Stability of Positive Switched Linear Systems: Weak Excitation and Robustness to Time-Varying Delay

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***Abstract*—This article investigates the stability of positive switched linear systems. We start from motivating examples and focus on the case when each switched subsystem is marginally stable (in the sense that all the eigenvalues of the subsystem matrix are in the closed left-half plane with those on the imaginary axis simple) instead of asymptotically stable. A weak excitation condition is first proposed such that the considered positive switched linear system is exponentially stable. An extension to the case without dwell time assumption is also presented. Then, we study the influence of time-varying delay on the stability of the considered positive switched linear system. We show that the proposed weak excitation condition for the delay-free case is also sufficient for the asymptotic stability of the positive switched linear system under unbounded time-varying delay. In addition, it is shown that the convergence rate is exponential if there exists an upper bound for the delay, irrespective of the magnitude of this bound. The motivating examples are revisited to illustrate the theoretical results.**

***Index Terms*—Eigenvalues, positive switched linear systems, time-varying delay.**

I. Introduction

Over the past few decades, huge efforts have been devoted to the study of switched systems as such systems are often encountered in practical applications [1], [2]. A switched system can give rise to rich dynamics as it switches between a family of subsystems and the switching signal can be arbitrary. Stability analysis has been one of the main focuses of the research on switched systems, e.g., [2], [3].

More recently, switched positive linear systems have attracted attention from many areas, e.g., economics, biology, sociology, and communication. A typical switched positive system is the virus mutation treatment model [4]. For a switched positive linear system, each subsystem is a positive linear system, so the state remains nonnegative when the initial condition is nonnegative. It was shown in [5] that Hurwitz stability of the convex hull of a set of Metzler matrices is necessary for the stability of the associated switched positive linear system under arbitrary switching, and that this condition becomes sufficient for two-dimensional systems. Furthermore, Fainshil *et al.* showed that the above condition is not sufficient for three- or higher-dimensional systems [6]. Some weaker condition was proposed in [7] for a class of continuous-time positive switched systems with subsystems described by Metzler matrices taking the form $A + bc_i^T$, $i \in \{1, 2, \dots, n\}$, where A is a diagonal $n \times n$ matrix. In addition, the stability of a class of nonlinear positive systems, i.e., subhomogeneous cooperative systems, was studied in [8].

In many biological and engineered systems, time delays occur in the system dynamics. In biological systems they arise in the signal transduction. Analogously, in engineered networked systems they are often caused by data transmissions via wired or wireless communication channels. The case of constant delay was studied in [9], [10] for some classes of nonlinear positive systems and delay-independent conditions for stability were established. The stability and input-output gain of linear systems with time delays and cone invariance were considered in [11]. Here, the cone invariance property can be viewed as a generalization of the nonnegativity property of a positive system. It was shown in [12] that a positive system is asymptotically stable for any bounded delay if and only if the sum of all the system matrices is a Hurwitz matrix. In certain applications, the delay can grow unbounded, which may have a significant impact on the system behavior. In [13], it was shown that a class of positive systems, whose vector fields are homogeneous and order-preserving, are insensitive to a general class of time delays that may be unbounded and time-varying. For switched positive systems, such a robustness property to unbounded delay still holds under certain conditions. In [14], some stability criteria were established for switched positive linear systems with unbounded time delay. However, the criteria were derived under conservative assumptions on the excitation conditions. In addition, it should be noted that in most of the above results for positive switched linear systems, it is assumed that each subsystem is asymptotically stable.

In this article, we focus on the stability of positive switched linear systems with and without delay. Each subsystem is assumed to be marginally stable. The motivation includes consensus and congestion control problems [15], [16]. Similar problems were also considered in analyzing the convergence of the products of stochastic matrices for discrete-time systems [17]–[20]. The main contributions of this article are conditions for the stability of positive switched linear systems with time-varying delay. To this end, we propose a weak excitation condition such that the considered system is exponentially stable and extend the discussion to the case without a dwell time assumption. In addition, it is shown that the proposed weak excitation condition for the delay-free case is sufficient for asymptotic stability under unbounded time-varying delay. It is further shown that the delayed positive switched linear system is exponentially stable when there exists an arbitrary upper bound for the time-varying delay.

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The structure of the article is as follows. We introduce the considered problems in Section II and provide two motivating examples in Section III. The exponential stability result for positive switched systems without delays is established in Section IV. Then, the robustness of the system to unbounded time-varying delay is studied in Section V together with the case of arbitrary bounded delay. Simulation results for the motivating examples are given in Section VI and a brief concluding remark in Section VII.

Notation: The set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{N}$. The nonnegative orthant of $n-$dimensional real space is represented by $\mathbb{R}^n_+$. $\mathbb{Z}^+_0$ denotes the nonnegative integer set. $\mathbf{0}$ and $\mathbf{1}$ denote the all-zero and all-one vectors with compatible dimensions, respectively. $\mathcal{C}([a, b], \mathbb{R}^n_+)$ denotes the space of all real-valued continuous functions on $[a, b]$ taking values in $\mathbb{R}^n_+$. $A \preceq B$ means that $a_{ij} \leq b_{ij}$ for all i, j, where a_{ij}, b_{ij} denote entry (i, j) of matrices A and B. Similarly, we define $A \succeq B$, $A \prec B$, and $A \succ B$. A matrix M is a Metzler matrix if the off diagonal entries of M are nonnegative. $\lceil x \rceil$ denotes the smallest integer larger than or equal to a real number x. $\|x\|$ denotes the Euclidean norm of a vector x.

II. Problem Statement

In this article, we first study the stability of delay-free switched positive linear systems. The dynamics is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \geq 0, \quad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $x_0 \in \mathbb{R}^n_+$ is a constant vector, and $\sigma(t) : [0, \infty) \to \mathcal{P}$ is a piecewise constant function, i.e., there exists a sequence of increasing time instants $\{t_l\}^\infty_0$ such that $\sigma(t)$ remains constant for $t \in [t_l, t_{l+1})$ and switches at $t = t_l$, $t_0 = 0$, and $\mathcal{P} = \{1, 2, \ldots, p\}$ is a finite set. We place a common assumption on the switching signal.

