma (2005); Tanner et al. (2007); Ren and Beard (2008); Cortés and Bullo (2005); Shi and Hong (2009); Shi et al. (2012).

In most cases all the agents in the system follow the same control protocols, but sometimes there are also *leaders* present whose role it is to guide the network, and they usually do not follow the same protocol as the remaining nodes, referred to as *followers*. This leads to the so-called leader-follower models in the study of multi-agent systems.

There have been considerable research interests in the study of leader-follower models. The controllability of leader-follower multi-agent systems was introduced in Tanner (2004), in which necessary and sufficient conditions were established for system controllability. The graph-theoretic characterizations of controllability for leader-follower multi-agent systems were further studied in Ji et al. (2006); Rahmani and Mesbahi (2006); Rahmani et al. (2009). In Hong et al. (2006), tracking control for multi-agent consensus with one single active leader was studied with a neighbor-based observer. In Gu and Wang (2009), a leader-follower flocking model was discussed, where only a few agents have the knowledge of a desired trajectory. The leader-to-formation stability was studied for formation control of multi-agent systems in Olfati-Saber and Murray (2004). There has also been work on multiple leaders, e.g., in Couzin et al. (2005), a simple model was given to simulate the effectiveness of leaders guiding a school of fishes to a particular food region. In Lin et al. (2005), a straight-line formation problem was presented, where all the agents' target was to converge to a line segment specified by two edge leaders. A containment control scheme was proposed, with fixed undirected interactions, in Ji et al. (2008), aiming to drive a group of agents to the polytope spanned by several stationary or moving leaders. Further, distributed control protocols were presented to drive a collection of mobile agents to stationary or moving leaders with connectivity-maintenance and collision-avoidance in Cao and Ren (2009). In Meng et al. (2012), swarm tracking problems with group dispersion and cohesion behaviors were discussed for a group of Lagrange systems. Shi and Hong (2009) studied multiple leaders aggregating the whole multi-agent group within a convex target set under mild connectivity and protocol assumptions. In Shi et al. (2012), set input-to-state stability (SISS) and set integral input-to-state stability (SiISS) were introduced for multi-agent network tracking of a set of moving leaders, and critical connectivity conditions were obtained for the system to be SISS or SiISS.

To the best of our knowledge, few works have considered the case when the follower agents are tracking an antagonistic leader (*evader*) rather than a cooperative leader. Related are the classical results of game theory for the so-called pursuit-evasion game, e.g., Basar and Olsder (1999); Ho et al. (1965), where the considered game consists of a pursuer who aims to capture the evader while the evader tries to prevent being captured. In this paper, we present a framework for a multi-agent systems tracking an evader. A motivating example is resilience for a multi-agent system against faults, where the worst case fault is modeled as the faulty agent trying to avoid being tracked by using positive feedback from the relative state of the follower agents. Trackability and escapability are introduced, and conditions are established on the system parameters for the multi-agent system to be trackable or escapable.

In Section 2, we introduce the multi-agent model, and define the trackability problem. In Section 3 the convergence analysis and main results are presented. In Section 4, we show the trackability and escapability characteristics with numerical simulations. Finally, some concluding remarks are given in Section 5.

⋆ This work has been supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council.

2. PROBLEM DEFINITION

Consider a multi-agent system consisting of one faulty agent (*leader/evader*) and n follower agents. The set of agents are denoted by $\mathcal{V} = \{0, 1, \ldots, n\}$, where 0 is the faulty agent, and the remaining follower agents are denoted by $\mathcal{V}^F \triangleq \{1, \ldots, n\}$.

The interaction topology of the multi-agent network is modeled as a switching topology, and can thus be described by a time-varying undirected graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$. Here, $\sigma : [0, +\infty) \to \mathcal{Q}$ is a piecewise constant function, where $\mathcal{Q}$ is a finite set indicating the possible undirected graphs, see Godsil and Royle (2001). $\mathcal{G}^F_{\sigma(t)} = (\mathcal{V}^F, \mathcal{E}^F_{\sigma(t)})$ denotes the induced communication graph among the follower agents. For any two nodes $i, j \in \mathcal{V}$, j is said to be a *neighbor* of node i at time t if there exist an edge between them in $\mathcal{G}_{\sigma(t)}$. Let $\mathcal{N}_i(\sigma(t))$ represent the set of agent i's neighbors in $\mathcal{G}_{\sigma(t)}$, for $i \in \mathcal{V}$ and time t, and $\mathcal{N}^F_i(\sigma(t))$ the set neighbors in $\mathcal{G}^F_{\sigma(t)}$, for $i \in \mathcal{V}^F$.

Each agent $i \in \mathcal{V}^F$ has a state denoted by $x_i \in \mathbb{R}$, and the state of the faulty agent 0 is denoted as $y \in \mathbb{R}$. The dynamics for the agents is described as follows:

(i) The goal of each follower agent $i \in \mathcal{V}^F$ is to reach consensus with every other agent in $\mathcal{V}$. The evolution of state $x_i(t)$ is given by
$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}^F_i(\sigma(t))} a_{ij}(x,t)\big(x_j(t) - x_i(t)\big) + b_i(t)\big(y(t) - x_i(t)\big)$$
where the function
$$b_i(t) = \begin{cases} b_*, & \text{if } i \text{ is connected to the faulty agent} \\ 0, & \text{otherwise.} \end{cases}$$
marks whether agent $i \in \mathcal{V}^F$ is connected to the faulty agent or not, with a given constant $b_* > 0$ and a piecewise continuous function $a_{ij}(x,t) > 0$, $i, j \in \mathcal{V}^F$ describing the weight of edge $\{i, j\}$.

(ii) The goal of the faulty agent is to escape from the followers. The evolution of the faulty agent's state is given by
$$\dot{y}(t) = -\sum_{j \in \mathcal{N}_0(\sigma(t))} \Big(x_j(t) - y(t)\Big).$$

The overall dynamics for the considered multi-agent systems can then be summarized as:
$$\begin{cases} \dot{y}(t) = \displaystyle\sum_{j \in \mathcal{N}_0(\sigma(t))} \big(y(t) - x_j(t)\big), \\ \dot{x}_i(t) = \displaystyle\sum_{j \in \mathcal{N}^F_i(\sigma(t))} a_{ij}(x,t)\big(x_j(t) - x_i(t)\big) \\ \qquad\qquad + b_i(t)\big(y(t) - x_i(t)\big), \quad i = 1, \ldots, n \end{cases} \tag{1}$$

Note that, different to most of the existing leader-follower models Hong et al. (2006); Tanner et al. (2004); Shi and Hong (2009), the faulty agent is observing the follower's states and then takes opposed actions in order to escape from being tracked. The interesting question is whether the faulty agent can be tracked, or if it will escape successfully.

