

# Formation Control for Multi-Agent Systems with Connectivity Preservation and Event-Triggered Controllers <sup>★</sup>

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**Abstract:** In this paper, event-triggered controllers and corresponding algorithms are proposed to establish the formation with connectivity preservation for multi-agent systems. Each agent needs to update its control input and to broadcast this control input together with the relative state information to its neighbors at its own triggering times, and to receive information at its neighbors' triggering times. Two types of system dynamics, single integrators and double integrators, are considered. As a result, all agents converge to the formation exponentially with connectivity preservation, and Zeno behavior can be excluded. Numerical simulations show the effectiveness of the theoretical results.

*Keywords:* Multi-agent systems, formation control, connectivity, event-triggered control.

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## 1. INTRODUCTION

In the last two decades, numerous contributions have been given in the study of distributed cooperative control, including consensus control and formation control, etc, for multi-agent systems, see e.g., Jadbabaie et al. (2003), Olfati-Saber and Murray (2004), Moreau (2004), Ren and Beard (2005), Liu et al. (2011). In these contributions, there are two common assumptions that are assumed expressly or implicitly. One is that all agents must maintain a (strongly or weakly) connected communication graph either for all time or in an average sense. Another one is that the information can be continuously transmitted between agents or each agent can continuously sense the relative information, such as relative positions. However, it is not straightforward to verify the connectivity since the states of agents change during the evolution of the system and thus the connection between two agents may change. Moreover, it may be also impractical to require continuous sensing or broadcasting in real applications.

Connectivity preservation is a fundamental problem in the study of many areas, such as mobile robots. Normally, mobile robots have limited communication, including sensing and broadcasting, capabilities and one robot can only communicate with other robots that are within close enough distance. It is natural to model the interaction between a group of mobile robots with limited communication radii as a multi-agent system and the corresponding communication graph is determined by their relative distances. Motivated by this, many researchers studied distributed cooperative control for multi-agent systems with connectivity preservation, see e.g., Spanos and Murray (2004), Dimarogonas and Kyriakopoulos (2007), Ji and Egerstedt

(2007), Zavlanos and Pappas (2007), Ajorlou et al. (2010), Kan et al. (2012), Boskos and Dimarogonas (2015). Different from this paper, in these papers continuous sensing or broadcasting is needed.

On the other hand, one common method to avoid continuous broadcasting and sensing is using periodical sampling. However the drawbacks of this method are that it normally requires a large amount of agents take action in a synchronous manner and it is not efficient. Motivated by the future trend that agents can be equipped with embedded microprocessors with limited resources to transmit and collect information, event-triggered control was introduced by Åström and Bernhardsson (1999), Tabuada (2007). Instead of using the continuous state to realize desired convergence properties, the control in the event-triggered control strategy is piecewise constant between the triggering times which need to be determined. A key challenge in event-triggered control is how to design the triggering conditions to determine the triggering times, and to exclude Zeno behavior. For continuous-time multi-agent systems, the Zeno behavior is the behavior that there are infinite number of triggers in a finite time interval, see Johansson et al. (1999). To name a few existing works on multi-agent systems with event-triggered controllers, see e.g., Dimarogonas et al. (2012), Seyboth et al. (2013), Meng et al. (2015), Nowzari and Cortés (2016). Different from this paper, these papers do not consider the connectivity preservation.

In this paper, we study formation control for multi-agent systems with connectivity preservation and event-triggered controller. We propose distributed conditions for agents to determine their triggering times and two corresponding algorithms for each agent to avoid continuous checking of its own triggering condition. Moreover, it is only at its triggering times that each agent needs to

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update its control input by sensing the relative positions between itself and its neighbors, to broadcast its triggering information, including current triggering time, the relative states and control input at this time, to its neighbors. In other words, all agents only need to do broadcasting and sensing at their triggering times and this occurs not in synchronous manner. As a result, all agents converge to a formation exponentially with connectivity preservation. In addition, Zeno behavior can be excluded by proving that the inter-event times are lower bounded by a positive constant. Two related existing papers are Yu and Antsaklis (2012), Fan and Hu (2015). However, Yu and Antsaklis (2012) didn't explicitly exclude the Zeno behavior, but it is well known that such behavior can cause problem, see Johansson et al. (1999). And it was under the assumption that no agent exhibits Zeno behavior, Fan and Hu (2015) proved asymptotic rendezvous can be achieved.

The rest of this paper is organized as follows. Section 2 introduces the preliminaries. The main results are stated in Section 3 and Section 4. Simulations are given in Section 5. Finally, the paper is concluded in Section 6.

**Notions:**  $\|\cdot\|$  represents the Euclidean norm for vectors or the induced 2-norm for matrices.  $|S|$  is the cardinality of set  $S$ .  $\mathbf{1}_n$  denotes the column vector with each component 1 and dimension  $n$ .  $I_n$  is the  $n$  dimension identity matrix.  $\rho(\cdot)$  stands for the spectral radius for matrices and  $\rho_2(\cdot)$  indicates the minimum positive eigenvalue for matrices having positive eigenvalues. Given two symmetric matrices  $M, N$ ,  $M \geq N$  means  $M - N$  is a positive semi-definite matrix. The notation  $A \otimes B$  means the Kronecker product of matrices  $A$  and  $B$ .

## 2. PRELIMINARIES

In this section we review some definitions and results from algebraic graph theory and formation control, for more details see Mesbahi and Egerstedt (2010).

### 2.1 Algebraic Graph Theory

For a *undirected graph*  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$  with  $n$  nodes (vertices), the set of nodes  $\mathcal{V} = \{1, \dots, n\}$ , set of edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ , and the *adjacency matrix*  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  given by  $a_{ij} = a_{ji} = 1$  if  $(i, j) \in \mathcal{E}$ , and  $a_{ij} = 0$  otherwise. Let  $N_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$  and  $\deg(i) = \sum_{j=1}^n a_{ij}$  denotes the *neighbors* and *degree* of agent  $i$ , respectively. The *degree matrix* of graph  $\mathcal{G}$  is defined as  $Deg = \text{diag}([\deg(1), \dots, \deg(n)])$ . The *Laplacian matrix* is defined as  $L = Deg - A$ .

