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A Lie Bracket Approximation for Extremum Seeking Vehicles

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Abstract: In this paper we propose a novel methodology for the analysis of autonomous vehicles seeking the extremum of an arbitrary smooth nonlinear map in the plane. By interpreting the extremum seeking schemes as input-affine systems with periodic excitations and by using the methodology of Lie brackets, we calculate a simplified system which approximates the qualitative behavior of the original one better than existing methods. By examining this approximate Lie bracket system, we are able to directly derive properties of the original one. Thus, by showing that the Lie bracket direction is directly related to the unknown gradient of the objective function we prove global uniform practical asymptotic stability of the extremum point for vehicles modeled as single integrators and non-holonomic unicycles. We illustrate the proposed method through simulations.

Keywords: Extremum Seeking, Autonomous Mobile Robots, Stability, Nonlinear Control Systems.

1. INTRODUCTION

We consider optimization problems, where the task is to steer a vehicle to an extremum of a physically measurable source such as an electromagnetic field or an acoustic noise emitted by a sender. In a stochastic setting, this problem can be translated to the task of finding the position of a noise source or the position with the highest signal to noise ratio. In many applications the analytic representation of the objective function is unknown and, in some cases, its measurements are disturbed by noise, so that it is not possible to calculate the gradient explicitly. Therefore, one is interested in a method suitable for control of autonomous vehicles without position measurements that will drive a system to an extremum point by using only online measurements of the objective function.

The extremum seeking feedback with periodic perturbations has been widely used for dealing with these problems. The authors of Zhang et al. (2007a) and Zhang et al. (2007b) analyzed local practical stability of two dimensional extremum seeking schemes for quadratic maps with decoupled coordinates, using averaging techniques. In Tan et al. (2006) semi-global practical stability was considered for certain scalar extremum seeking systems. In Stanković and Stipanović (2009) and Stanković and Stipanović (2010) the schemes were modified by introducing vanishing gains, such that almost sure convergence was achieved even in the presence of measurement noise. We propose, in this paper, a methodology that allows to deduce properties of the extremum seeking schemes in a novel and intuitive way, using the Lie bracket methodology instead of a pure averaging analysis. This methodology has been widely used for the analysis of input-affine, non-holonomic systems which allows to construct inputs in order to steer the system from an initial point to an end point. The main idea is to apply a specific switching input signal which will drive a system in directions that are not directly accessible but turn out to be the directions of the Lie brackets of the existing vector fields. By calculating inputs for an extended system consisting of virtual inputs assigned to the Lie bracket directions, it is possible to construct inputs for the original system such that the trajectories are always in a region close to the extended one. This procedure was extensively analyzed in e.g. Li and Canny (1992), Isidori (1989) and Sastry (1999) from different viewpoints. It has been shown that the switching input signals, which steer the system in the Lie bracket directions, can be chosen to be periodic such as sinusoidal functions with high frequency.

In this paper, we show that the extremum seeking can be interpreted as the Lie bracket motion, where the external perturbing sinusoids are considered as inputs which drive the system in the Lie bracket direction. We prove that this Lie bracket direction is the direction of the gradient of the unknown objective function together with some additional terms which do not appear when using the pure averaging method, even with decoupled quadratic maps. These terms contribute to a better approximation of the

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qualitative behavior of the original system compared to existing methods. We analyze convergence properties of the here proposed dynamical systems consisting of a vehicle having some specific motion dynamics in connection with the extremum seeking feedback, with arbitrary smooth nonlinear objective function. We prove global practical uniform asymptotic stability (see, e.g., Teel et al. (1998) and Chaillet and Loria (2006)) using the proposed Lie bracket system approximation and the results presented in Moreau and Aeyels (2000), where global practical stability of classes of systems depending on a small parameter was analyzed. This small parameter, which, in our case, is inversely proportional to the frequency of the perturbing sinusoids, determines the size of a region around the extremum to which the system converges. By letting the frequency go to infinity, the region contracts to a single point. We demonstrate wide applicability of the proposed methodology by applying it to the analysis of two dimensional extremum seeking schemes involving velocity actuated vehicles and non-holonomic unicycles.

This paper is structured as follows. In Section 2 we recall the mathematical preliminaries that we are using throughout the paper. In Section 3 we show how the extremum seeking systems with single integrator and unicycle motion dynamics can be approximated by their corresponding Lie bracket systems and we prove their global practical stability. In Section 4 we present simulation results. Conclusions and future work are mentioned in Section 5.