Let $(x(t), y(t)) = (x_1(t), \ldots, x_n(t), y(t))^T \in \mathbb{R}^{n+1}$ denote the solution of (1) with initial value $x^0 = x(0) \in \mathbb{R}^n$ and $y^0 = y(0) \in \mathbb{R}$. Define
$$\Upsilon(t) = \max_{i=1,\ldots,n} \big|x_i(t) - y(t)\big|$$
as the tracking measure. We introduce the following notations.

Definition 1.

(i) System (1) is *trackable* for initial value $x^0 = x(0) \in \mathbb{R}^n$ and $y^0 = y(0) \in \mathbb{R}$ if
$$\lim_{t \to \infty} \Upsilon(t) = 0.$$
(ii) System (1) is *globally trackable* if it is trackable for all initial values.
(iii) System (1) is *escapable* for initial value $x^0 = x(0) \in \mathbb{R}^n$ and $y^0 = y(0) \in \mathbb{R}$ if
$$\lim_{t \to \infty} \Upsilon(t) = \infty.$$

3. MAIN RESULTS

In this section, we present the main results and the convergence analysis. We first study fixed communication graphs, and then investigate time-varying communication graphs.

3.1 Fixed Graphs

This subsection focuses on time-invariant graphs, with the following assumption.

Assumption 2. (Fixed Topology). The communication graph $\mathcal{G}_{\sigma(t)}$ and the functions a_{ij}, b_i are time-invariant, i.e., independent of t.

Hence, in this section we will drop the time parameter t from the topology.

Denote the state difference as $\xi_i(t) = x_i(t) - y(t)$. System (1) can then be written as:
$$\dot{\xi}_i = \sum_{j \in \mathcal{N}^F_i} a_{ij}(\xi_j - \xi_i) - b_i \xi_i + \sum_{j \in \mathcal{N}_0} \xi_j \tag{2}$$
Let $L_F = D - A$ be the Laplacian matrix of the follower graph, given by $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ and $D = \mathrm{diag}(d_1 \ldots d_n)$, where $d_i = \sum_{j=1, j \neq i}^n a_{ij}$ is the node degree. Let $B = \mathrm{diag}(b_1, \ldots, b_n)$ denote the connections from the followers to the faulty agent, and $E = \mathbf{1}e^T$ with $e^T = (e_1 \ldots e_n) \in \mathbb{R}^n$ denote the connections from the faulty agent to the followers, where $e_i = 1$ if $i \in \mathcal{N}_0$ and $e_i = 0$ otherwise. Denoting $\xi = (\xi_1, \ldots, \xi_n)^T$, we can rewrite (2) with the compact form
$$\dot{\xi} = -G\xi, \tag{3}$$
where $G = L_F + B - E$.

Noticing that global tracking for System (1) is equivalent with stability of system (3), the following conclusion follows directly.

Theorem 3. Suppose Assumption 2 holds, then

(i) System (1) is globally trackable if and only if $-G$ is a Hurwitz matrix.
(ii) There exist initial values for which System (1) is escapable if and only if $-G$ has at least one eigenvalue with strictly positive real part.

In fact, if $-G$ has an eigenvalue λ with corresponding eigenvector β_λ which has strictly positive real part, then, for every initial value (x^0, y^0) with $x^0 - y^0\mathbf{1}$ not orthogonal to β_λ, System (1) is escapable.

Although Theorem 3 gives a clear description of the trackability of System (1), we still need simple conditions which only rely on the structure of the communication graph. The following lemma can be found in Hong et al. (2006).

Lemma 4. Suppose Assumption 2 holds, and $\mathcal{G}$ is connected. Then $L_F + B$ is a positive definite matrix.

According to Lemma 4, we can denote the eigenvalues of $-(L_F + B)$ as $\lambda_n^* \leq \cdots \leq \lambda_1^* < 0$, and then the following conclusion holds.

Theorem 5. Suppose Assumption 2 holds and $\mathcal{G}$ is connected. System (1) is globally trackable if $\lambda_1^* < -\sqrt{|\mathcal{N}_0|}$, where $|\mathcal{N}_0|$ represents the number of elements in $\mathcal{N}_0$.

The proof of Theorem 5 relies on the following lemma on the perturbation of eigenvalues (Quarteroni et al. (2000)):

Lemma 6. Given an eigenvalue λ and the matrix Λ consisting of eigenvectors of $C \in \mathbb{R}^{n\times n}$. Let μ be an eigenvalue of matrix $C + P \in \mathbb{R}^{n\times n}$, then

$$\min_{\lambda\in\sigma_C} \left|\lambda - \mu\right| \leq \|\Lambda\|_2 \|\Lambda^{-1}\|_2 \|P\|_2 \tag{4}$$

where σ_C denotes the spectrum of C.

We are now ready to present the proof of Theorem 5.

Proof of Theorem 5. Applying Lemma 6 on matrix $(-L_F - B) + E$, we have

$$\min_{\lambda_i^*} \left|\lambda_i^* - \mu\right| \leq \|E\|_2 \tag{5}$$

for any eigenvalue μ of $-G$, because we can select eigenvectors of $-L_F - B$ which forms an orthogonal matrix. Moreover, noticing that

$$\|E\|_2 = \|\mathbf{1}e^T\|_2 \leq \|\mathbf{1}\|_2\|e\|_2 = \sqrt{|\mathcal{N}_0|},$$

we obtain

$$\mathrm{Re}(\mu) \leq \lambda_1^* + \|E\|_2 \leq \lambda_1^* + \sqrt{|\mathcal{N}_0|} < 0$$

for any eigenvalue μ of $-G$ when $\lambda_1^* < -\sqrt{|\mathcal{N}_0|}$. □

Theorem 5 gives us a sufficient condition for global trackability, and in the next theorem we give a necessary condition for global trackability.