Assumption 1 (Dwell Time): There exists a lower bound $\tau_d > 0$, such that $\inf_l(t_{l+1} - t_l) \geq \tau_d$.

Remark 1: The dwell time assumption is extensively used in the analysis of convergence of switched systems, e.g., [2]. The lower bound τ_d can be arbitrarily small. Therefore, Assumption 1 can be easily satisfied in many practical applications. The relaxation of Assumption 1 will be discussed in Section IV-B.

In practice, the existence of delays in control systems are often inevitable. This motivates us to study the influence of a time-varying delay on the stability of the considered switched positive linear system. The system is described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - \tau(t)), \quad t \geq 0 \tag{2a}$$

$$x(t) = \phi(t), \quad -\tau_0 \leq t \leq 0 \tag{2b}$$

where $\tau(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $\tau_0 \geq 0$, $\phi(t) \in \mathcal{C}([-\tau_0, 0], \mathbb{R}^n_+)$ is a vector-valued function specifying the initial state of the system. We impose the following mild assumption on $\tau(t)$.

Assumption 2 (Unbounded Delay): $\lim_{t\to+\infty}(t - \tau(t)) = +\infty$. Assumption 2 means that as t increases, $\tau(t)$ should grow slower. It is not hard to check that all bounded delays satisfy this assumption. However, the converse is not true in general:

Example 1: Let $\tau(t) = t/\zeta$, where $\zeta > 1$ is a constant. It is clear that $\lim_{t\to+\infty}(t - \tau(t)) = \lim_{t\to+\infty}((\zeta - 1)/\zeta)t = +\infty$ and therefore Assumption 2 holds. However, $\tau(t)$ is not bounded.

We next define positivity and introduce a necessary assumption that guarantees positivity.

Definition 1: System (1) or (2) is said to be positive if for any switching signal σ and any nonnegative initial condition $x(0) \in \mathbb{R}^n_+$ (for (1)), or $\phi(t) \in \mathcal{C}([-\tau_0, 0], \mathbb{R}^n_+)$ (for (2)), the state trajectory $x(t) \in \mathbb{R}^n_+$, for all $t \geq 0$.

Assumption 3 (Positivity):

- For system (1), A_p is a Metzler matrix, for all $p \in \mathcal{P}$.
- For system (2), A_p is a Metzler matrix and $B_p \succeq 0$, for all $p \in \mathcal{P}$.

Remark 2: Assumption 3 is a necessary and sufficient condition to guarantee that system (1) or (2) is positive (Lemma 3, [21]).

The objective of this article is to establish stability conditions for systems (1) and (2).

III. Motivating Examples

In this section, we present two motivating examples on vehicle formation control. We show that these examples fit (1) or (2) proposed in Section II.

Consider that there are four vehicles to maintain a stable formation. Let $x_1, \ldots, x_4$ represent the position of each vehicle. The dynamics of the vehicles are described by [22]

$$\dot{x}_1(t) = -x_1(t) + l_{13}(\sigma(t))(x_3(t) - x_1(t)) \tag{3a}$$

$$\dot{x}_2(t) = l_{21}(\sigma(t))(x_1(t) - x_2(t)) + l_{23}(\sigma(t))(x_3(t) - x_2(t)) \tag{3b}$$

$$\dot{x}_3(t) = l_{32}(\sigma(t))(x_2(t) - x_3(t)) + l_{34}(\sigma(t))(x_4(t) - x_3(t)) \tag{3c}$$

$$\dot{x}_4(t) = -4x_4(t) + l_{43}(\sigma(t))(x_3(t) - x_4(t)) \tag{3d}$$

where vehicles 1 and 4 can maintain stable positions on their own, but vehicles 2 and 3 rely on the position information of their neighboring vehicles for stabilization. The parameters $l_{ij}(\sigma(t))$ represent the position adjustment based on neighboring position information. Due to possible communication failure, $\sigma(t) : [0, \infty) \to \mathcal{P}$ is a switching signal specifying when the communication topology changes. For all $p \in \mathcal{P}$, and $i, j \in \{1, 2, 3, 4\}$, $\mathrm{l_{ij}}(p)$ is positive if the information is successfully delivered and zero, otherwise.

We can easily show that (3) can be written in the form of (1) with $A_{\sigma(t)} = \begin{bmatrix} -1 - l_{13} & 0 & l_{13} & 0 \\ l_{21} & -l_{21} - l_{23} & l_{23} & 0 \\ 0 & l_{32} & -l_{32} - l_{34} & l_{34} \\ 0 & 0 & l_{43} & -4 - l_{43} \end{bmatrix}$, where $\mathrm{l_{ij}} := \mathrm{l_{ij}}(\sigma(t))$. It is not hard to show that A_p is Metzler for all $p \in \mathcal{P}$ since $\mathrm{l_{ij}}(p)$ is nonnegative for all $p \in \mathcal{P}$, and $i, j \in \{1, 2, 3, 4\}$.

We further consider the scenario that there exists a communication delay when sending information to neighboring vehicles. Then, the dynamics of the vehicles become

$$\dot{x}_1(t) = -x_1(t) + l_{13}(\sigma(t))(x_3(t - \tau(t)) - x_1(t)) \tag{4a}$$

$$\begin{aligned}\dot{x}_2(t) = {} & l_{21}(\sigma(t))(x_1(t - \tau(t)) - x_2(t)) \\ & + l_{23}(\sigma(t))(x_3(t - \tau(t)) - x_2(t))\end{aligned} \tag{4b}$$

$$\begin{aligned}\dot{x}_3(t) = {} & l_{32}(\sigma(t))(x_2(t - \tau(t)) - x_3(t)) \\ & + l_{34}(\sigma(t))(x_4(t - \tau(t)) - x_3(t))\end{aligned} \tag{4c}$$

$$\dot{x}_4(t) = -4x_4(t) + l_{43}(\sigma(t))(x_3(t - \tau(t)) - x_4(t)) \tag{4d}$$

where $\tau(t)$ represents the communication delay.