2. PRELIMINARIES

We make use of the following notation.

A function f is said to belong to the class C^∞ if it is smooth, or infinitely continuously differentiable (see also Khalil (2002)).

A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

The Jacobian of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ with components $f(x) = (f_1(x), \ldots, f_m(x))^\top$ and each $f_i : \mathbb{R}^n \to \mathbb{R}$, is denoted by

$$\frac{\partial f(x)}{\partial x} := \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}.$$

The gradient of a continuously differentiable function $J : \mathbb{R}^n \to \mathbb{R}$ with respect to x is denoted by $\nabla_x J(x) := \left(\frac{\partial J(x)}{\partial x_1}, \ldots, \frac{\partial J(x)}{\partial x_n}\right)^\top$ and $(\nabla_x J)^2$ stands for $\nabla_x J(x)^\top \nabla_x J(x)$.

The norm $||\cdot||_{C[0,T]}$ denotes $||y||_{C[0,T]} = \max_{t\in[0,T]} |y(t)|$.

The Lie bracket (cf. Sastry (1999)) of two vector fields f and g is defined as $[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$.

Before we state our main results, some mathematical preliminaries are introduced. One of the main ideas used in the upcoming proofs is to introduce the Lie bracket approximation of a system excited with periodic inputs. We will show that stability of this approximative system implies practical stability of the original one. The following theorem gives conditions on how an input-affine system can be approximated by an extended system consisting of vector-fields calculated from Lie brackets.

Consider the following system

$$\dot{x} = \sum_{i=1}^{m} b_i(x) u_i^\epsilon, \quad x \in \mathbb{R}^n, \quad b_i(x) \in C^\infty : \mathbb{R}^n \to \mathbb{R}^n \quad (1)$$

with inputs $u_i^\epsilon = \bar{u}_i(t) + \frac{1}{\sqrt{\epsilon}} \tilde{u}_i(t,\theta)$, $\epsilon > 0$, where $\tilde{u}_i$ is 2π-periodic in $\theta = t/\epsilon$, and has zero average, *i.e.*, $\int_0^{2\pi} \tilde{u}_i(t,\theta) d\theta = 0$.

Consider also the system

$$\dot{z} = \sum_{i=1}^{m} b_i(z)\bar{u}_i + \frac{1}{2\pi} \sum_{i<j} [b_i, b_j] \nu_{i,j}, \quad z(0) = x(0), \quad (2)$$

where

$$\nu_{i,j} = \int_0^{2\pi} \int_0^{\theta} \tilde{u}_i(t,\tau) \tilde{u}_j(t,\theta) d\tau d\theta. \quad (3)$$

The following lemma states the connection between these two systems in terms of the difference in their trajectories, by giving a bound that tends to zero as ϵ tends to zero.

Lemma 1. (Thm. 2.1 in Li and Canny (1992) p. 68). For sufficiently small $\epsilon > 0$, the trajectory of the system (1), is bounded by the solution of the system (2) in the sense that

$$||x - z||_{C[0,2\pi]} \le \Delta_\epsilon \quad (4)$$

where Δ_ϵ is a parameter that tend to zero as $\epsilon \to 0$.

From Lemma 1 we can prove that the trajectories of system (1) converge uniformly on compact time intervals to the trajectories of (2) as $\epsilon \to 0$. We omit the proof at this point, because a similar result for a system with two inputs can be found in the original paper of Moreau and Aeyels (2000). Under these conditions the following holds:

Lemma 2. (cf. Moreau and Aeyels (2000)). If the origin is a globally uniformly asymptotically stable equilibrium point of system (2), then for sufficiently small $\epsilon > 0$ the origin of system (1) is practically globally uniformly asymptotically stable.

By performing a change of variables the result can be extended to any point in the state space.

3. MAIN RESULTS

We divide our results in two subsections. The first one is devoted to the single integrator model of the vehicle, while the second one treats the case of unicycle dynamics. The presented results can be extended to other dynamical systems such as double-integrator dynamics (see Dürr (2010)). By adding low-pass compensators and some additional assumptions, the practical stability can be proved in a similar way.