Similarly, we can define the *weighted graph*  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Omega)$  with  $\Omega = (\omega_{ij}) \in \mathbb{R}^{n \times n}$  is a nonnegative symmetric matrix and  $\omega_{ij} > 0$  if  $(i, j) \in \mathcal{E}$ . And we denote the corresponding *Laplacian matrix* as  $L_\omega$ .

A *path* of length  $k$  between agent  $i$  and agent  $j$  is a subgraph with distinct agents  $i_0 = i, \dots, i_k = j \in \mathcal{V}$  and edges  $(i_j, i_{j+1})$ ,  $j = 0, \dots, k-1$ . A graph is *connected* if there is a path between any two agents.

Let  $m$  denotes the number of edges in  $\mathcal{G}$ , i.e.,  $m = |\mathcal{E}(\mathcal{G})|$  and label the edges in  $\mathcal{G}$  as  $e_1, \dots, e_m$ . And denote  $W = \text{diag}\{\omega(e_1), \dots, \omega(e_m)\}$ , where  $\omega(e_k) = \omega_{ij}$  with  $e_k$  being

the label of edge  $(i, j)$ . After assigning a direction to each edge, the  $n \times m$  *incidence matrix*  $D = (d_{ij})$  is defined as

$$d_{ij} = \begin{cases} -1 & \text{if agent } i \text{ is the tail of edge } e_j, \\ 1 & \text{if agent } i \text{ is the head of edge } e_j, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

*Lemma 2.1.*  $DD^\top$  is independent of the labels and orientations given to  $\mathcal{G}$ , and  $DD^\top = L$ ,  $DWD^\top = L_\omega$ .

*Lemma 2.2.* (Yi et al. (2016)) Let  $K_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$  and assume  $\mathcal{G}$  is connected, then  $K_n L_\omega = L_\omega$ ,  $\rho_2(DD^\top) > 0$ ,  $\rho(K_n) = 1$  and

$$0 \leq \rho_2(DD^\top)K_n \leq DD^\top. \quad (2)$$

### 2.2 Formation Control Problem

Given a connected graph  $\mathcal{G}$ , let  $d_{ij} \in \mathbb{R}^p$  denotes the *desired internode displacement* of edge  $(i, j) \in \mathcal{E}(\mathcal{G})$ . Denote  $\Phi = \{(\tau_1^\top, \dots, \tau_n^\top)^\top \in \mathbb{R}^{np} | \tau_i - \tau_j = d_{ij}, \text{ for all } (i, j) \in \mathcal{E}(\mathcal{G})\}$ . We call the set of desired internode displacements  $\{d_{ij}, (i, j) \in \mathcal{E}(\mathcal{G})\}$  a formation associated with  $\mathcal{G}$  and we say it is *feasible* if  $\Phi \neq \emptyset$ .

Consider a multi-agent system with  $n$  agents corresponding to the  $n$  nodes in this graph. Let  $x_i(t) \in \mathbb{R}^p$  denotes the position of agent  $i$  at time  $t \geq 0$ . A group of agents are said to converge to a *formation* if

$$\lim_{t \rightarrow \infty} x_i(t) - x_j(t) = d_{ij}$$

for all  $(i, j) \in \mathcal{E}(\mathcal{G})$ .

In practice, agents normally have limited communication capabilities. For simplicity we assume all agents have the same communication radius  $\Delta > 0$ . We say the graph  $\mathcal{G}$  and the multi-agent system are *consistent* if  $\|x_i(t) - x_j(t)\| \leq \Delta$  for all  $(i, j) \in \mathcal{E}(\mathcal{G})$  and all time  $t \geq 0$ . Namely, the communication channels are kept for all time. Notice here we assume the following.

*Assumption 2.3.*  $\{d_{ij}, (i, j) \in \mathcal{E}(\mathcal{G})\}$ , a formation associated with  $\mathcal{G}$ , is feasible and  $\|d_{ij}\| < \Delta$ ,  $\forall (i, j) \in \mathcal{E}(\mathcal{G})$ .

*Definition 2.4.* A group of agents associated with a graph  $\mathcal{G}$  are said to converge to a formation with connectivity preservation if they converge to a formation while the graph  $\mathcal{G}$  remains consistent with their dynamics.

## 3. FORMATION CONTROL WITH CONNECTIVITY PRESERVATION: SINGLE INTEGRATORS

In this section, we consider the case when the dynamics of agents can be modeled as single integrators given by

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{V}. \quad (3)$$

From Assumption 2.3, we know  $\Phi \neq \emptyset$ . Choose  $(\tau_1^\top, \dots, \tau_n^\top)^\top \in \Phi$  and let  $y_i(t) = x_i(t) - \tau_i$  for  $i = 1, \dots, n$ . Then, we can rewrite the above multi-agent system as

$$\dot{y}_i(t) = u_i(t), \quad i \in \mathcal{V}. \quad (4)$$

For  $\|y_i - y_j\| < \Delta - \|d_{ij}\|$ , the edge-tension function  $\nu_{ij}$  (introduced first in Ji and Egerstedt (2007)) is defined as

$$\nu_{ij}(\Delta, y) = \begin{cases} \frac{\|y_i - y_j\|^2}{\Delta - \|d_{ij}\| - \|y_i - y_j\|}, & \text{if } (i, j) \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

with

$$\frac{\partial \nu_{ij}(\Delta, y)}{\partial y_i} = \frac{2\Delta - 2\|d_{ij}\| - \|y_i - y_j\|}{(\Delta - \|d_{ij}\| - \|y_i - y_j\|)^2} (y_i - y_j),$$

if  $(i, j) \in \mathcal{E}(\mathcal{G})$ , and  $\frac{\partial \nu_{ij}(\Delta, y)}{\partial y_i} = 0$  otherwise.