It is not hard to show that (4) can be written in the form of (2) with $A_{\sigma(t)} = \begin{bmatrix} -1 - l_{13} & 0 & 0 & 0 \\ 0 & -l_{21} - l_{23} & 0 & 0 \\ 0 & 0 & -l_{32} - l_{34} & 0 \\ 0 & 0 & 0 & -4 - l_{43} \end{bmatrix}$,

$B_{\sigma(t)} = \begin{bmatrix} 0 & 0 & l_{13} & 0 \\ l_{21} & 0 & l_{23} & 0 \\ 0 & l_{32} & 0 & l_{34} \\ 0 & 0 & l_{43} & 0 \end{bmatrix}$, where $l_{\mathrm{ij}} := l_{\mathrm{ij}}(\sigma(t))$. It is not hard to show that A_p is Metzler and $B_p \succeq 0$ for all $p \in \mathcal{P}$.

Assumption 3 is thus satisfied for these examples and systems (3) and (4) can be represented by the models (1) and (2).

IV. Exponential Stability Without Delay

Most works on switched positive linear systems assume that each element of $\{A_p\}_{p\in\mathcal{P}}$ is a Hurwitz matrix [21], [23], [24]. In this article, however, we focus on the situation that each element is only marginally stable. We first in Section IV-A state a weak excitation condition and show that system (1) is exponentially stable under this assumption. Then, in Section IV-B, we consider the case when the dwell time assumption (Assumption 1) does not hold.

Before moving on, we first define Dini derivatives and give a useful lemma. Let $D^+V(t, x(t))$ be the upper Dini derivative of $V(t, x(t))$ with respect to t, i.e., $D^+V(t, x(t)) = \limsup_{\varepsilon\to 0^+}((V(t+\varepsilon, x(t+\varepsilon)) - V(t, x(t)))/\varepsilon)$.

Lemma 1 [25]: Suppose for each $i \in \mathcal{N}$, $V_i : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ is continuously differentiable. Let $V(t, x) = \max_{i\in\mathcal{N}} V_i(t, x)$, and let $\widetilde{\mathcal{N}}(t) = \{i \in \mathcal{N} : V_i(t, x(t)) = V(t, x(t))\}$ be the set of indices where the maximum is reached at time t. Then $D^+V(t, x(t)) = \max_{i\in\widetilde{\mathcal{N}}(t)} \dot{V}_i(t, x(t))$.

A. Weak Excitation

Assumption 4 (Weak Excitation): There exists a vector $\mathbf{v} \succ \mathbf{0}$ such that $A_p\mathbf{v} \preceq \mathbf{0}$, for all $p \in \mathcal{P}$. In addition, there exist a time sequence $\{t^*_m\}_0^\infty$ and a positive constant T with $t^*_{m+1} - t^*_m < T$ such that $\sum_{z\in\{l\in\mathbb{Z}_0^+|t^*_m\le t_l<t^*_{m+1}\}} A_{\sigma(t_z)}\mathbf{v} \prec \mathbf{0}$ for all $m \in \mathbb{Z}_0^+$.

Remark 3: Firstly, the time sequenc. $\{t^*_m\}_0^\infty$ in Assumption IV can be different from the switching sequence $\{t_l\}_0^\infty$. Secondly, the existence of a common vector $\mathbf{v}$ is not restrictive as it seems to be. In fact, it corresponds to the existence of a common Lyapunov function for the stability analysis of the switched system (e.g., [1]) and can be realized in many applications (soon to be verified in Section VI). The extension to the case of multiple vectors $\mathbf{v}_p$ is possible and can be derived easily by following a similar analysis as in the proof of Theorem 1. Thirdly, the existence of the positive constant T for the weak excitation condition is indispensable for the states of the system to asymptotically converge to zero. Note that T can be arbitrarily large and therefore the exciting frequency can be very small. In addition, the set of excitation is not necessarily periodic and a non-periodic example is given as follows.

Example 2: Consider system (1) with $A_{\sigma(t)}$ switching between A_1 and A_2, where $A_1 = \begin{bmatrix} 0 & 0 \\ 1.5 & -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -2.5 & 1 \\ 0 & 0 \end{bmatrix}$. The nonperiodic signal $\sigma(t)$ switches at time instants $\kappa_1, \kappa_1+\kappa_2, \kappa_1+\kappa_2+\kappa_3, \ldots$, where κ_h, $h = 1, 2, \ldots$, is randomly chosen from the uniform distribution on the interval (1, 2) with $\kappa_0 = 0$. In addition, for each $h = 0, 2, 4, \ldots$, $\sigma(t) = 1$ for $t \in [\sum_{i=0}^h \kappa_i, \sum_{i=0}^{h+1}\kappa_i)$ and $\sigma(t) = 0$, for $t \in [\sum_{i=0}^{h+1}\kappa_i, \sum_{i=0}^{h+2}\kappa_i)$ if $\kappa_h \le 1.5$; or $\sigma(t) = 0$ for $t \in [\sum_{i=0}^h \kappa_h, \sum_{i=0}^{h+1}\kappa_i)$ and $\sigma(t) = 1$, for $t \in [\sum_{i=0}^{h+1}\kappa_i, \sum_{i=0}^{h+2}\kappa_i)$ if $\kappa_h > 1.5$. It is clear that $\sigma(t)$ is not periodic but Assumptions 1 and 4 are still satisfied with $\mathbf{v} = [1, 2]^{\mathrm{T}}$ and $T = 4$.

Theorem 1: Suppose that Assumptions 1, 3, and 4 hold. Then, system (1) is exponentially stable.