Consider a nonlinear map $J(x)$ satisfying the following assumptions for the rest of the document:

A.1 $J \in C^\infty$
A.2 there exists a unique x^* such that $\frac{\partial J(x)}{\partial x}|_{x^*} = 0$

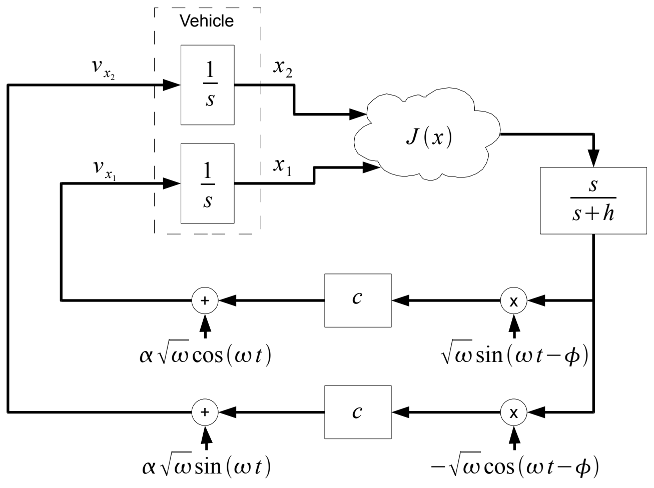


Fig. 1. Single-Integrator Extremum Seeking

A.3 $J(x) \to -\infty$ if $||x|| \to \infty$.

These assumptions imply that upper and lower bounds on $J(x)$ exist such that $\gamma_1(|x-x^*|) \leq -J(x)+J(x^*) \leq \gamma_2(|x-x^*|)$ for some functions $\gamma_1(|x|), \gamma_2(|x|) \in \mathcal{K}_\infty$.

3.1 Single-Integrator Dynamics

We will analyze the feedback in Figure 1 involving a vehicle with single-integrator dynamics. The function $J(x)$ is evaluated at the position of the vehicle denoted by $x(t) = (x_1(t), x_2(t))^\top$. We replace the filter $\frac{s}{s+h}$ with an equivalent state-space representation

$$\begin{aligned}\dot{e} &= -eh + u\\ y &= -eh + u,\end{aligned}$$

and denote the trajectory of the filter state by $e(t)$. The state-space representation of the overall system is given by

$$\begin{aligned}\dot{x}_1 &= c(J(x) - eh)\sqrt{\omega}\sin(\omega t - \phi) + \alpha\sqrt{\omega}\cos(\omega t)\\ \dot{x}_2 &= -c(J(x) - eh)\sqrt{\omega}\cos(\omega t - \phi) + \alpha\sqrt{\omega}\sin(\omega t)\\ \dot{e} &= -he + J(x).\end{aligned} \tag{5}$$

We can now state our first result.

Lemma 3. Consider the extremum seeking feedback in Equation (5) and the system

$$\begin{aligned}\dot{\bar{x}}_1 =& \frac{1}{2}(c\alpha\nabla_{\bar{x}_1}J(\bar{x})\cos(\phi) + c\alpha\nabla_{\bar{x}_2}J(\bar{x})\sin(\phi)\\ &- c^2\nabla_{\bar{x}_2}J(\bar{x})(J(\bar{x}) - \bar{e}h)))\\ \dot{\bar{x}}_2 =& \frac{1}{2}(c\alpha\nabla_{\bar{x}_2}J(\bar{x})\cos(\phi) - c\alpha\nabla_{\bar{x}_1}J(\bar{x})\sin(\phi)\\ &+ c^2\nabla_{\bar{x}_1}J(\bar{x})(J(\bar{x}) - \bar{e}h)))\\ \dot{\bar{e}} =& -\bar{e}h + J(\bar{x})\\ (\bar{x}, \bar{e})^\top|_0 =& (x, e)^\top|_0.\end{aligned} \tag{6}$$

For sufficiently large ω the trajectory of the original system (5) is bounded by solution of the system (6), such that

$$||(x, e)^\top - (\bar{x}, \bar{e})^\top||_{C[0,2\pi]} \leq \Delta_\epsilon$$

where $\epsilon = 1/\omega$ and Δ_ϵ is a parameter such that $\lim_{\epsilon\to 0}\Delta_\epsilon = 0$.