Theorem 7. Suppose Assumption 2 holds and $\mathcal{G}$ is connected. If System (1) is globally trackable, then the fault threshold $b_* \geq |\mathcal{N}_0|$ is satisfied.

Proof of Theorem 7. Theorem 3 implied that

$$\mathrm{Re}\left(\lambda(L_F + B - E)\right) \geq 0$$

if the system is globally trackable. But if the system is trackable, then so is also the system where $b_i = b_*$, $\forall i \in \mathcal{V}^F$, hence

$$\mathrm{Re}\left(\lambda(L_F - E)\right) \geq -b_*.$$

Notice that $\mathbf{1}$ is an eigenvector to E with eigenvalue $|\mathcal{N}_0|$, but also an eigenvector of L_F with eigenvalue 0. Thus,

$$0 - |\mathcal{N}_0| \geq -b_* \quad \Rightarrow \quad b_* \geq |\mathcal{N}_0|$$

□

3.2 Time-varying Graphs

This subsection discusses time-varying graphs. As usual in the literature, e.g., Jadbabaie et al. (2003); Lin et al. (2007); Shi and Hong (2009), an assumption is imposed for the switching signal $\sigma(t)$.

Assumption 8. (Dwell Time). There exist a lower bound $\tau_D > 0$ between two switching instances of $\sigma(t)$.

We also impose bounds on the weight functions, $a_{ij}(x,t)$:

Assumption 9. (Weights Rule). There exists $a^* > 0$ and $a_* > 0$ such that

$$a_* \leq a_{ij}(x,t) \leq a^*, \quad t \in \mathbb{R}^+, x \in \mathbb{R}^n.$$

The joint graph of $\mathcal{G}_{\sigma(t)}$ in time interval $[t_1, t_2)$ with $t_1 < t_2 \leq +\infty$ is denoted as $\mathcal{G}([t_1,t_2)) = \cup_{t\in[t_1,t_2)}\mathcal{G}(t) = (\mathcal{V}, \cup_{t\in[t_1,t_2)}\mathcal{E}_{\sigma(t)})$. The joint follower graph is similarly defined as $\mathcal{G}^F([t_1,t_2))$. Another assumption is given on the connectivity of the joint communication graphs:

Assumption 10. (Joint Connectivity). There exists $T > 0$ such that both $\mathcal{G}([t,t+T))$ and $\mathcal{G}^F([t,t+T))$ are connected for all t.

For time-varying communication graphs, we have the following main results.

Theorem 11. Suppose Assumptions 8, 9, and 10 hold, then there exist initial values for which System (1) is escapable if $b_* < 1$.

Theorem 12. Suppose Assumptions 8, 9, and 10 hold. System (1) is globally trackable if the system parameters b_*, a_*, a^*, T, τ_D satisfy

$$0 < \left(e^{n(n+1)T_0} - w_*\left(\varrho_0 e^{-(n^2-1)a^*T_0}\right)^n\right) < 1 \tag{6}$$

where $T_0 = T + \tau_D$ and

$$w_* = \frac{b_* + (e^{\tau_D} - 1)\left(e^{nT_0} - 1\right)\left(e^{-b_*T_0} - 1\right)}{(n-1)a^* + b_*} \cdot e^{-(n-1)a^*(n+1)T_0};$$

$$\varrho_0 = \frac{\left(1 - e^{-((n-2)a^*+a_*)\tau_D}\right)a_*}{(n-2)a^* + a_*}.$$

It is not hard to see that parameters meeting the requirement of Theorem 12 can always be found as long as we choose T_0^{-1} and a^* sufficiently large. In the rest of this subsection, we first establish several lemmas which are useful for the convergence analysis, and then prove Theorem 11 and 12.

Key Lemmas Since we are analyzing piecewise continues functions, we recall the Dini derivatives. Let a and b $(> a)$ be two real numbers and consider a function $h : (a,b) \to \mathbb{R}$ and a point $t \in (a,b)$. The upper Dini derivative of h at t is defined as

$$D^+h(t) = \limsup_{s\to 0^+} \frac{h(t+s) - h(t)}{s}.$$

It is well known that when h is continuous on (a,b), h is non-increasing on (a,b) if and only if $D^+h(t) \leq 0$ for every $t \in (a,b)$ Clarke et al. (1998). The next result is given for the calculation of Dini derivative Danskin (1966); Lin et al. (2007).

Lemma 13. Let $V_i(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ $(i = 1, \ldots, n)$ be C^1 and $V(t, x) = \max_{i=1,\ldots,n} V_i(t, x)$. If $\mathcal{I}(t) = \{i \in \{1, 2, \ldots, n\} : V(t, x(t)) = V_i(t, x(t))\}$ is the set of indices where the maximum is reached at t, then $D^+V(t, x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t, x(t))$.

Introduce

$$m(t) = \min_{i\in\mathcal{V}^F} x_i(t); \quad M(t) = \max_{i\in\mathcal{V}^F} x_i(t).$$

The following lemma holds.

Lemma 14. Suppose $y(t) \in [m(t), M(t)]$ for $t \geq 0$. Then $D^+m(t) \geq 0$ and $D^+M(t) \leq 0$ for all $t \geq 0$.

Proof. We only prove $D^+m(t) \geq 0$, the other case follows by a symmetric argument. Denoting $\mathcal{I}_0(t)$ as the index set consisting of all the follower nodes which reaches the minimal value at time t. Let $i \in \mathcal{I}_0(t)$, then we have

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i^F(\sigma(t))} a_{ij}(x, t)\big(x_j(t) - x_i(t)\big) + b_i(t)\big(y(t) - x_i(t)\big) \geq 0$$

because $x_j(t) \geq x_i(t) = m(t)$ for all $j \in \mathcal{N}_i^F(\sigma(t))$ and $y(t) \geq x_i(t) = m(t)$. Therefore, according to Lemma 13,

$$D^+m(t) = \max_{i\in\mathcal{I}_0(t)} \dot{x}_i(t) \geq 0.$$

□

Then the following lemma indicates that System (1) is trackable in a special case.