Proof: Define $V(x(t)) = \max_{i\in\mathcal{N}} x_i(t)/v_i$, where $v_i > 0$ denotes entry i of $\mathbf{v}$ in Assumption 4. As $x_i(t) \ge 0$, for all $t \ge 0$ and for all $i \in \mathcal{N}$ from Remark 2, $V(x(t)) \ge 0$ for all $t \ge 0$. Let $\widetilde{\mathcal{N}}(t) = \{i \in \mathcal{N} : x_i(t)/v_i = V(x(t))\}$ be the set of indices where the maximum is reached at time t. Then

$$D^+V = \max_{i\in\widetilde{\mathcal{N}}(t)} \frac{\dot{x}_i}{v_i} = \max_{i\in\widetilde{\mathcal{N}}(t)} \frac{\sum_{j=1}^n a^{ij}_{\sigma(t)} x_j(t)}{v_i}$$
$$= \max_{i\in\widetilde{\mathcal{N}}(t)} \frac{a^{ii}_{\sigma(t)} x_i(t) + \sum_{j\neq i} a^{ij}_{\sigma(t)} x_j(t)}{v_i}$$

where $a^{ij}_{\sigma(t)}$ denotes entry (i, j) of $A_{\sigma(t)}$ and x_i denotes entry i of x. Note that for the above equation, $(x_j/v_j) \le (x_i/v_i)$, for all $j \in \mathcal{N}$. We thus know that

$$D^+V \le \max_{i\in\widetilde{\mathcal{N}}(t)} \frac{a^{\mathrm{ii}}_{\sigma(t)} x_i(t) + \sum_{j\neq i} v_j \frac{v_i}{a}{}^{\mathrm{ij}}_{\sigma(t)} x_i(t)}{v_i}$$
$$\le \max_{i\in\widetilde{\mathcal{N}}(t)} \frac{\left(v_i a^{\mathrm{ii}}_{\sigma(t)} + \sum_{j\neq i} v_j a^{\mathrm{ij}}_{\sigma(t)}\right) x_i(t)}{v_i^2}$$

where we have used Assumption 3 and the fact that $x_i(t) \ge 0$. It thus follows from the first part of Assumption 4 that $D^+V(x(t)) \le 0$, for all $t \ge 0$.

We next show that $V(x(t))$ decreases exponentially with respect to time. Firstly, it is easy to check that the cardinality of the set $\{l \in \mathbb{Z}_0^+ | t^*_m \le t_l < t^*_{m+1}\}$ is finite and bounded by $\lceil (T/\tau_d)\rceil$ for all $m \in \mathbb{Z}_0^+$. Since $\mathcal{P}$ is finite, we know that $\sum_{z\in\{l\in\mathbb{Z}_0^+|t^*_m\le t_l<t^*_{m+1}\}} A_{\sigma(t_z)}$ takes values in a finite set of matrices for all $m \in \mathbb{Z}_0^+$. It follows from Assumption 4 that there exists a constant $\delta > 0$ such that $\sum_{z\in\{l\in\mathbb{Z}_0^+|t^*_m\le t_l<t^*_{m+1}\}} A_{\sigma(t_z)}\mathbf{v} \preceq -\delta\mathbf{1}$ for all $m \in \mathbb{Z}_0^+$. We will also need the fact that there exists a constant $a^* > 0$ such that $|a^{\mathrm{ij}}_p| \le a^*$ for all $i, j \in \mathcal{N}$ and all $p \in \mathcal{P}$, since $\mathcal{P}$ and $\mathcal{N}$ are finite.

It is not hard to show that for any $t \in [0, \infty)$, $[t, t + 2T)$ contains at least one interval $[t^*_m, t^*_{m+1})$ for some m, where T is the constant from Assumption 4. In addition, in view of the first part of Assumption 4, it is not hard to check that $\sum_{z\in\{l\in\mathbb{Z}_0^+|t\le t_l<t+2T\}} A_{\sigma(t_z)}\mathbf{v} \preceq -\delta\mathbf{1}$. Now, fix any $t_0 \ge 0$. Let $\{t_{l_2}, t_{l_3}, \ldots, t_{l_q}\}$ be the subsequence of $\{t_l\}_0^\infty$ containing all the switching points of $\sigma(t)$ during $(t_0, t_0 + 2T)$. Add t_0 and $t_0 + 2T$ into this subsequence forming the new subsequence $\{t_{l_1}, t_{l_2}, \ldots, t_{l_q}, t_{l_{q+1}}\}$ with $t_{l_1} = t_0$ and $t_{l_{q+1}} = t_0 + 2T$. Note that $q \le \lceil (2T/\tau_d)\rceil$. Let $\sigma(t) = p_z$ for $t \in [t_{l_z}, t_{l_{z+1}})$, where $z = 1, 2\ldots, q$. Based on the fact that $D^+V(x(t)) \le 0$, for all $t \ge 0$, it follows that $x_i(t) \le v_i V_0$, for all $i \in \mathcal{N}$ and all $t \ge t_0$, where $V_0 := V(x(t_0))$.

For all $t \ge t_0$, it then follows that

$$\dot{x}_i(t) = a^{\mathrm{ii}}_{\sigma(t)} x_i(t) + \sum_{j\neq i} a^{\mathrm{ij}}_{\sigma(t)} x_j(t)$$
$$\le a^{\mathrm{ii}}_{\sigma(t)} (x_i(t) - v_i V_0) + \left(a^{\mathrm{ii}}_{\sigma(t)} + \sum_{j\neq i} a^{\mathrm{ij}}_{\sigma(t)} \frac{v_j}{v_i}\right) v_i V_0$$
$$\le -a^* (x_i(t) - v_i V_0) + \alpha^i_{\sigma(t)} v_i V_0$$

where $\alpha^i_{\sigma(t)} = a^{\mathrm{ii}}_{\sigma(t)} + \sum_{j\neq i} a^{\mathrm{ij}}_{\sigma(t)}(v_j/v_i)$. Thus, $\sum_{z=1}^q \alpha^i_{p_z} \le -(\delta/v_i)$ for each $i \in \mathcal{N}$. It follows from the comparison principle (see Lemma 3.4 of [26]) that for all $i \in \mathcal{N}$,