Proof. By using the identities $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$ and $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$, (5) yields to

$$\begin{aligned}\dot{x}_1 =& c(J - eh)\sqrt{\omega}\sin(\omega t)\cos(\phi)\\ &- c(J - eh)\sqrt{\omega}\cos(\omega t)\sin(\phi) + \alpha\sqrt{\omega}\cos(\omega t)\\ \dot{x}_2 =& -c(J - eh)\sqrt{\omega}\cos(\omega t)\cos(\phi)\\ &- c(J - eh)\sqrt{\omega}\sin(\omega t)\sin(\phi) + \alpha\sqrt{\omega}\sin(\omega t)\\ \dot{e} =& -eh + J\end{aligned}$$

where we omitted the argument of $J(x)$. Writing the above equation as an input-affine system with inputs u_1 and u_2, yields

$$\begin{aligned}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{e}\end{pmatrix} =& \underbrace{\begin{pmatrix}c(J - eh)\cos(\phi)\\ \alpha - c(J - eh)\sin(\phi)\\ 0\end{pmatrix}}_{f}\underbrace{\sqrt{\omega}\sin(\omega t)}_{u_1}\\ &+ \underbrace{\begin{pmatrix}\alpha - c(J - eh)\sin(\phi)\\ -c(J - eh)\cos(\phi)\\ 0\end{pmatrix}}_{g}\underbrace{\sqrt{\omega}\cos(\omega t)}_{u_2}\\ &+ \begin{pmatrix}0\\ 0\\ -eh + J\end{pmatrix}\underbrace{1}_{u_0}.\end{aligned}$$

The Lie bracket of the vector fields f and g can be calculated as follows

$$\begin{aligned}[f, g] =& \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g\\ =& \begin{pmatrix}-c\nabla_{\bar{x}_1}J\sin(\phi) & -c\nabla_{\bar{x}_2}J\sin(\phi) & ch\sin(\phi)\\ -c\nabla_{\bar{x}_1}J\cos(\phi) & -c\nabla_{\bar{x}_2}J\cos(\phi) & ch\cos(\phi)\\ 0 & 0 & 0\end{pmatrix}\\ &\cdot\begin{pmatrix}c(J - eh)\cos(\phi)\\ \alpha - c(J - eh)\sin(\phi)\\ 0\end{pmatrix}\\ &- \begin{pmatrix}c\nabla_{\bar{x}_1}J\cos(\phi) & c\nabla_{\bar{x}_2}J\cos(\phi) & -ch\cos(\phi)\\ -c\nabla_{\bar{x}_1}J\sin(\phi) & -c\nabla_{\bar{x}_2}J\sin(\phi) & ch\sin(\phi)\\ 0 & 0 & 0\end{pmatrix}\\ &\cdot\begin{pmatrix}\alpha - c(J - eh)\sin(\phi)\\ -c(J - eh)\cos(\phi)\\ 0\end{pmatrix}\\ =& \begin{pmatrix}-c\alpha\nabla_{\bar{x}_1}J\cos(\phi) - c\alpha\nabla_{\bar{x}_2}J\sin(\phi) + c^2\nabla_{\bar{x}_2}J(J - eh))\\ -c\alpha\nabla_{\bar{x}_2}J\cos(\phi) + c\alpha\nabla_{\bar{x}_1}J\sin(\phi) - c^2\nabla_{\bar{x}_1}J(J - eh))\\ 0\end{pmatrix}.\end{aligned}$$

By using Lemma 1 and $\theta := \frac{t}{\epsilon} = \omega t$, we obtain for the approximative system

$$\begin{pmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\\ \dot{\bar{e}}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -\bar{e}h + J\end{pmatrix} + \frac{1}{2\pi}\nu_{1,2}[f, g]$$

where $\nu_{1,2}$ is defined as

$$\nu_{1,2} = \int_0^{2\pi}\int_0^{\theta}\sin(\tau)\cos(\theta)d\tau d\theta = -\pi,$$

which is the same as (6). Therefore, according to Lemma 1, the result follows.

We deduced the approximate system given by Equation (6) that is easier to analyze than the original system (5) as the following theorem will show. Let us make the following assumptions on the parameters of the system

B.1 $h > 0$
B.2 $\alpha > 0$
B.3 $c > 0$
B.4 $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

Theorem 4. Under the Assumptions A.1–A.3, B.1–B.4 and sufficiently large ω, the point x^* is practically globally uniformly asymptotically stable for the system in Eq. (5).

Proof. Consider the extended system in Equation (6) and divide the system into two interconnected subsystems $\dot{\bar{x}} = (\dot{\bar{x}}_1, \dot{\bar{x}}_2)^\top = f_1(\bar{x}, \bar{e})$ and $\dot{\bar{e}} = f_2(\bar{x}, \bar{e})$.