Lemma 15. Suppose Assumptions 8, 9, and 10 hold, and $y(t) \in [m(t), M(t)]$ for $t \geq 0$. Then System (1) is trackable.

Proof. Take $t_0 \geq 0$. We divide the proof into three steps.

Step 1. Suppose node $i_0 \in \mathcal{V}^F$ reaches the minimal value at time t_0, i.e., $x_{i_0}(t_0) = m(t_0)$. We bound $x_{i_0}(t)$ in this step.

Based on Lemma 14, we have

$$m(t) \geq m(t_0); \quad M(t) \leq M(t_0) \tag{7}$$

for all $t \geq t_0$. As a result, with Assumption 9, if $y(t) \in [m(t), M(t)]$ for $t \geq 0$, we obtain

$$\begin{aligned}\dot{x}_{i_0}(t) &= \sum_{j\in\mathcal{N}_{i_0}^F(\sigma(t))} a_{i_0j}(x, t)\big(x_j(t) - x_{i_0}(t)\big) + b_i(t)\big(y(t) - x_{i_0}(t)\big)\\ &\leq \Big(\sum_{j\in\mathcal{N}_{i_0}^F(\sigma(t))} a_{i_0j}(x, t)\Big)\Big(M(t) - x_{i_0}(t)\Big) + b_i(t)\Big(M(t) - x_{i_0}(t)\Big)\\ &\leq (n-1)a^*\big(M(t_0) - x_{i_0}(t)\big) + b_*\big(M(t_0) - x_{i_0}(t)\big)\\ &= -\big((n-1)a^* + b_*\big)\big(x_{i_0}(t) - M(t_0)\big), \quad t \geq t_0.\end{aligned} \tag{8}$$

Thus, by Grönwall's inequality, we further conclude that

$$\begin{aligned}x_{i_0}(t) &\leq e^{-\big((n-1)a^*+b_*\big)(t-t_0)}x_{i_0}(t_0) + \Big(1 - e^{-\big((n-1)a^*+b_*\big)(t-t_0)}\Big)M(t_0)\\ &= e^{-\big((n-1)a^*+b_*\big)(t-t_0)}m(t_0) + \Big(1 - e^{-\big((n-1)a^*+b_*\big)(t-t_0)}\Big)M(t_0), \quad t \geq t_0,\end{aligned}$$

which implies

$$\begin{aligned}x_{i_0}(t) &\leq e^{-\big((n-1)a^*+b_*\big)(n-1)T_0}m(t_0) + \Big(1 - e^{-\big((n-1)a^*+b_*\big)(n-1)T_0}\Big)M(t_0)\\ &= d_0m(t_0) + (1 - d_0)M(t_0)\\ &\doteq \phi_0\end{aligned} \tag{9}$$

for all $t \in [t_0, t_0 + (n-1)T_0]$, where $d_0 = e^{-\big((n-1)a^*+b_*\big)(n-1)T_0}$ and $T_0 = T + \tau_D$.

Step 2. According to the joint connectivity Assumption 10, there exists one node i_1 such that i_1 is connected to i_0 in the graph $\mathcal{G}_{\sigma(\hat{t}_1)}$ for some $\hat{t}_1 \in [t_0, t_0 + T)$. We bound $x_{i_1}(t)$ in this step.

There are two cases.

- There exists some $s \in [\hat{t}_1, \hat{t}_1 + \tau_D]$ such that
$$x_{i_1}(s) \leq \phi_0 = d_0m(t_0) + (1 - d_0)M(t_0). \tag{10}$$
- For all $t \in [\hat{t}_1, \hat{t}_1 + \tau_D]$, it holds that $x_{i_1}(t) \geq \phi_0$. Then we see from (9) that
$$\begin{aligned}\dot{x}_{i_1}(t) &= \sum_{j\in\mathcal{N}_{i_1}^F(\sigma(t))} a_{i_1j}(x, t)\big(x_j(t) - x_{i_1}(t)\big) + b_i(t)\big(y(t) - x_{i_1}(t)\big)\\ &\leq a_{i_1i_0}(t)\big(x_{i_0}(t) - x_{i_1}(t)\big) + \Big(M(t) - x_{i_1}(t)\Big)\sum_{j\in\mathcal{N}_{i_1}^F(\sigma(t))\setminus\{i_0\}} a_{ij}(x, t) + b_i(t)\Big(M(t) - x_{i_1}(t)\Big)\\ &\leq a_*\big(\phi_0 - x_{i_1}(t)\big) + (n-2)a^*\big(M(t_0) - x_{i_1}(t)\big) + b_*\big(M(t_0) - x_{i_1}(t)\big)\\ &= -\big((n-2)a^* + b_* + a_*\big) \cdot \Big(x_{i_1}(t) - \frac{M(t_0)\big((n-2)a^* + b_*\big) + a_*\phi_0}{(n-2)a^* + b_* + a_*}\Big), \quad t \in [\hat{t}_1, \hat{t}_1 + \tau_D].\end{aligned}$$
This implies
$$\begin{aligned}x_{i_1}(\hat{t}_1 + \tau_D) &\leq \delta_0x_{i_1}(\hat{t}_1) + (1 - \delta_0)\Big(\frac{M(t_0)\big((n-2)a^* + b_*\big) + a_*\phi_0}{(n-2)a^* + b_* + a_*}\Big)\\ &\leq \delta_0M(t_0) + (1 - \delta_0)\Big(\frac{M(t_0)\big((n-2)a^* + b_*\big) + a_*\phi_0}{(n-2)a^* + b_* + a_*}\Big)\\ &= \frac{a_*(1 - \delta_0)d_0}{(n-2)a^* + b_* + a_*}m(t_0) + \Big(1 - \frac{a_*(1 - \delta_0)d_0}{(n-2)a^* + b_* + a_*}\Big)M(t_0)\\ &\doteq (1 - \delta_0)d_0\lambda_0m(t_0) + \Big(1 - (1 - \delta_0)d_0\lambda_0\Big)M(t_0),\end{aligned}$$
after some simple algebra, where
$$\delta_0 \doteq e^{-\big((n-2)a^*+b_*+a_*\big)\tau_D};$$
$$\lambda_0 \doteq \frac{a_*}{(n-2)a^* + b_* + a_*}.$$

Consequently, either of the cases leads to the existence of $\tilde{t}_1 \in [t_0, t_0 + T_0]$ such that

$$x_{i_1}(\tilde{t}_1) \le (1-\delta_0)d_0\lambda_0 m(t_0) + \Big(1-(1-\delta_0)d_0\lambda_0\Big)M(t_0).$$

Noticing that inequality (8) also holds for i_1, we can similarly obtain

$$x_{i_1}(t) \le (1-\delta_0)d_0^2\lambda_0 m(t_0) + \Big(1-(1-\delta_0)d_0^2\lambda_0\Big)M(t_0)$$

for all $t \in [t_0 + T_0, t_0 + (n-1)T_0]$.