$$\begin{aligned} & x_i\left(t_{l_{q+1}}\right) \\ & \leq \left(1 - e^{-a^*\left(t_{l_{q+1}} - t_{l_1}\right)}\right) v_i V_0 + e^{-a^*\left(t_{l_{q+1}} - t_{l_1}\right)} x_i(t_0) \\ & \quad + v_i V_0 \int_{t_{l_1}}^{t_{l_2}} e^{-a^*\left(t_{l_{q+1}} - s\right)} \alpha_{p_1}^i \mathrm{d}s \\ & \quad + v_i V_0 \int_{t_{l_2}}^{t_{l_3}} e^{-a^*\left(t_{l_{q+1}} - s\right)} \alpha_{p_2}^i \mathrm{d}s \\ & \quad + \cdots + v_i V_0 \int_{t_{l_q}}^{t_{l_{q+1}}} e^{-a^*\left(t_{l_{q+1}} - s\right)} \alpha_{p_q}^i \mathrm{d}s \\ & \leq v_i V_0 + v_i V_0 e^{-a^*\left(t_{l_{q+1}} - t_{l_2}\right)} \left(1 - e^{-a^*\left(t_{l_2} - t_{l_1}\right)}\right) \frac{\alpha_{p_1}^i}{a^*} \\ & \quad + v_i V_0 e^{-a^*\left(t_{l_{q+1}} - t_{l_3}\right)} \left(1 - e^{-a^*\left(t_{l_3} - t_{l_2}\right)}\right) \frac{\alpha_{p_2}^i}{a^*} + \cdots \\ & \quad + v_i V_0 \left(1 - e^{-a^*\left(t_{l_{q+1}} - t_{l_q}\right)}\right) \frac{\alpha_{p_q}^i}{a^*}. \end{aligned}$$

Moreover,

$$\begin{aligned} x_i\left(t_{l_{q+1}}\right) & \leq v_i V_0 + v_i V_0 \frac{e^{-a^*\left(t_{l_{q+1}} - t_{l_1}\right)} \left(1 - e^{-a^* \tau_d}\right)}{a^*} \\ & \qquad \times \left(\alpha_{p_1}^i + \alpha_{p_2}^i + \cdots + \alpha_{p_q}^i\right) \\ & \leq v_i V_0 + v_i V_0 \frac{e^{-2a^*T} \left(1 - e^{-a^* \tau_d}\right)}{a^*} \sum_{z=1}^{q} \alpha_{p_z}^i \\ & \leq v_i V_0 - v_i V_0 \frac{e^{-2a^*T} \left(1 - e^{-a^* \tau_d}\right)}{a^*} \frac{\delta}{v_i}. \end{aligned}$$

This implies that $V(x(t_0+2T)) = \max_{i\in\mathcal{N}}(x_i(t_{l_{q+1}})/v_i) \leq (1-\eta)V_0$, where $\eta = (e^{-2a^*T}\left(1 - e^{-a^*\tau_d}\right)\delta / a^* \max_{i\in\mathcal{N}}\{v_i\})$ describes how much V can be decreased after $2T$. Consider the time interval $[t_0 + 2T, t_0 + 4T]$. Based on that $D^+V(x(t)) \leq 0$, $\forall t \geq 0$, we can similarly show that $V(x(t_0 + 4T)) \leq (1 - \eta)V(x(t_0 + 2T))$. Let N be the smallest positive integer such that $t \leq t_0 + 2NT$. It then follows that $V(x(t)) \leq (1 - \eta)^{N-1}V_0 \leq \beta e^{-\gamma(t-t_0)}V_0$, where

$$\beta = \frac{a^* \max_{i\in\mathcal{N}}\{v_i\}}{a^* \max_{i\in\mathcal{N}}\{v_i\} - e^{-2a^*T}\left(1 - e^{-a^*\tau_d}\right)\delta}$$

is a constant and

$$\gamma = \frac{1}{2T} \ln \frac{a^* \max_{i\in\mathcal{N}}\{v_i\}}{a^* \max_{i\in\mathcal{N}}\{v_i\} - e^{-2a^*T}\left(1 - e^{-a^*\tau_d}\right)\delta} \tag{5}$$

describes the decreasing speed of V. By choosing $t_0 = 0$, we obtain exponential stability for any switching signal σ and any nonnegative initial condition $x(0) = x_0 \in \mathbb{R}_+^n$. ■

Remark 4: In addition to proving the exponential stability of system (1), we also explicitly derive the convergence rate. From (5), we know that a smaller T will induce a faster convergence rate. This fact agrees with the intuition because a smaller T means the excitation is more frequent.

B. No Dwell Time Assumption

Note that Theorem 1 relies on Assumption 1 on the existence of the dwell time. We show in the following proposition that Assumption 1 can be removed if the weak excitation condition of Assumption 4 is changed to an integral excitation condition.

Assumption 5 (Integral Weak Excitation): There exists a vector $\mathbf{v} \succ \mathbf{0}$ such that $A_p\mathbf{v} \preceq \mathbf{0}$, for all $p \in \mathcal{P}$. In addition, there exist a time sequence $\{t_m^*\}_0^\infty$ and positive constants ξ and T with $t_{m+1}^* - t_m^* < T$ such that $(\int_{t_m^*}^{t_{m+1}^*} A_{\sigma(s)}\mathrm{d}s)\mathbf{v} \preceq -\xi\mathbf{1}$ for all $m \in \mathbb{Z}_0^+$.

Proposition 1: Suppose that Assumptions 3 and 5 hold. Then, system (1) is exponentially stable.

Proof: Define the same Lyapunov function $V(x(t)) = \max_{i\in\mathcal{N}} x_i(t)/v_i$ as given in the proof of Theorem 1. We can show that $D^+V(x(t)) \leq 0$, for all $t \geq 0$ based on the first part of Assumption 5. We also know that there exists a constant $a^* > 0$ such that $|a_p^{\mathrm{ij}}| \leq a^*$ for all $i, j \in \mathcal{N}$ and all $p \in \mathcal{P}$. For all $t \geq t_0$, it follows that $\dot{x}_i(t) \leq a_{\sigma(t)}^{ii}(x_i(t) - v_iV_0) + \alpha_{\sigma(t)}^i v_i V_0$, where $\alpha_{\sigma(t)}^i$ and V_0 are defined in the proof of Theorem 1.