Taking the Lyapunov function candidate $V(\bar{x}) = -J(\bar{x}) + J(x^*)$, that is under the Assumptions A.1-A.3 a valid Lyapunov function, we obtain for the first subsystem

$$\begin{aligned}\dot{V} =& -\nabla_{\bar{x}_1} J \dot{\bar{x}}_1 - \nabla_{\bar{x}_2} J \dot{\bar{x}}_2 \\ =& - c\alpha(\nabla_{\bar{x}_1} J)^2 \cos(\phi) - c\alpha(\nabla_{\bar{x}_2} J)^2 \cos(\phi) \\ &- c^2 J(\bar{x} - eh)\nabla_{\bar{x}_1} J \nabla_{\bar{x}_2} J \\ &+ c^2 J(\bar{x} - eh)\nabla_{\bar{x}_1} J \nabla_{\bar{x}_2} J \\ &- c\alpha \nabla_{\bar{x}_1} J \nabla_{\bar{x}_2} J \sin(\phi) + c\alpha \nabla_{\bar{x}_2} J \nabla_{x_1} J \sin(\phi) \\ =& - c\alpha(\nabla_{\bar{x}_1} J)^2 \cos(\phi) - c\alpha(\nabla_{\bar{x}_2} J)^2 \cos(\phi) \\ &< 0 \quad \forall \bar{x} \neq x^*.\end{aligned}$$

We can conclude that the first part of the approximate system is globally uniformly asymptotically stable, independently of $\bar{e}$.

The subsystem $\dot{\bar{e}} = f_2(\bar{x}, \bar{e})$ is input-to-state stable with respect to $J(\bar{x})$ as input.

The feedback connection of globally uniformly asymptotically stable system and an input-to-state stable system is globally uniformly asymptotically stable. Using Lemma 2 we conclude that the original system (5) is practically globally uniformly asymptotically stable, where $\bar{x}(t) \to x^*$ and $\bar{e} \to J(x^*)/h_i$ for $t \to \infty$.

3.2 Unicycle Dynamics

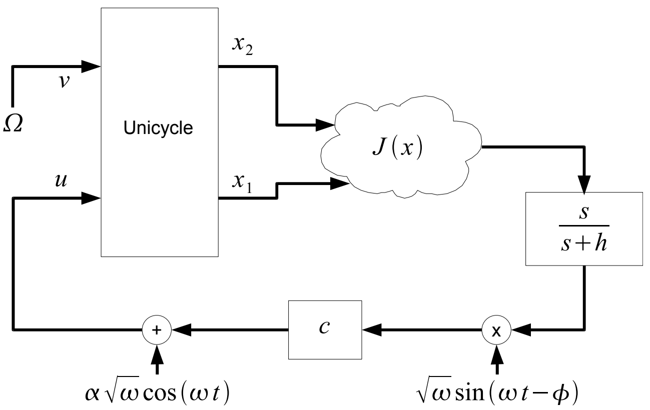


Fig. 2. Extremum Seeking for the Unicycle Model

Further investigations showed that even more complicated, nonholonomic systems such as the unicycle model can be analyzed using Lie brackets. One possibility to do the feedback is shown in Figure 2. The unicycle model is given by the equations

$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}u\cos\theta \\ u\sin\theta \\ v\end{pmatrix} \tag{7}$$

as usual in the literature.

Lemma 5. Consider the Unicycle Model in Equation (7) with extremum seeking feedback

$$u = (J(x) - eh)c\sqrt{\omega}\sin(\omega t - \phi) + \alpha\sqrt{\omega}\cos(\omega t),$$

where e denotes the state of the filter $\frac{s}{s+h}$, $h > 0$, and $v = \Omega = const.$ a constant input.