We denote as  $\omega_{ij}(t)$  the weight coefficient of the partial derivative of  $\nu_{ij}$  with respect to  $y_i$  as above, i.e.,

$$\omega_{ij}(t) = \begin{cases} \frac{2\Delta - 2\|d_{ij}\| - \|y_i(t) - y_j(t)\|}{(\Delta - \|d_{ij}\| - \|y_i(t) - y_j(t)\|)^2}, & \text{if } (i, j) \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Note that  $\omega_{ij}(t)$  can also be written as a function of  $x_i(t)$  and  $x_j(t)$  since  $y_i(t) - y_j(t) = x_i(t) - x_j(t) - d_{ij}$ .

In order to reduce the overall need of sensing and system updates, we use the following event-triggered controller

$$u_i(t) = \sum_{j \in N_i} -\omega_{ij}(t_{k_i}^i(t)) (y_i(t_{k_i}^i(t)) - y_j(t_{k_i}^i(t))) \quad (7)$$

$$= \sum_{j \in N_i} -\omega_{ij}(t_{k_i}^i(t)) (x_i(t_{k_i}^i(t)) - x_j(t_{k_i}^i(t)) - d_{ij}), \quad (8)$$

where  $k_i(t) = \arg \max_k \{t_{k_i}^i \leq t\}$  with the increasing time sequence  $\{t_{k_i}^i\}_{k=1}^{\infty}$ , which is named as *triggering times*. We assume  $t_1^i = 0$ , for all  $i \in \mathcal{V}$ . One can see that the control input (8) only updates at the triggering times.

In the following theorem, we will give triggering conditions to determine the triggering times such that the formation with connectivity preservation can be established and the Zeno behavior can be excluded.

**Theorem 3.1.** Given a graph  $\mathcal{G}$  which is undirected and connected and a formation associated with  $\mathcal{G}$  which satisfies Assumption 2.3. Consider the multi-agent system (3) with event-triggered controller (8) associated with  $\mathcal{G}$ . Assume that at the initial time  $t = 0$ ,

$$\|x_i(0) - x_j(0) - d_{ij}\| = \|y_i(0) - y_j(0)\| < \Delta - \|d_{ij}\|, \quad (9)$$

for all  $(i, j) \in \mathcal{E}(\mathcal{G})$ . Given  $\alpha > 0$ , and  $0 < \beta < \beta_0$  with  $\beta_0 = \frac{\rho_2(DD^\top)}{\Delta_0}$  and  $\Delta_0 = \max_{(i,j) \in \mathcal{E}(\mathcal{G})} \Delta - \|d_{ij}\|$ , and given the first triggering time  $t_1^i = 0$ , agent  $i$  determines the triggering times  $\{t_{k_i}^i\}_{k=2}^{\infty}$  by

$$t_{k+1}^i = \max_{r \geq t_k^i} \{r : \|e_i(t)\| \leq \alpha e^{-\beta t}, \forall t \in [t_k^i, r]\} \quad (10)$$

where

$$e_i(t) = \sum_{j \in N_i} \omega_{ij}(t) (x_i(t) - x_j(t) - d_{ij}) - \sum_{j \in N_i} \omega_{ij}(t_{k_i}^i(t)) (x_i(t_{k_i}^i(t)) - x_j(t_{k_i}^i(t)) - d_{ij}).$$

Then the multi-agent system (3) with event-triggered controller (8) converges to the formation exponentially with connectivity preservation, and there is no Zeno behavior.

**Proof.** This theorem holds if we can prove that

- (1)  $\|x_i(t) - x_j(t)\| \leq \Delta, \forall (i, j) \in \mathcal{E}(\mathcal{G}), \forall t \geq 0$ ;
- (2)  $\lim_{t \rightarrow \infty} x_i(t) - x_j(t) = d_{ij}, \forall (i, j) \in \mathcal{E}(\mathcal{G})$ , exponentially;
- (3) there exists  $\xi_i > 0$ , such that  $t_{k+1}^i - t_k^i \geq \xi_i, \forall i \in \mathcal{V}$  and  $\forall k = 1, 2, \dots$

(i) We define the total tension energy of  $\mathcal{G}$  as

$$\nu(\Delta, y) = \frac{1}{2} \sum_{i=1}^n \sum_{j \in N_i} \nu_{ij}(\Delta, y), \quad (11)$$

where  $y(t) = (y_1^\top(t), \dots, y_n^\top(t))^\top$ . The time derivative of  $\nu(\Delta, y(t))$  along the trajectories of the multi-agent system (4) with event-triggered controller (7) is

$$\begin{aligned} \dot{\nu}(\Delta, y(t)) &= \sum_{i=1}^n \sum_{j \in N_i} \left[ \frac{\partial \nu_{ij}(\Delta, y)}{\partial y_i} \right]^\top \Big|_{y=y(t)} \dot{y}_i(t) \\ &= \sum_{i=1}^n \sum_{j \in N_i} [\omega_{ij}(t)(y_i(t) - y_j(t))]^\top \\ &\quad \times [e_i(t) - \sum_{j \in N_i} \omega_{ij}(t)(y_i(t) - y_j(t))] \\ &\leq -\|L_\omega y(t)\|^2 + a\|L_\omega y(t)\|^2 + \frac{1}{4a} \sum_{i=1}^n \|e_i(t)\|^2, \end{aligned} \quad (12)$$

for all  $0 < a < 1$ . From (10), we know that

$$\|e_i(t)\| \leq \alpha e^{-\beta t}, \forall t \geq 0. \quad (13)$$

Hence

$$\dot{\nu}(\Delta, y(t)) \leq \frac{n\alpha^2}{4a} e^{-2\beta t}, \forall t \geq 0.$$

Thus

$$\begin{aligned} \nu(\Delta, y(t)) &\leq \nu(\Delta, y(0)) + \frac{n\alpha^2}{8a\beta} [1 - e^{-2\beta t}] \\ &\leq k_\nu, \forall t \geq 0, \end{aligned} \quad (14)$$

where

$$\begin{aligned} k_\nu &= \nu(\Delta, y(0)) + \frac{n\alpha^2}{8a\beta} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \in N_i} \frac{\|x_i(0) - x_j(0) - d_{ij}\|^2}{\Delta - \|d_{ij}\| - \|x_i(0) - x_j(0) - d_{ij}\|} + \frac{n\alpha^2}{8a\beta}. \end{aligned} \quad (15)$$