It is not hard to check that for all $t \in [0, \infty)$, $(\int_t^{t+2T} A_{\sigma(s)}\mathrm{d}s)\mathbf{v} \preceq -\xi\mathbf{1}$ from Assumption 5. Fix any $t_0 \geq 0$. Same as in the proof of Theorem 1, we define the subsequence $\{t_{l_1}, t_{l_2}, \ldots, t_{l_q}, t_{l_{q+1}}\}$ of $\{t_l\}_0^\infty$, where $t_{l_1} = t_0$ and $t_{l_{q+1}} = t_0 + 2T$. It then follows from the comparison principle that for all $i \in \mathcal{N}$,

$$\begin{aligned} x_i\left(t_{l_{q+1}}\right) & \leq v_i V_0 + v_i V_0 \int_{t_{l_1}}^{t_{l_2}} e^{a_{p_1}^{\mathrm{ii}}\left(t_{l_{q+1}} - s\right)} \alpha_{p_1}^i \mathrm{d}s \\ & \quad + v_i V_0 \int_{t_{l_2}}^{t_{l_3}} e^{a_{p_2}^{\mathrm{ii}}\left(t_{l_{q+1}} - s\right)} \alpha_{p_2}^i \mathrm{d}s + \cdots \\ & \quad + v_i V_0 \int_{t_{l_q}}^{t_{l_{q+1}}} e^{a_{p_q}^{\mathrm{ii}}\left(t_{l_{q+1}} - s\right)} \alpha_{p_q}^i \mathrm{d}s \\ & \leq v_i V_0 + v_i V_0 e^{-2a^*T} \int_{t_{l_1}}^{t_{l_2}} \alpha_{p_1}^i \mathrm{d}s + v_i V_0 e^{-2a^*T} \\ & \quad \times \int_{t_{l_2}}^{t_{l_3}} \alpha_{p_2}^i \mathrm{d}s \cdots + v_i V_0 e^{-2a^*T} \int_{t_{l_q}}^{t_{l_{q+1}}} \alpha_{p_q}^i \mathrm{d}s \\ & \leq v_i V_0 + v_i V_0 e^{-2a^*T} \int_{t_{l_1}}^{t_{l_{q+1}}} \alpha_{\sigma(s)}^i \mathrm{d}s \\ & \leq v_i V_0 - v_i V_0 e^{-2a^*T} \frac{\xi}{v_i}. \end{aligned}$$

Then, following a similar analysis as previously, we know that $V(x(t)) \leq \beta e^{-\gamma(t-t_0)} V(x(0))$, where $\beta = (\max_{i\in\mathcal{N}}\{v_i\}/(\max_{i\in\mathcal{N}}\{v_i\} - e^{-2a^*T}\xi))$ is a constant and $\gamma = (1/2T)\ln(\max_{i\in\mathcal{N}}\{v_i\}/(\max_{i\in\mathcal{N}}\{v_i\} - e^{-2a^*T}\xi))$ describes the decreasing speed of V. This implies that exponential stability is achieved for any switching signal σ and any nonnegative initial condition $x(0) = x_0 \in \mathbb{R}_+^n$ for system (1). ■

V. Asymptotic Stability With Time-Varying Delay

In this section, we focus on asymptotic stability of the time-varying system (2). Firstly, since $A_{\sigma(t)}$ and $B_{\sigma(t)}$ are piecewise continuous functions with respect to time, and $\phi(\cdot)$ and $\tau(\cdot)$ are continuous functions, it follows that there exists a unique $x(t)$ defined on $[-\tau_0, \infty)$ that coincides with $\phi(\cdot)$ on $[-\tau_0, 0]$ and satisfies (2) for all $t \geq 0$ (pp. 401–426, [27]).

A. Unbounded Time-Varying Delay

Assumption 2 allows the time-varying delay to be unbounded, but that there exists a constant $T_1 = \sup\{t \geq 0 : t - \tau(t) < 0\}$. Therefore, we can define the constant $\tau_0 = -\inf_{0\leq t\leq T_1}\{t - \tau(t)\}$. Therefore $\phi(\cdot)$ is defined on a bounded interval $[-\tau_0, 0]$.

Assumption 6 (Weak Excitation): There exists a vector $\mathbf{v} \succ \mathbf{0}$ such that $(A_p + B_p)\mathbf{v} \preceq \mathbf{0}$, for all $p \in \mathcal{P}$. In addition, there exist a time sequence $\{t_m^*\}_0^\infty$ and a positive constant T with $t_{m+1}^* - t_m^* < T$ such that $\sum_{z \in \{l \in \mathbb{Z}_0^+ | t_m^* \le t_l < t_{m+1}^*\}} (A_{\sigma(t_z)} + B_{\sigma(t_z)})\mathbf{v} \prec \mathbf{0}$ for all $m \in \mathbb{Z}_0^+$.

Theorem 2: Suppose that Assumptions 1, 3, and 6 hold. Then, system (2) is asymptotically stable for all time delay satisfying Assumption 2.

Proof: We use $V(x(t)) = \max_{i\in\mathcal{N}} x_i(t)/v_i$ as in the proof of Theorem 1. Firstly, we prove the following fact using a contradiction argument.

Fact I: $V(x(t)) \le V_0^*$, for all $t \ge 0$, where $V_0^* := \max_{-\tau_0 \le \theta \le 0} \max_{i\in\mathcal{N}} (\phi_i(\theta)/v_i)$.