Consider furthermore the system

$$\begin{aligned}\dot{\bar{x}}_1 =& \frac{1}{2}(c\alpha\nabla_{\bar{x}_1} J \cos(\phi)\cos^2(\Omega t) \\ &+ c\alpha\nabla_{\bar{x}_2} J \cos(\phi)\cos(\Omega t)\sin(\Omega t))) \\ \dot{\bar{x}}_2 =& \frac{1}{2}(c\alpha\nabla_{\bar{x}_2} J \cos(\phi)\sin^2(\Omega t) \\ &+ c\alpha\nabla_{\bar{x}_1} J \cos(\phi)\cos(\Omega t)\sin(\Omega t))) \\ \dot{\bar{e}} =& -\bar{e}h + J \\ (\bar{x}, \bar{e})^\top|_0 =& (x, e)^\top|_0.\end{aligned} \tag{8}$$

For sufficiently large ω, the trajectory of the original system (7) is bounded by solutions of the reduced Lie bracket system (8), such that

$$||(x, e)^\top - (\bar{x}, \bar{e})^\top||_{C[0,2\pi]} \leq \Delta_\epsilon,$$

where $\epsilon = 1/\omega$ and Δ_ϵ is a parameter such that $\lim_{\epsilon\to 0} \Delta_\epsilon = 0$.

Proof. The proof follows the same procedure as before. The system (7) with given input is written as an input-affine system. By calculating the Lie bracket and applying Lemma 1 the result follows.

Let us make the following assumptions on the parameters

D.1 $h > 0$,
D.2 $\alpha > 0$,
D.3 $c > 0$,
D.4 $\Omega \neq 0$,
D.5 $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

Theorem 6. Under the Assumptions A.1–A.3, D.1–D.5, with inputs $u = (J(x) - eh)c\sqrt{\omega}\sin\omega t + \alpha\sqrt{\omega}\cos\omega t$, $v = \Omega$ and sufficiently large ω, the point x^* is practically globally uniformly asymptotically stable for the system in Eq. (7).

Proof. We take the Lyapunov function candidate $V(\bar{x}) = -J(\bar{x}) + J(x^*)$ that is under the assumptions on $J(\bar{x})$, a valid Lyapunov function. The derivative of V along the trajectories of the first subsystem $\bar{x}$ yields

$$\begin{aligned}\dot{V} =& -\nabla_{\bar{x}_1} J \dot{\bar{x}}_1 - \nabla_{\bar{x}_2} J \dot{\bar{x}}_2 \\ =& -\frac{1}{2}\cos(\phi)\alpha C(\nabla_{\bar{x}_1} J)^2 \cos^2 \Omega t \\ &- \frac{1}{2}\cos(\phi)\alpha C 2\nabla_{\bar{x}_1} J \nabla_{\bar{x}_2} J \sin\Omega t \cos\Omega t \\ &- \frac{1}{2}\cos(\phi)\alpha C(\nabla_{\bar{x}_2} J)^2 \sin^2 \Omega t \\ =& -\frac{1}{2}\cos(\phi)\alpha C\left(\nabla_{\bar{x}_1} J \cos\Omega t + \nabla_{\bar{x}_2} J \sin\Omega t\right)^2 \\ \leq& \ 0.\end{aligned}$$

This calculation shows that $\dot{V}$ is only negative semi-definite. This is due to the fact that the system is time-varying, and there are singular points in the state-space, where $\dot{\bar{x}} = 0$, but which are not steady-states for the system.

Injecting $\nabla_{\bar{x}_1} J \cos\Omega t + \nabla_{\bar{x}_2} J \sin\Omega t = 0$ into the differential equation of $\dot{\bar{x}} = f(\bar{x}, t)$ yields

$$\begin{aligned}\dot{\bar{x}}_1 =& \alpha\nabla_{\bar{x}_1} JC\cos^2\Omega t + \alpha\nabla_{\bar{x}_1} JC\cos\Omega t\cos\Omega t = 0\\ \dot{\bar{x}}_2 =& \alpha\nabla_{\bar{x}_2} JC\sin^2\Omega t + \alpha\nabla_{\bar{x}_2} JC\sin\Omega t\sin\Omega t = 0.\end{aligned}$$

One can deduce that if $\bar{x}_1 = const.$ and $\bar{x}_2 = const.$ then $\nabla_{\bar{x}_1} J$ and $\nabla_{\bar{x}_1} J$ are also constant. But as there are no constant values such that $\nabla_{\bar{x}_1} J\cos\Omega t + \nabla_{\bar{x}_2} J \sin\Omega t = 0, \quad \forall t \in \mathbb{R}_+$ except $\nabla_{\bar{x}_1} J = \nabla_{\bar{x}_2} J = 0$, as one can see by injecting $t = 0$ that implies $\nabla_{\bar{x}_1} J = 0$, and $t = \frac{\pi}{\Omega}$ that leads to $\nabla_{\bar{x}_2} J = 0$. As $V(\bar{x}(t))$ is monotonically decreasing and bounded from below, it must go to zero for $t \to \infty$. Therefore the system approaches the maximum with $t \to \infty$ for all initial conditions.