Step 3. By the joint connectivity Assumption 10, we can proceed the analysis in time intervals $[t_0 + T_0), \dots, [t_0 + (n-2)T_0, t_0 + (n-1)T_0)$, and $i_2, i_3, \dots, i_{n-1} \in \mathcal{V}^F$ can be found such that

$$x_{i_s}(t) \le \big[(1-\delta_0)d_0\lambda_0\big]^s d_0 m(t_0) + \Big(1-\big[(1-\delta_0)d_0\lambda_0\big]^s d_0\Big)M(t_0)$$

for all $t \in [t_0 + sT_0, t_0 + (n-1)T_0]$, which immediately yields

$$M\big(t_0 + (n-1)T_0\big) \le \big[(1-\delta_0)d_0\lambda_0\big]^{n-1} d_0 m(t_0) + \Big(1-\big[(1-\delta_0)d_0\lambda_0\big]^{n-1} d_0\Big)M(t_0).$$

Thus, according to Lemma 14, we eventually obtain

$$M\big(t_0 + (n-1)T_0\big) - m\big(t_0 + (n-1)T_0\big) \le \Big(1-\big[(1-\delta_0)d_0\lambda_0\big]^{n-1} d_0\Big)\Big(M(t_0) - m(t_0)\Big). \quad (11)$$

Since t_0 is chosen arbitrarily, (11) implies

$$\lim_{t\to\infty}\Big[M(t) - m(t)\Big] = 0,$$

and thus

$$\lim_{t\to\infty}\xi(t) = 0$$

as long as $y(t) \in [m(t), M(t)]$ for $t \ge 0$. This completes the proof.

□

For the case when there exists some $t_* \ge 0$ such that $y(t_*) \notin [m(t_*), M(t_*)]$. The following lemma holds.

Lemma 16. (i) If there exists some $t_* \ge 0$ such that $y(t_*) > M(t_*)$. Then $y(t) > M(t)$ for all $t \ge t_*$.

(ii) If there exists some $t_* \ge 0$ such that $y(t_*) < m(t_*)$. Then $y(t) < m(t)$ for all $t \ge t_*$.

Proof. We just focus on (i), and then (ii) holds symmetrically.

Since $y(t_*) > M(t_*)$ and the differential equation (1) is with piecewise continuous right-hand side, there exists $\varepsilon > 0$ such that $y(t) > M(t)$ for $t \in [t_*, t_* + \varepsilon)$. Consequently, by a similar analysis as Lemma 14, we have

$$D^+M(t) \le b_*\big(y(t) - M(t)\big); \quad D^+y(t) \ge 0$$

for $t \in [t_*, t_* + \varepsilon]$. This leads to

$$y(t_* + \varepsilon) - M(t_* + \varepsilon) > e^{-b_*\varepsilon}\big(y(t_*) - M(t_*)\big) > 0. \quad (12)$$

Take

$$\varepsilon_0 = \sup\big\{\varepsilon \ge 0 : \ y(t) > M(t) \text{ for } t \in [t_*, t_* + \varepsilon)\big\}.$$

Then (12) implies it is impossible for ε_0 to be finite, which yields $y(t) > M(t)$ for all $t \ge t_*$. This completes the proof. □

Proof of Theorem 11. Take initial value (x^0, y^0) with $y^0 > M(0)$. Then, Lemma 16 implies that $y(t) > M(t)$ for all $t > 0$. Therefore, we have

$$D^+M(t) \le 0; \quad D^+y(t) = 0$$

when there is no follower agent connecting to the faulty agent, and

$$D^+M(t) \le b_*\big(y(t) - M(t)\big); \quad D^+y(t) \ge y(t) - M(t)$$

when at least one follower agent connects to the faulty agent. This leads to

$$D^+\big[y(t) - M(t)\big] \ge \begin{cases} 0, & \text{if no follower is connected to the faulty agent at time } t \\ (1-b_*)\big[y(t) - M(t)\big], & \text{otherwise.}\end{cases}$$

It is straightforward to see that $\lim_{t\to\infty}\big[y(t) - M(t)\big] = \infty$. □

Proof of Theorem 12. Based on Lemma 16, we just need to prove Theorem 12 for the cases when $\exists t_* \ge 0$ such that $y(t_*) > M(t_*)$ or $y(t_*) < m(t_*)$. We focus on the first case, since the proof for the second case can be obtained by a symmetric argument.

Suppose $y(t_*) > M(t_*)$ for some $t_* > 0$. Then, Lemma 16 suggests that $y(t) > M(t)$ for all $t \ge t_*$. Choose $t_0 \ge t_*$, we divide the rest of the proof into four steps.

Step 1. We bound $y(t)$ in this step. Similar to Lemma 14, since $y(t) > M(t)$, we have $D^+m(t) \ge 0$ for all $t \ge t_*$. Noticing that

$$\dot{y}(t) = \sum_{j\in\mathcal{N}_0(\sigma(t))}\big(y(t) - x_j(t)\big) \le n\big(y(t) - m(t)\big) \le n\big(y(t) - m(t_0)\big)$$

for all $t \ge t_0$, we obtain

$$y(t) \le e^{n(t-t_0)}y(t_0) + \big(1 - e^{n(t-t_0)}\big)m(t_0), \quad t \ge t_0. \quad (13)$$

This implies

$$y(t) \le e^{n(n+1)T_0}y(t_0) + \big(1 - e^{n(n+1)T_0}\big)m(t_0), \quad t \in [t_0, t_0 + (n+1)T_0],$$

where $T_0 = T + \tau_D$.