(Proof of Fact I): Let $z_i(t) = (x_i(t)/v_i) - V_0^*$. It is obvious that $z_i(0) \le 0$, for all $i \in \mathcal{N}$. We next show that $z_i(t) \le 0$, for all $t \ge 0$ and $i \in \mathcal{N}$. Suppose this is not true. Then, based on the continuity of $z_i(t)$, there exists a $d \in \mathcal{N}$ and a $t^* \ge 0$ such that $z_i(t) \le 0$ for $t \in [0, t^*]$ and $i \in \mathcal{N}$, $z_d(t^*) = 0$, and $\dot{z}_d(t)|_{t=t^*} > 0$. Then, we know that for all $t \ge 0$, $\dot{x}_i(t) = a_{\sigma(t)}^{ii} x_i(t) + \sum_{j\ne i} a_{\sigma(t)}^{ij} x_j(t) + \sum_{j=1}^n b_{\sigma(t)}^{ij} x_j(t-\tau(t))$, where $b_{\sigma(t)}^{ij}$ denotes entry (i,j) of matrix $B_{\sigma(t)}$. If $t^* - \tau(t^*) \in [0, t^*]$, then $z_i(t^* - \tau(t^*)) \le 0$. This implies that $x_i(t^* - \tau(t^*)) \le v_i V_0^*$. Otherwise, if $t^* - \tau(t^*) \in [-\tau_0, 0]$, we still have $x_i(t^* - \tau(t^*)) \le v_i V_0^*$. This implies that $\dot{x}_i(t) \le a_{\sigma(t)}^{ii}(x_i(t) - v_i V_0^*) + \overline{\alpha}_{\sigma(t)}^i v_i V_0^*$, where $\overline{\alpha}_{\sigma(t)}^i = a_{\sigma(t)}^{ii} + \sum_{j\ne i} a_{\sigma(t)}^{ij}(v_j/v_i) + \sum_j b_{\sigma(t)}^{ij}(v_j/v_i)$. It then follows from Assumption 6 that $\dot{x}_d(t)|_{t=t^*} \le \overline{\alpha}_{\sigma(t^*)}^d v_d V_0^* \le 0$. This shows a contradiction and verifies Fact I.

Secondly, we prove the following fact.

Fact II: $V(t-\tau(t)) \le V_0^*$, for all $t \ge T_1$.

(Proof of Fact II): Based on the definition of T_1, we know that $t - \tau(t) \ge 0$ for all $t \ge T_1$. Thus, it follows from Fact I that $V(t - \tau(t)) \le V_0^*$, for all $t \ge T_1$.

Thirdly, we prove the following fact.

Fact III: $V(x(t)) \le (1-\eta)V_0^*$, for all $t \ge T_1 + 2T$, where η is given in the proof of Theorem 1.

(Proof of Fact III): For all $t \ge T_1$, we know that

$$\begin{aligned}\dot{x}_i(t) &= a_{\sigma(t)}^{\mathrm{ii}} x_i(t) + \sum_{j\ne i} a_{\sigma(t)}^{\mathrm{ij}} x_j(t) + \sum_{j=1}^n b_{\sigma(t)}^{\mathrm{ij}} x_j\left(t-\tau(t)\right)\\ &\le a_{\sigma(t)}^{\mathrm{ii}} x_i(t) + \sum_{j\ne i} a_{\sigma(t)}^{\mathrm{ij}} x_j(t) + \sum_{j=1}^n b_{\sigma(t)}^{\mathrm{ij}} v_j V\left(t-\tau(t)\right)\\ &\le a_{\sigma(t)}^{\mathrm{ii}}\left(x_i(t) - v_i V_0^*\right) + \overline{\alpha}_{\sigma(t)}^i v_i V_0^*\end{aligned}$$

where Fact II has been used for the second inequality. Then, by fixing any $t \ge T_1$ and following a similar analysis as for Theorem 1, we know that $V(x(t+2T)) \le (1-\eta)V(x(t)) \le (1-\eta)V_0^*$. This verifies Fact III.

We next define $T_2 = \sup\{t \ge T_1 + 2T : t - \tau(t) < T_1 + 2T\}$. From Assumption 2, T_2 is finite. Therefore, we can similarly show that $V(x(t)) \le (1-\eta)^2 V_0^*$, for all $t \ge T_2 + 2T$. Repeating this procedure, we can show that $V(x(t)) \le (1-\eta)^k V_0^*$, for all $t \ge T_k + 2T$, where T_k can be iteratively obtained. As $k \to \infty$, it follows that $V(x(t)) \to 0$. This in turn shows that $\lim_{t\to\infty} x(t) = 0$ for any nonnegative initial condition $\phi(t) \in \mathcal{C}([-\tau_0, 0], \mathbb{R}_+^n)$. ■

Remark 5: Theorem 2 shows that the weak excitation condition for the system without delay is sufficient also for the stability of the system with any time-varying delay satisfying Assumption 2. In addition, there are no restrictions on the derivative of $\tau(t)$ (so arbitrarily fast varying delays are allowed) and the upper bound of $\tau(t)$ (so unbounded varying delays are allowed). This result does not hold for general linear delayed systems; the positiveness of the system plays an indispensable role as shown in the proof.

B. Exponential Stability for Arbitrary Bounded Delay

In this section, we study system (2) with arbitrary bounded delay, with the delay being a subclass of the time-varying delay satisfying Assumption 2. We show that in this case the convergence of (2) is not only asymptotic, but even exponential.

Assumption 7 (Bounded Delay): There exists a constant $\tau_0 > 0$ such that for all $t \ge 0$, $0 \le \tau(t) \le \tau_0$.

Theorem 3: Suppose that Assumptions 1, 3, 6 hold. Then, system (2) is exponentially stable for all bounded delay satisfying Assumption 7.

Proof: Let again $V(x(t)) = \max_{i\in\mathcal{N}} x_i(t)/v_i$. Since Fact I in the proof of Theorem 2 holds for any time delay satisfying Assumption 2, it is also true for bounded delay. Thus we have $V(x(t)) \le V_0^*$ for all $t \ge 0$.

If $t - \tau(t) \le 0$, it is clear from the definition of V_0^* in the proof of Theorem 2 that $V(t-\tau(t)) \le V_0^*$. Since $V(t-\tau(t)) \le V_0^*$ also holds when $t - \tau(t) \ge 0$, we conclude that $V(t-\tau(t)) \le V_0^*$, for all $t \ge 0$.