Then, for any  $(i, j) \in \mathcal{E}(\mathcal{G})$  and  $t \geq 0$ , we have

$$\nu_{ij}(\Delta, y(t)) \leq 2\nu(\Delta, y(t)) \leq 2k_\nu.$$

Hence

$$\|y_i(t) - y_j(t)\| \leq k_{ij}, \quad (16)$$

where

$$k_{ij} = -k_\nu + \sqrt{k_\nu^2 + 2k_\nu(\Delta - \|d_{ij}\|)} < \Delta - \|d_{ij}\|. \quad (17)$$

Then, we have

$$\begin{aligned} \|x_i(t) - x_j(t)\| &= \|x_i(t) - \tau_i - (x_j(t) - \tau_j) + d_{ij}\| \\ &\leq \|y_i(t) - y_j(t)\| + \|d_{ij}\| \leq k_{ij} + \|d_{ij}\| < \Delta, \end{aligned} \quad (18)$$

and thus connectivity maintenance is established.

(ii) Let  $e(t) = (e_1^\top(t), \dots, e_n^\top(t))^\top$ ,  $\bar{y}(t) = \frac{1}{n} \sum_{i=1}^n y_i(t)$  and  $\delta(t) = y(t) - \mathbf{1}_n \otimes \bar{y}(t) = (K_n \otimes I_p)y(t)$ . We consider the Lyapunov candidate

$$V(t) = \frac{1}{2} \delta^\top(t) \delta(t) = \frac{1}{2} y^\top(t) (K_n \otimes I_p) y(t). \quad (19)$$

Then its derivative along the trajectories of the multi-agent system (4) with event-triggered controller (7) is

$$\begin{aligned}
\dot{V}(t) &= y^\top(t)(K_n \otimes I_p)\dot{y}(t) \\
&= y^\top(t)(K_n \otimes I_p)[-(DWD^\top \otimes I_p)y(t) + e(t)] \\
&= -y^\top(t)(DWD^\top \otimes I_p)y(t) + \delta^\top(t)e(t) \\
&\leq -\frac{2}{\Delta_0}y^\top(t)(DD^\top \otimes I_p)y(t) + \beta_0\delta^\top(t)\delta(t) + \frac{1}{4\beta_0}\|e(t)\|^2 \\
&\leq -2\beta_0y^\top(t)(K_n \otimes I_p)y(t) + \beta_0\delta^\top(t)\delta(t) + \frac{1}{4\beta_0}\|e(t)\|^2 \\
&\leq -2\beta_0V(t) + \frac{n\alpha^2}{4\beta_0}e^{-2\beta t}, \tag{20}
\end{aligned}$$

where the first inequality holds since  $W$  is diagonal and its diagonal elements are all not smaller than  $\frac{2}{\Delta_0}$  and the second inequality holds from Lemma 2.2. Hence

$$V(t) \leq V(0)e^{-2\beta_0 t} + \frac{n\alpha^2}{8\beta_0(\beta_0 - \beta)}[e^{-2\beta t} - 1] \tag{21}$$

$$< k_V e^{-2\beta t}, \tag{22}$$

where

$$k_V = V(0) + \frac{n\alpha^2}{8\beta_0(\beta_0 - \beta)}. \tag{23}$$

Thus

$$\begin{aligned}
\|y_i(t) - y_j(t)\|^2 &\leq 2\|y_i(t) - \bar{y}(t)\|^2 + 2\|\bar{y}(t) - y_j(t)\|^2 \\
&\leq 4V(t) < 4k_V e^{-2\beta t}. \tag{24}
\end{aligned}$$

Hence

$$\lim_{t \rightarrow \infty} x_i(t) - x_j(t) = \lim_{t \rightarrow \infty} y_i(t) - \tau_i - (y_j(t) - \tau_j) = d_{ij}, \tag{25}$$

exponentially.

(iii) For  $(i, j) \in \mathcal{E}(\mathcal{G})$ , define

$$f_{ij}(l) = \frac{2\Delta - 2\|d_{ij}\| - l}{(\Delta - \|d_{ij}\| - l)^2}, l \in [0, \Delta - \|d_{ij}\|], \tag{26}$$

$$g_{ij}(l) = \frac{2(\Delta - \|d_{ij}\|)^2}{(\Delta - \|d_{ij}\| - l)^3}, l \in [0, \Delta - \|d_{ij}\|], \tag{27}$$

$$h_{ij}(l) = \frac{3\Delta - 3\|d_{ij}\| - l}{(\Delta - \|d_{ij}\| - l)^3}, l \in [0, \Delta - \|d_{ij}\|]. \tag{28}$$

We can easily check that both  $f_{ij}(l)$ ,  $g_{ij}(l)$  and  $h_{ij}(l)$  are increasing functions on  $[0, \Delta - \|d_{ij}\|]$ . Then from (16), we have

$$0 < \omega_{ij}(t) < f_{ij}(k_{ij}), \forall t \geq 0. \tag{29}$$

Then

$$\begin{aligned}
\|\dot{y}_i(t)\| &= \|e_i(t) - \sum_{j \in N_i} \omega_{ij}(t)(y_i(t) - y_j(t))\| \\
&\leq \|e_i(t)\| + \sum_{j \in N_i} \omega_{ij}(t)\|y_i(t) - y_j(t)\| \\
&< \alpha e^{-\beta t} + \sum_{j \in N_i} 2f_{ij}(k_{ij})\sqrt{k_V}e^{-\beta t}. \tag{30}
\end{aligned}$$