On the other hand, (13) implies

$$D^+M(t) \le b_*\big(y(t) - M(t)\big) \le -b_*\Big(M(t) - e^{nT_0}y(t_0) - \big(1 - e^{nT_0}\big)m(t_0)\Big), \quad t \in [t_0, t_0 + T_0],$$

which yields

$$M(t) \le e^{-b_*T_0}M(t_0) + \big(1 - e^{-b_*T_0}\big) \cdot \Big(e^{nT_0}y(t_0) + \big(1 - e^{nT_0}\big)m(t_0)\Big), \quad t \in [t_0, t_0 + T_0]. \quad (14)$$

Since $\mathcal{G}\big([t_0, t_0 + T)\big)$ is connected, there exists $\hat{t}_1 \in [t_0, t_0 + T)$ such that the faulty agent is connected to some follower agent at time $\hat{t}_1$. As a result, (14) leads to

$$\begin{aligned}\dot{y}(t) &= \sum_{j\in\mathcal{N}_0(\sigma(t))}\big(y(t)-x_j(t)\big)\\ &\geq y(t)-M(t)\\ &\geq y(t)-e^{-b_*T_0}M(t_0)-\big(1-e^{-b_*T_0}\big)\\ &\quad\cdot\Big(e^{nT_0}y(t_0)+\big(1-e^{nT_0}\big)m(t_0)\Big) \qquad (15)\end{aligned}$$

for $t\in[\hat{t}_1,\hat{t}_1+\tau_D]$ with $\hat{t}_1+\tau_D\leq T_0$, which implies

$$\begin{aligned}y(\hat{t}_1+\tau_D) &\geq e^{\tau_D}y(t_0)+(1-e^{\tau_D})\cdot\\ &\quad\Big[e^{-b_*T_0}M(t_0)+\big(1-e^{-b_*T_0}\big)\\ &\quad\Big(e^{nT_0}y(t_0)+\big(1-e^{nT_0}\big)m(t_0)\Big)\Big] \qquad (16)\end{aligned}$$

Let $0<\chi\leq 1$ be the constant satisfying $y(t_0)-M(t_0)=\chi\big[y(t_0)-m(t_0)\big]$. Noticing that $y(t)$ is strictly increasing for $t>t_*$, we see from (16) that

$$\begin{aligned}y(t) &\geq y(\hat{t}_1+\tau_D)\\ &\geq y(t_0)+\Big(e^{\tau_D}-1\Big)\\ &\quad\cdot\Big(\big(e^{nT_0}-1\big)\big(e^{-b_*T_0}-1\big)+\chi e^{-b_*T_0}\Big)\\ &\quad\cdot\Big(y(t_0)-m(t_0)\Big)\\ &= y(t_0)+p_0\Big(y(t_0)-m(t_0)\Big), \qquad (17)\end{aligned}$$

for all $t\geq t_0+T_0$ after some simple algebra, where

$$p_0 \doteq \big(e^{\tau_D}-1\big)\Big(\big(e^{nT_0}-1\big)\big(e^{-b_*T_0}-1\big)+\chi e^{-b_*T_0}\Big). \qquad (18)$$

Step 2. We give a lower bound for $m\big(t_0+(n+1)T_0\big)$ in this step.

Because $\mathcal{G}\big([t_0+T_0,t_0+2T)\big)$ is connected, there exists at least one follower node $i_0\in\mathcal{V}^F$ and $\hat{t}_2\in[t_0+T_0,t_0+2T)$ such that i_0 is connected to the faulty agent at time $\hat{t}_2$. Therefore, with (17), we have

$$\begin{aligned}\dot{x}_{i_0}(t) &= \sum_{j\in\mathcal{N}^F_{i_0}(\sigma(t))}a_{i_0j}(x,t)\big(x_j(t)-x_{i_0}(t)\big)\\ &\qquad +b_*\big(y(t)-x_{i_0}(t)\big)\\ &\geq (n-1)a^*\big(m(t_0)-x_{i_0}(t)\big)\\ &\quad +b_*\Big(y(t_0)+p_0\big(y(t_0)-m(t_0)\big)-x_{i_0}(t)\Big),\end{aligned}$$

for $t\in[\hat{t}_2,\hat{t}_2+\tau_D]$, which implies

$$\begin{aligned}x_{i_0}(\hat{t}_2+\tau_D) &\geq e^{-((n-1)a^*+b_*)\tau_D}x_{i_0}(\hat{t}_2)\\ &\quad+\big(1-e^{-((n-1)a^*+b_*)\tau_D}\big)\\ &\quad\cdot\tfrac{(n-1)a^*m(t_0)+b_*\left[y(t_0)+p_0\left(y(t_0)-m(t_0)\right)\right]}{(n-1)a^*+b_*}\\ &\geq e^{-((n-1)a^*+b_*)\tau_D}m(t_0)\\ &\quad+\big(1-e^{-((n-1)a^*+b_*)\tau_D}\big)\\ &\quad\cdot\tfrac{(n-1)a^*m(t_0)+b_*\left[y(t_0)+p_0\left(y(t_0)-m(t_0)\right)\right]}{(n-1)a^*+b_*}\\ &= \frac{b_*+p_0}{(n-1)a^*+b_*}y(t_0)\\ &\quad+\Big(1-\frac{b_*+p_0}{(n-1)a^*+b_*}\Big)m(t_0).\end{aligned}$$

Next, for $t\in[\hat{t}_2+\tau_D,t_0+(n+1)T_0]$, we have

$$\begin{aligned}\dot{x}_{i_0}(t) &= \sum_{j\in\mathcal{N}^F_{i_0}(\sigma(t))}a_{i_0j}(x,t)\big(x_j(t)-x_{i_0}(t)\big)\\ &\qquad+b_i(t)\big(y(t)-x_{i_0}(t)\big)\\ &\geq (n-1)a^*\big(m(t_0)-x_{i_0}(t)\big),\end{aligned}$$

and thus,

$$\begin{aligned}x_{i_0}(t) &\geq e^{-(n-1)a^*(n+1)T_0}x_{i_0}(\hat{t}_2+\tau_D)\\ &\quad+\big(1-e^{-(n-1)a^*(n+1)T_0}\big)m(t_0)\\ &\geq \frac{b_*+p_0}{(n-1)a^*+b_*}e^{-(n-1)a^*(n+1)T_0}y(t_0)\\ &\quad+\Big(1-\frac{(b_*+p_0)e^{-(n-1)a^*(n+1)T_0}}{(n-1)a^*+b_*}\Big)m(t_0)\\ &\doteq w_0y(t_0)+(1-w_0)m(t_0)\end{aligned}$$

for all $t\in[t_0+2T_0,t_0+(n+1)T_0]$, where $w_0=\frac{b_*+p_0}{(n-1)a^*+b_*}e^{-(n-1)a^*(n+1)T_0}$.