Similar to the proof of Fact III in the proof of Theorem 2, we can show that for all $t \ge 2T$, $V(x(t)) \le (1-\eta)V_0^*$. Since $\tau(t) \le \tau_0$, when $t \ge 2T + \tau_0$, $t - \tau(t) \ge 2T$. We have that $V(t-\tau(t)) \le (1-\eta)V_0^*$ for all $t \ge 2T + \tau_0$. Repeating the arguments in Fact II of the proof of Theorem 2, one has that for all $t \ge 4T + \tau_0$, $V(x(t)) \le (1-\eta)^2 V_0^*$. Continuing this process, we derive that $V(x(t)) \le (1-\eta)^k V_0^*$, for $t \ge (k-1)(2T+\tau_0) + 2T$ and $k \ge 1$. Let N be the smallest positive integer such that $t \le N(2T+\tau_0) + 2T$. It then follows that for all $t \ge 2T$, $V(x(t)) \le (1-\eta)^N V_0^* \le e^{-\alpha(t-2T)} V_0^*$, where $\alpha = -(\ln(1-\eta)/(2T+\tau_0))$. This completes the proof. ■

VI. Motivating Examples Revisited

In this section, we revisit the motivating examples of Section III and illustrate the results of Theorems 1 and 2. The initial states are arbitrarily chosen positive constants. We first assume that $\sigma(t) \to \{1,2\}$ is switching periodically at time instants $t_l = l$, $l = 1, 2, \ldots$. For the vehicle formation without communication delay (3), the subsystems are given by

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -4 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & -2 & 1\\ 0 & 0 & 1 & -5 \end{bmatrix}$$

where A_1 represents that the communication for vehicle 3 fails and A_2 represents that the communication for vehicle 2 fails. Note that A_1 and A_2 are not asymptotically stable but Assumption 4 is satisfied with $\mathbf{v} = [1, 2, 2, 1]^{\mathrm{T}}$ and $T = 2$. Fig. 1 shows the trajectories of x and $\ln(\|x\|)$. We can see that x converges to the origin while all the states remain positive. Also, since the trajectory of $\ln(\|x\|)$ can be upper bounded by a decreasing straight line, the convergence speed is exponential, which agrees with the conclusion of Theorem 1.

We next consider the vehicle formation with communication delay (4). We still assume $\sigma(t) \to \{1,2\}$ is switching periodically at time instants $t_l = l$, $l = 1, 2, \ldots$. The subsystems are given by

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -4 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -5 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

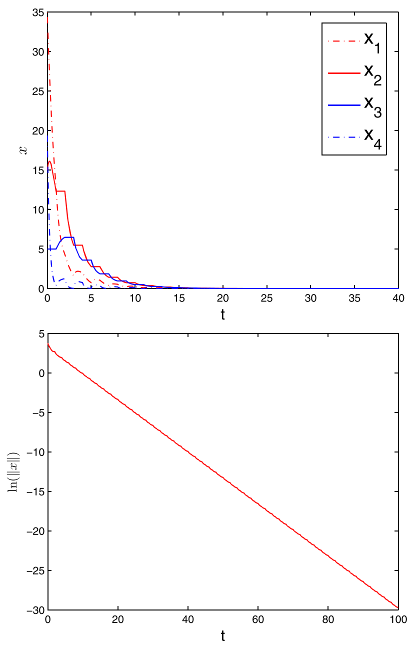

Fig. 1. State convergence for system (3).

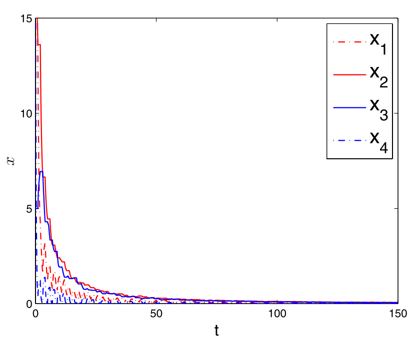

Fig. 2. State convergence for system (4) for the case of unbounded delay.

It is clear that $A_1 + B_1$ and $A_2 + B_2$ are not asymptotically stable, but Assumption 5 is satisfied with $\mathbf{v} = [1, 2, 2, 1]^{\mathrm{T}}$ and $T = 2$. We first consider the case of unbounded delay, where $\tau(t) = t/4$. We see from Fig. 2 that x converges to the origin. This agrees with the implication of Theorem 2.

We finally consider the case when the time-varying delay is bounded. The delay is chosen as $\tau(t) = \sin t$. The switching signal, the system matrices, and the initial states are the same as previously. We see from Fig. 3 that x converges to the origin for system (4) and the convergence speed is exponential. In addition, Fig. 4 shows the trajectories of $\ln(\|x\|)$ for both unbounded and bounded delays. It is clear that the case of bounded delay has a faster convergence speed.

VII. Conclusion

We have investigated the stability of positive switched linear systems under the relaxed assumption that each switched system is marginally stable. A weak excitation condition was proposed to guarantee the exponential stability of the system without time delay. We also considered positive switched linear systems with time-varying delay. It was shown that the proposed weak excitation condition for the delay-free case is sufficient for asymptotic stability under unbounded time-varying delay. The convergence rate was proven to be exponential when the delay is bounded.

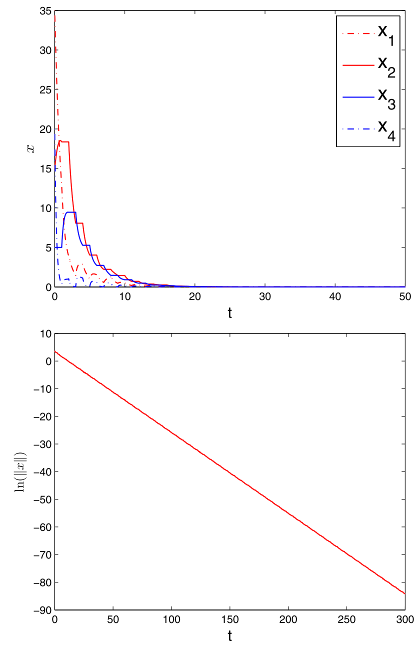

Fig. 3. State convergence of system (4) for the case of bounded delay.

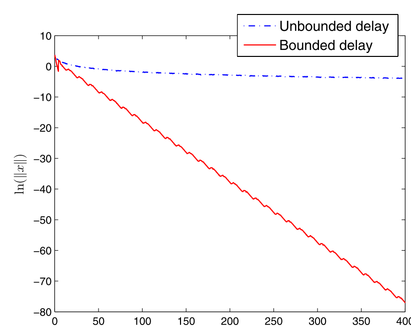

Fig. 4. Convergence speed comparison between a system with unbounded delay and one with bounded delay.

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