From

$$\begin{aligned}
&\dot{e}_i(t) \\
&= \sum_{j \in N_i} [\dot{\omega}_{ij}(t)(y_i(t) - y_j(t)) + \omega_{ij}(t)(\dot{y}_i(t) - \dot{y}_j(t))] \\
&= \sum_{j \in N_i} \left\{ h_{ij}(\|y_i(t) - y_j(t)\|) \frac{(y_i(t) - y_j(t))^\top}{\|y_i(t) - y_j(t)\|} (\dot{y}_i(t) - \dot{y}_j(t)) \right. \\
&\quad \left. \times (y_i(t) - y_j(t)) + \omega_{ij}(t)(\dot{y}_i(t) - \dot{y}_j(t)) \right\}, \tag{31}
\end{aligned}$$

we have

$$\begin{aligned}
\frac{d\|e_i(t)\|}{dt} &\leq \|\dot{e}_i(t)\| \\
&\leq \sum_{j \in N_i} g_{ij}(\|y_i(t) - y_j(t)\|)(\|\dot{y}_i(t)\| + \|\dot{y}_j(t)\|) \\
&\leq \sum_{j \in N_i} g_{ij}(k_{ij})(\|\dot{y}_i(t)\| + \|\dot{y}_j(t)\|) < c_i e^{-\beta t}, \tag{32}
\end{aligned}$$

where

$$\begin{aligned}
c_i &= \sum_{j \in N_i} g_{ij}(k_{ij}) \left[ 2\alpha + \sum_{l \in N_i} 2f_{il}(k_{il})\sqrt{k_V} \right. \\
&\quad \left. + \sum_{l \in N_j} 2f_{jl}(k_{jl})\sqrt{k_V} \right]. \tag{33}
\end{aligned}$$

Thus, a sufficient condition to guarantee

$$\alpha e^{-\beta t} \geq \|e_i(t)\| = \int_{t_k^i}^t \frac{d\|e_i(s)\|}{ds} ds, \forall t \in [t_k^i, t_{k+1}^i),$$

is

$$\begin{aligned}
\alpha e^{-\beta t} &\geq \int_{t_k^i}^t c_i e^{-\beta s} ds = \frac{c_i}{\beta} [e^{-\beta t_k^i} - e^{-\beta t}] \\
&\Leftrightarrow (c_i + \alpha\beta)e^{-\beta t} \geq c_i e^{-\beta t_k^i} \Leftrightarrow (c_i + \alpha\beta)e^{-\beta(t-t_k^i)} \geq c_i \\
&\Leftrightarrow (c_i + \alpha\beta)[1 - \beta(t-t_k^i)] \geq c_i \Leftrightarrow t - t_k^i \leq \xi_i,
\end{aligned}$$

where

$$\xi_i = \frac{\alpha}{c_i + \alpha\beta} > 0. \tag{34}$$

In other words, for all  $t \in [t_k^i, t_k^i + \xi_i]$ ,  $\|e_i(t)\| < \alpha e^{-\beta t}$  holds. Hence  $t_{k+1}^i > t_k^i + \xi_i$ .

*Remark 3.2.* In order to determine  $\beta$ ,  $\beta_0$  should be known first. However  $\beta_0$  is a global parameter since it relates to  $\rho_2(DD^\top)$  and  $\Delta_0$ . We can lower bound  $\beta_0$  by  $\frac{4}{n(n-1)\Delta}$  since  $\Delta_0 < \Delta$  and  $\rho_2(DD^\top) \geq \frac{4}{n(n-1)}$ , see Mohar (1991).

Apparently, in order to check the inequality in the triggering condition (10), continuous sensing of the relative positions between neighbors is needed, which is a drawback. In the following we will give an algorithm to avoid this. In other words, the following algorithm is an implementation of Theorem 3.1 but it only requires agents to sense and to broadcast, at the triggering times. We name this algorithm as *Event-Triggered Algorithm-1*, which is illustrated as follows.

For any agent  $i$ , at any time  $s \geq 0$ , it knows its last triggering time  $t_{k_i(s)}^i$  and its control input  $u_i(s) = u_i(t_{k_i(s)}^i)$  which is a constant until it determines its next triggering time. If agent  $i$  also knows the relative position  $x_i(s) - x_j(s)$  and  $u_j(s) = u_j(t_{k_j(s)}^j)$  which is a constant until agent  $j$  determines its next triggering time, for  $j \in N_i$ . Then agent  $i$  can predict

$$\begin{aligned}
x_i(t) - x_j(t) &= x_i(s) - x_j(s) \\
&\quad + (t-s)[u_i(t_{k_i(s)}^i) - u_j(t_{k_j(s)}^j)], \tag{35}
\end{aligned}$$

until  $t \leq \min\{t_{k_i(s)+1}^i, t_{k_j(s)+1}^j\}$ . This means continuous sensing or broadcasting are not needed any more. Here, we name  $\{t_{k_i(t)}^i, u_i(t_{k_i(t)}^i), x_i(t_{k_i(t)}^i) - x_j(t_{k_i(t)}^i)\}$  as the *triggering information* that agent  $i$  sends to agent  $j$  for  $j \in N_i$ . In conclusion, we propose the following algorithm.

*Event-Triggered Algorithm-1 (ETA-1):*

- 1: Choose  $\alpha > 0$  and  $0 < \beta < \beta_0$ ;
- 2: Initialize  $t_1^i = 0$  and  $k = 1$ ;
- 3: At time  $s = t_k^i$ , agent  $i$  uses (35) to predict the relative position  $x_i(t) - x_j(t), \forall j \in N_i$ ;
- 4: Agent  $i$  substitutes these relative positions into  $e_i(t)$  and finds out  $\tau_{k+1}^i$  which is the smallest solution of equation  $\|e_i(t)\| = \alpha e^{-\beta t}$ ;
- 5: **if** there exists  $j \in N_i$  such that agent  $j$  broadcasts its triggering information at  $t_0 \in (s, \tau_{k+1}^i)$  **then**
- 6: agent  $i$  updates  $s = t_0$  and goes back to Step 3;
- 7: **else**
- 8: agent  $i$  determines  $t_{k+1}^i = \tau_{k+1}^i$ , updates its control input  $u_i(t_{k+1}^i)$  by sensing the relative positions between itself and its neighbors, broadcasts its triggering information to its neighbors, resets  $k = k + 1$ , and goes back to Step 3;
- 9: **end if**

When applying above algorithm, although continuous sensing and broadcasting are avoided, each agent still needs to continuously listen for incoming information from its neighbors since the triggering times are not known in advance. Our future work will focus on proposing self-triggered algorithms to avoid this.