Step 3. Since $\mathcal{G}\big([t_0+2T_0,t_0+2T_0+T)\big)$ is connected, there exists at least one follower node $i_1\in\mathcal{V}^F$ and $\hat{t}_3\in[t_0+2T_0,t_0+2T_0+T)$ such that i_1 is connected to the faulty agent, or to the follower agent i_0, at time $\hat{t}_3$. Similar to the proof of Lemma 15, we have

$$\begin{aligned}\dot{x}_{i_1}(t) &= \sum_{j\in\mathcal{N}^F_{i_1}(\sigma(t))}a_{i_1j}(x,t)\big(x_j(t)-x_{i_1}(t)\big)\\ &\qquad+b_i(t)\big(y(t)-x_{i_1}(t)\big)\\ &\geq a_{i_1i_0}(t)\big(x_{i_0}(t)-x_{i_1}(t)\big)\\ &\quad+\Big(M(t)-x_{i_1}(t)\Big)\sum_{j\in\mathcal{N}^F_{i_1}(\sigma(t))\setminus\{i_0\}}a_{ij}(x,t)\\ &\geq a_*\big(w_0y(t_0)+(1-w_0)m(t_0)-x_{i_1}(t)\big)\\ &\quad+(n-2)a^*\big(m(t_0)-x_{i_1}(t)\big),\\ &\qquad\qquad t\in[\hat{t}_3,\hat{t}_3+\tau_D]\end{aligned}$$

where we assume $x_{i_1}(t)\leq w_0y(t_0)+(1-w_0)m(t_0),t\in[\hat{t}_3,\hat{t}_3+\tau_D]$, without loss of generality. As a result, we have

$$\begin{aligned}x_{i_1}(\hat{t}_3+\tau_D) &\geq e^{-((n-2)a^*+a_*)\tau_D}m(t_0)\\ &\quad+\big(1-e^{-((n-2)a^*+a_*)\tau_D}\big)\\ &\quad\cdot\tfrac{a_*\left(w_0y(t_0)+(1-w_0)m(t_0)\right)+(n-2)a^*m(t_0)}{(n-2)a^*+a_*}\\ &= \frac{\big(1-e^{-((n-2)a^*+a_*)\tau_D}\big)a_*w_0}{(n-2)a^*+a_*}y(t_0)\\ &\quad+\Big(1-\tfrac{\left(1-e^{-((n-2)a^*+a_*)\tau_D}\right)a_*w_0}{(n-2)a^*+a_*}\Big)m(t_0)\\ &\doteq \varrho_0w_0y(t_0)+(1-\varrho_0w_0)m(t_0),\end{aligned}$$

where $\varrho_0=\frac{\left(1-e^{-((n-2)a^*+a_*)\tau_D}\right)a_*}{(n-2)a^*+a_*}$. This immediately implies

$$x_{i_1}(t) \geq \varrho_0 w_0 e^{-(n-1)a^*(n+1)T_0} y(t_0) + \Big(1 - \varrho_0 w_0 e^{-(n-1)a^*(n+1)T_0}\Big) m(t_0)$$

for all $t \in [t_0 + 3T_0, t_0 + (n+1)T_0]$.

Step 4. Continuing the analysis, estimates for follower nodes $i_2, \ldots, i_n$ can be made similarly, and we will eventually arrive at

$$x_i\big(t_0+(n+1)T_0\big) \geq w_0\big(\varrho_0 e^{-(n-1)a^*(n+1)T_0}\big)^n y(t_0) + \Big(1 - w_0\big(\varrho_0 e^{-(n-1)a^*(n+1)T_0}\big)^n\Big) m(t_0)$$

for all $i = 1 \ldots, n$, and thus

$$m\big(t_0+(n+1)T_0\big) \geq w_0\big(\varrho_0 e^{-(n-1)a^*(n+1)T_0}\big)^n y(t_0) + \Big(1 - w_0\big(\varrho_0 e^{-(n-1)a^*(n+1)T_0}\big)^n\Big) m(t_0). \quad (19)$$

As a result, (13) and (19) lead to

$$\begin{aligned}
&\Big[y\big(t_0 + (n+1)T_0\big) - m\big(t_0 + (n+1)T_0\big)\Big] \\
&\leq e^{n(n+1)T_0} y(t_0) + \big(1 - e^{n(n+1)T_0}\big) m(t_0) \\
&\quad - w_0\big(\varrho_0 e^{-(n-1)a^*(n+1)T_0}\big)^n y(t_0) \\
&\quad - \Big(1 - w_0\big(\varrho_0 e^{-(n-1)a^*(n+1)T_0}\big)^n\Big) m(t_0) \\
&= \Big(e^{n(n+1)T_0} - w_0\big(\varrho_0 e^{-(n^2-1)a^*T_0}\big)^n\Big) \cdot \Big[y(t_0) - m(t_0)\Big]. \quad (20)
\end{aligned}$$

By denoting $\Psi(t) = y(t) - m(t)$, (20) implies

$$\Psi\big(t_0 + (n+1)T_0\big) \leq \Big(e^{n(n+1)T_0} - w_0\big(\varrho_0 e^{-(n^2-1)a^*T_0}\big)^n\Big)\Psi(t_0) \quad (21)$$

for all $t_0 \geq t_*$. According to the definition of p_0 in (18), w_0 increases as long as χ increases. Then we see from (21) that

$$\Psi\big(t_0 + (n+1)T_0\big) \leq \Big(e^{n(n+1)T_0} - w_*\big(\varrho_0 e^{-(n^2-1)a^*T_0}\big)^n\Big)\Psi(t_0)$$

where

$$w_* = \frac{b_* + \big(e^{\tau_D} - 1\big)\big(e^{nT_0} - 1\big)\big(e^{-b_*T_0} - 1\big)}{(n-1)a^* + b_*} \cdot e^{-(n-1)a^*(n+1)T_0}.$$

When the given parameter condition holds, $0 < \Big(e^{n(n+1)T_0} - w_*\big(\varrho_0 e^{-(n^2-1)a^*T_0}\big)^n\Big) < 1$, the desired conclusion follows.