#### 4. FORMATION CONTROL WITH CONNECTIVITY PRESERVATION: DOUBLE INTEGRATORS

In this section, we consider the case when the dynamics of agents can be modeled as double integrators given by

$$\begin{cases} \dot{x}_i(t) = q_i(t), \\ \dot{q}_i(t) = u_i(t), \quad i \in \mathcal{V}, \end{cases} \quad (36)$$

where  $x_i(t) \in \mathbb{R}^p$  still denotes the position of agent  $i$  at time  $t$ ,  $q_i(t) \in \mathbb{R}^p$  denotes the speed and  $u_i(t) \in \mathbb{R}^p$  is the control input.

Denote

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, z_i(t) = \begin{bmatrix} y_i(t) \\ q_i(t) \end{bmatrix}.$$

One can easily check that  $(C, B)$  is controllable and  $(I_2, C)$  is observable. Hence, from Kucera (1972), we know that there exist positive constants  $k_0, k_1$  and  $k_2$  such that

$$P > 0, \quad \frac{1}{2}(PC + C^T P) - \beta_1 P B B^T P + 2I_2 \leq 0, \quad (37)$$

with  $P = \begin{bmatrix} k_0 & k_1 \\ k_1 & k_2 \end{bmatrix}$  and  $0 < \beta_1 \leq \beta_0$ .

Similar to the event-triggered controller (8), we use the following event-triggered controller

$$\begin{aligned} u_i(t) &= -k_1 \sum_{j \in N_i} \omega_{ij}(t_{k_i}^i) (x_i(t_{k_i}^i) - x_j(t_{k_i}^i) - d_{ij}) \\ &\quad - k_2 \sum_{j \in N_i} \omega_{ij}(t_{k_i}^i) (q_i(t_{k_i}^i) - q_j(t_{k_i}^i)) - k_3 q_i(t), \end{aligned} \quad (38)$$

<sup>1</sup> This kind of situation can only occur at most finite times during  $(s, \tau_{k+1}^i)$  since  $|N_i|$  is finite and inter-event times of each agent are lower bounded by a positive constant.

where  $k_3$  is a constant which will be determined later. Here we should highlight that this controller needs continuous feedback because of the item  $k_3 q_i(t)$ .

Similar to Theorem 3.1, we have the following results.

**Theorem 4.1.** Given a graph  $\mathcal{G}$  which is undirected and connected and a formation associated with  $\mathcal{G}$  which satisfies Assumption 2.3. Given  $0 < \beta \leq \beta_0$ , determine  $P$  by (37). Consider the multi-agent system (36) with event-triggered controller (38) associated with  $\mathcal{G}$ . Assume the initial position condition is also satisfied (9) for all  $(i, j) \in \mathcal{E}(\mathcal{G})$ . Given  $0 < k_3 < \frac{4}{k_2 + \sqrt{k_1^2 + k_2^2}}$ , and given  $\alpha_d > 0$ , and

$0 < \beta_d < (2 - k_4)\rho_2(P)$  with  $k_4 = k_3 \frac{k_2 + \sqrt{k_1^2 + k_2^2}}{2} < 2$ , and given the first triggering time  $t_1^i = 0$ , agent  $i$  determines the triggering times  $\{t_k^i\}_{k=2}^\infty$  by

$$t_{k+1}^i = \max_{r \geq t_k^i} \{r : \|E_i(t)\| \leq \alpha_d e^{-\beta_d t}, \forall t \in [t_k^i, r]\} \quad (39)$$

where

$$\begin{aligned} E_i(t) &= k_1 e_i(t) + k_2 e_{qi}(t), \\ e_{qi}(t) &= \sum_{j \in N_i} \omega_{ij}(t) (q_i(t) - q_j(t)) \\ &\quad - \sum_{j \in N_i} \omega_{ij}(t_{k_i}^i) (q_i(t_{k_i}^i) - q_j(t_{k_i}^i)). \end{aligned}$$

Then the multi-agent system (36) with event-triggered controller (38) converges to the formation exponentially with connectivity preservation, and there is no Zeno behavior.

**Proof.** Due to the length constraints, the proof is omitted here. Detailed proof can be found in Yi et al. (2016).

Similar to the analysis after Theorem 3.1, in order to check the inequality in the triggering condition (39), continuous sensing of the relative positions and speeds between neighbors is needed. In the following we will give an algorithm to implement Theorem 4.1 and at the same time to avoid continuous sensing by using the same idea as ETA-1. Denote

$$\begin{aligned} u_i^d(t) &= -k_1 \sum_{j \in N_i} \omega_{ij}(t_{k_i}^i) (x_i(t_{k_i}^i) - x_j(t_{k_i}^i) - d_{ij}) \\ &\quad - k_2 \sum_{j \in N_i} \omega_{ij}(t_{k_i}^i) (q_i(t_{k_i}^i) - q_j(t_{k_i}^i)), \end{aligned}$$

which is piecewise constant.

Similar to (35), agent  $i$  can predict

$$\begin{aligned} x_i(t) - x_j(t) &= \frac{1}{k_3} [1 - e^{k_3(s-t)}] [q_i(s) - q_j(s)] \\ &\quad + [k_3(t-s) - 1 + e^{k_3(s-t)}] \frac{u_i^d(t_{k_i}^i(s)) - u_j^d(t_{k_j}^j(s))}{(k_3)^2}, \end{aligned} \quad (40)$$

$$\begin{aligned} q_i(t) - q_j(t) &= e^{k_3(s-t)} [q_i(s) - q_j(s)] \\ &\quad + [1 - e^{k_3(s-t)}] \frac{u_i^d(t_{k_i}^i(s)) - u_j^d(t_{k_j}^j(s))}{k_3}, \end{aligned} \quad (41)$$

And, we propose the following algorithm.

*Event-Triggered Algorithm-2 (ETA-2):*

- 1: Choose  $0 < \beta_1 \leq \beta_0$  and determine  $P$  by (37);

- 2: Choose  $0 < k_3 < \frac{4}{k_2 + \sqrt{k_1^2 + k_2^2}}$ ,  $\alpha_d > 0$  and  $0 < \beta_d < (2 - k_4)\rho_2(P)$ ;
- 3: Initialize  $t_1^i = 0$  and  $k = 1$ ;
- 4: At time  $s = t_k^i$ , agent  $i$  uses (40) and (41) to predict the relative position  $x_i(t) - x_j(t)$  and the relative speed  $q_i(t) - q_j(t)$ ,  $\forall j \in N_i$ ;
- 5: Agent  $i$  substitutes these into  $E_i(t)$  and finds out  $\tau_{k+1}^i$  which is the smallest solution of equation  $\|E_i(t)\| = \alpha_d e^{-\beta_d t}$ ;
- 6: **if** there exists  $j \in N_i$  such that agent  $j$  broadcasts its triggering information at  $t_0 \in (s, \tau_{k+1}^i)$  **then**
- 7:     agent  $i$  updates  $s = t_0$  and goes back to Step 4;
- 8: **else**
- 9:     agent  $i$  determines  $t_{k+1}^i = \tau_{k+1}^i$ , updates  $u_i(t_{k+1}^i)$  by sensing the relative positions and relative speeds between itself and its neighbors, broadcasts its triggering information to its neighbors, resets  $k = k + 1$ , and goes back to Step 4;
- 10: **end if**

As noted earlier, each agent still needs to continuously listen for incoming information from its neighbors, which will be our future work.

## 5. SIMULATIONS

In this section, two numerical examples are given to demonstrate the effectiveness of the presented results.

Consider a network of  $n = 3$  agents in  $\mathbb{R}^2$  whose Laplacian matrix is given by

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The three agents are trying to establish a right triangle formation with

$$d_{12} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, d_{13} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, d_{23} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

The communication radius is  $\Delta = 4$ . We have  $\beta_0 = 1.5$ .

Firstly, we consider the situation that the three agents are modeled as single integrators. The initial positions of agents can be randomly selected as long as the initial condition (9) can be satisfied. Here, the initial position of agents are given by

$$x_1(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_2(0) = \begin{pmatrix} 3.5 \\ 7 \end{pmatrix}, x_3(0) = \begin{pmatrix} 4.5 \\ 5.5 \end{pmatrix}.$$

One can easily check that both Assumption 2.3 and initial condition (9) hold. Choose  $\alpha = 10$  and  $\beta = 1$ , by applying the ETA-1, we get the evolution of the formation shown in Fig. 1 (a), where “circles” denote the initial positions and “triangle” denotes the desired formation, and the triggering times of each agent shown in Fig. 1 (b), respectively.

Secondly, we consider the situation that the three agents are modeled as double integrators. The initial positions of agents are given as before. The initial speeds of agents can be randomly selected and here we choose

$$q_1(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, q_2(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, q_3(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

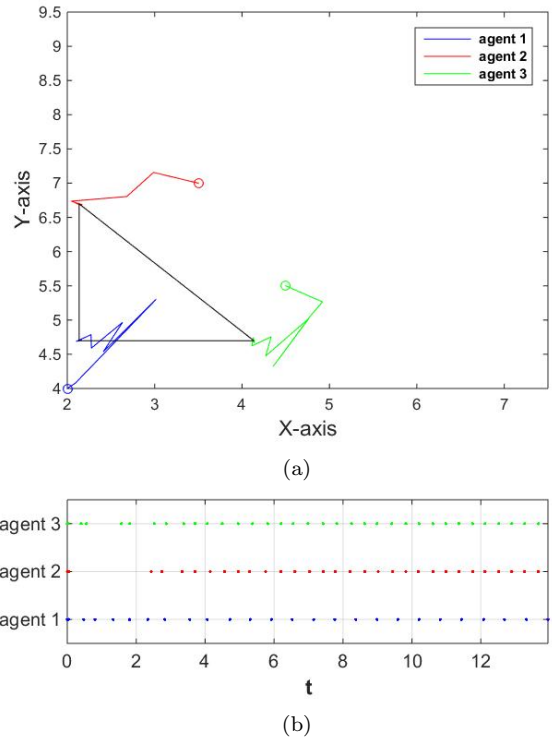


Fig. 1. (a) The trajectories evolution of the multi-agent system (3) with event-triggered controller (8) when performing ETA-1. (b) The triggering times of each agent.

We have  $P = \begin{bmatrix} 5.0237 & 1.1547 \\ 1.1547 & 1.4502 \end{bmatrix}$ ,  $k_1 = 1.1547$ ,  $k_2 = 1.4502$ , and  $\rho_2(P) = 1.1096$ . Choose  $k_3 = \frac{2}{k_2 + \sqrt{k_1^2 + k_2^2}} = 0.6053$ ,  $\alpha_d = 10$ , and  $\beta_d = 1$ , by applying the ETA-2, we get the evolution of the formation shown in Fig. 2 (a), where “circles” denote the initial positions and “triangle” denotes the desired formation, and the triggering times of each agent shown in Fig. 2 (b), respectively. It can be seen that double integrators have more smooth trajectories compared with single integrators.

## 6. CONCLUSIONS

In this paper, formation control for multi-agent systems with limited sensing and broadcasting radius is addressed. We first consider the situation that agents are modeled as single integrators. An event-triggered controller and a corresponding algorithm, to avoid continuous sensing and broadcasting, are proposed. Each agent only needs to sense and to broadcast at its triggering times, and to receive at its neighbors’ triggering times. As a result, the desired formation can be established exponentially with connectivity preservation and exclusion of Zeno behavior. Then, these results are extended to double integrators. The drawback of our approach is that each agent still needs to continuously listen for incoming information from its neighbors. Future research directions of this work include developing self-triggered algorithm to avoid this.