□

Remark 17. Here we only considered systems with a single faulty agent, since systems with multiple faulty agents always yield an escapable system.

4. ILLUSTRATION OF RESULTS

In Fig. 1 we examine both the line graph and the complete graph of different sizes, and select an arbitrary node with the fault. The minimal value of b_* which guarantees global trackability by Theorem 3 is computed and shown. As expected, the threshold is independent of the agents position in the symmetric complete graph, but it is also independent of the agents position for the line graph. Furthermore, the simulation indicates that the tight threshold for b_* is $b_* \geq n$ for global trackability (compared to $b_* \geq |\mathcal{N}_0|$ in Theorem 7)

In Fig. 2, 3 and 4 we show the agents' state evolution on a line graph with 5 nodes, and b_* selected as 3,4 and 5 respectively. In Fig. 2, where $b_* < n$, the state errors are diverging. In Fig. 3, where $b_* = n$, the state error remains constant, and in Fig. 4, where $b_* > n$, the state errors are diminishing.

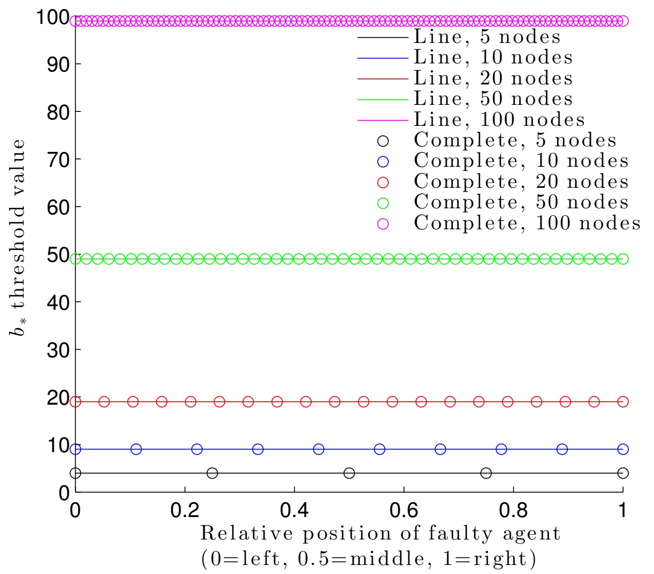

Fig. 1. Threshold value for b_* for line and complete graphs. The threshold value $b_* \geq n$ seems to be tight, and independent of the agents position and the topology.

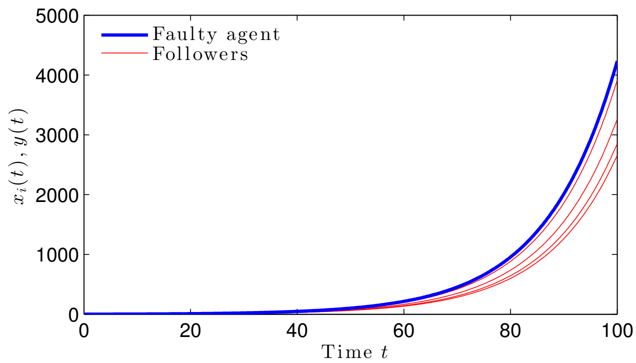

Fig. 2. Simulation of a multi agent system with a line topology consisting of 5 nodes, $n = 4$ and $b_* = 3$.

5. CONCLUSIONS

This paper presented a framework for a multi-agent system tracking a faulty agent. Different from most existing works, the faulty agent was acting against the follower agents using positive feedback from the relative state of the followers. Trackability and escapability concepts were introduced, and sufficient conditions and necessary conditions were established for the multi-agent system to be trackable and escapable under both fixed and time-varying communication topologies. Numerical simulations indicate that a tight condition might be possible. Further challenges

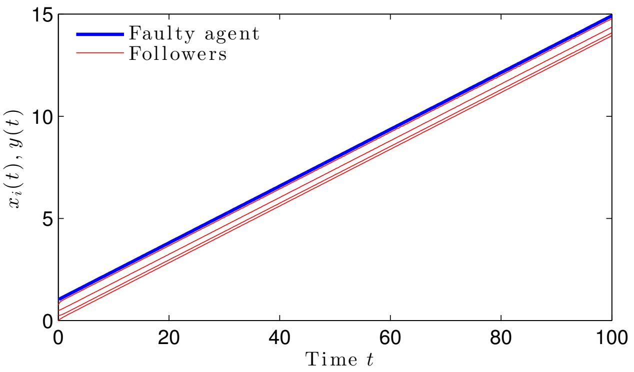

Fig. 3. Simulation of a multi agent system with a line topology consisting of 5 nodes, $n = 4$ and $b_* = 4$.

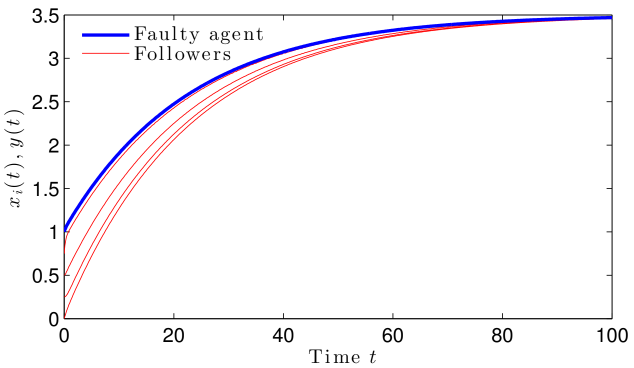

Fig. 4. Simulation of a multi agent system with a line topology consisting of 5 nodes, $n = 4$ and $b_* = 5$.

lie in designing the worst case behavior for the faulty agent, and optimal tracking protocols for the followers.

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