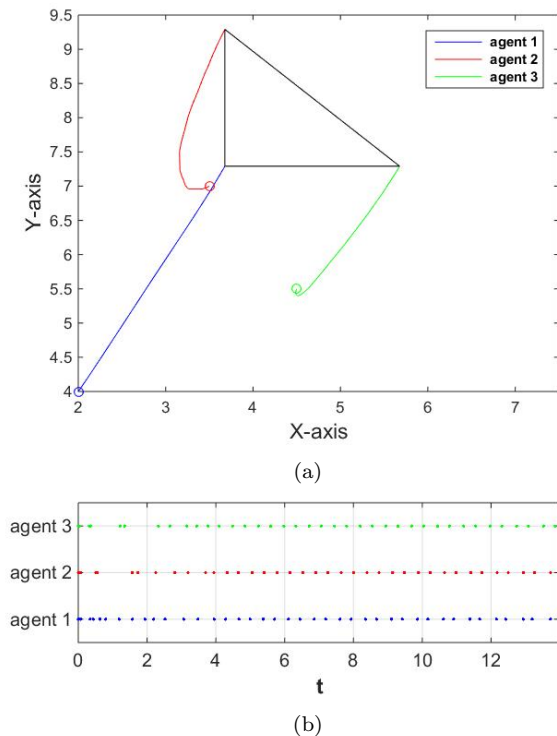


Fig. 2. (a) The trajectories evolution of the multi-agent system (36) with event-triggered controller (38) when performing ETA-2. (b) The triggering times of each agent.

## REFERENCES

- Ajorlou, A., Momeni, A., and Aghdam, A.G. (2010). A class of bounded distributed control strategies for connectivity preservation in multi-agent systems. *IEEE Transactions on Automatic Control*, 55(12), 2828–2833.
- Åström, K.J. and Bernhardsson, B. (1999). Comparison of periodic and event based sampling for first-order stochastic systems. In *Proceedings of the 14th IFAC World congress*, volume 11, 301–306. Citeseer.
- Boskos, D. and Dimarogonas, D.V. (2015). Robust connectivity analysis for multi-agent systems. In *2015 54th IEEE Conference on Decision and Control (CDC)*, 6767–6772.
- Dimarogonas, D.V., Frazzoli, E., and Johansson, K.H. (2012). Distributed event-triggered control for multi-agent systems. *IEEE Transactions on Automatic Control*, 57(5), 1291–1297.
- Dimarogonas, D.V. and Kyriakopoulos, K.J. (2007). Connectivity preserving state agreement for multiple unicycles. In *2007 American Control Conference*, 1179–1184.
- Fan, Y. and Hu, G. (2015). Connectivity-preserving rendezvous of multi-agent systems with event-triggered controllers. In *2015 54th IEEE Conference on Decision and Control (CDC)*, 234–239.
- Jadbabaie, A., Lin, J., and Morse, A.S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on automatic control*, 48(6), 988–1001.
- Ji, M. and Egerstedt, M.B. (2007). Distributed coordination control of multi-agent systems while preserving connectedness.
- Johansson, K.H., Egerstedt, M., Lygeros, J., and Sastry, S. (1999). On the regularization of zeno hybrid automata. *Systems & Control Letters*, 38(3), 141–150.
- Kan, Z., Dani, A.P., Shea, J.M., and Dixon, W.E. (2012). Network connectivity preserving formation stabilization and obstacle avoidance via a decentralized controller. *IEEE Transactions on Automatic Control*, 57(7), 1827–1832.
- Kucera, V. (1972). A contribution to matrix quadratic equations. *IEEE Transactions on Automatic Control*, 17(3), 344–347.
- Liu, B., Lu, W., and Chen, T. (2011). Consensus in networks of multiagents with switching topologies modeled as adapted stochastic processes. *SIAM Journal on Control and Optimization*, 49(1), 227–253.
- Meng, X., Xie, L., Soh, Y.C., Nowzari, C., and Pappas, G.J. (2015). Periodic event-triggered average consensus over directed graphs. In *Decision and Control (CDC), 2015 IEEE 54th Annual Conference on*, 4151–4156. IEEE.
- Mesbahi, M. and Egerstedt, M. (2010). *Graph theoretic methods in multiagent networks*. Princeton University Press.
- Mohar, B. (1991). Eigenvalues, diameter, and mean distance in graphs. *Graphs and combinatorics*, 7(1), 53–64.
- Moreau, L. (2004). Stability of continuous-time distributed consensus algorithms. In *2004 43rd IEEE Conference on Decision and Control (CDC)*, 3998–4003.
- Nowzari, C. and Cortés, J. (2016). Distributed event-triggered coordination for average consensus on weight-balanced digraphs. *Automatica*, 68, 237–244.
- Olfati-Saber, R. and Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9), 1520–1533.
- Ren, W. and Beard, R.W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5), 655–661.
- Seyboth, G.S., Dimarogonas, D.V., and Johansson, K.H. (2013). Event-based broadcasting for multi-agent average consensus. *Automatica*, 49(1), 245–252.
- Spanos, D.P. and Murray, R.M. (2004). Robust connectivity of networked vehicles. In *2004 43rd IEEE Conference on Decision and Control (CDC)*, 2893–2898.
- Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9), 1680–1685.
- Yi, X., Wei, J., Dimarogonas, D.V., and Johansson, K.H. (2016). Formation control for multi-agent systems with connectivity preservation and event-triggered controllers. *arXiv:1611.03105*.
- Yu, H. and Antsaklis, P.J. (2012). Formation control of multi-agent systems with connectivity preservation by using both event-driven and time-driven communication. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 7218–7223.
- Zavlanos, M.M. and Pappas, G.J. (2007). Potential fields for maintaining connectivity of mobile networks. *IEEE Transactions on Robotics*, 23(4), 812–816.