As before, the same argument for the filter state $\bar{e}$ can be used to prove global uniform asymptotic stability of the whole system (8) and conclude using Lemma 2 the result of the theorem.

In the theorem formulation we claim that ω needs to be *sufficiently large*, which implies that $\omega \gg \Omega$, as this assures that there are two different time-scales in the system. Therefore, it is not necessary to mention this explicitly as a condition for Theorem 6.

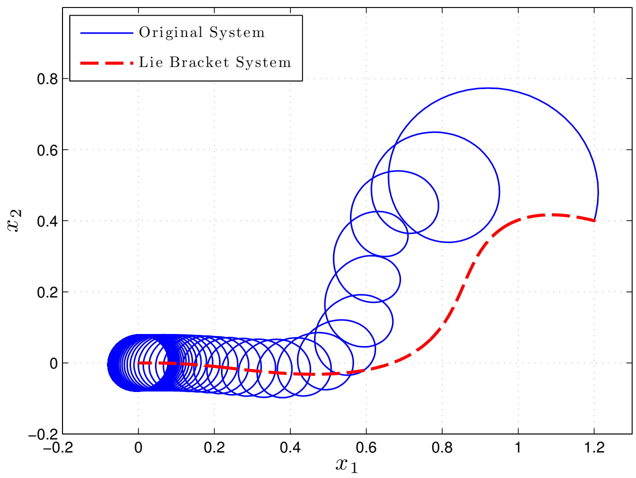
4. SIMULATION RESULTS

To get a better understanding of the proposed method, we will illustrate our results with the help of simulations. We also show that the proposed schemes can be used locally when Assumption A.3 is violated by choosing appropriate initial conditions such that the extremum seeking is initialized in the region of attraction of the extremum point. In this case all the results can be directly extended for showing local uniform practical asymptotic stability.

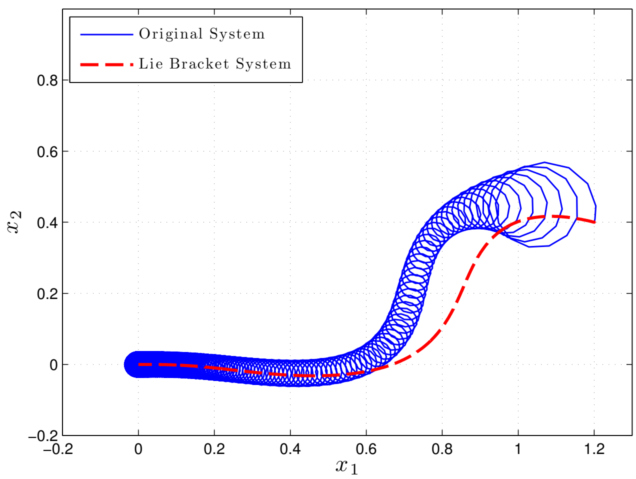
For the objective function, we choose $J(x) = e^{-x_1^2 - 5x_2^2} - 1$ that is a function *not* fulfilling Assumption A.3, since it approaches -1 as $||x|| \to \infty$. We compare through simulations the trajectories of the original extremum seeking system for the single integrator and unicycle models and the trajectories of their approximative Lie bracket systems, for $\omega = 10$ and $\omega = 50$.

For the single integrator case, we use the parameters $\alpha = 0.25$, $c = 1$, $\phi = 0$ and $h = 1$. The trajectories of the original system given by Eq. (5) and the Lie bracket system given by Eq. (6) are compared in Fig. 3.

For the unicycle model, we use the parameters $\alpha = 0.25$, $c = 1$, $\phi = 0$, $h = 1$ and $\Omega = 1$. To obtain a good approximation of the trajectories, it is necessary to choose Ω much smaller than ω. But even for larger values of Ω the stability is assured, as long as $\omega \gg \Omega$. In Fig. 4 the trajectories of the systems (7) and (8) are compared. Note that the derivative of the Lyapunov function along the trajectories of the Lie bracket system for the unicycle model, was only negative semi-definite. This is due to the fact that the state equations are zero at certain points. These points are visible as *spikes* in the trajectories (cf. Fig. 4) of both, the original and the Lie bracket system,



(a) $\omega = 10$



(b) $\omega = 50$

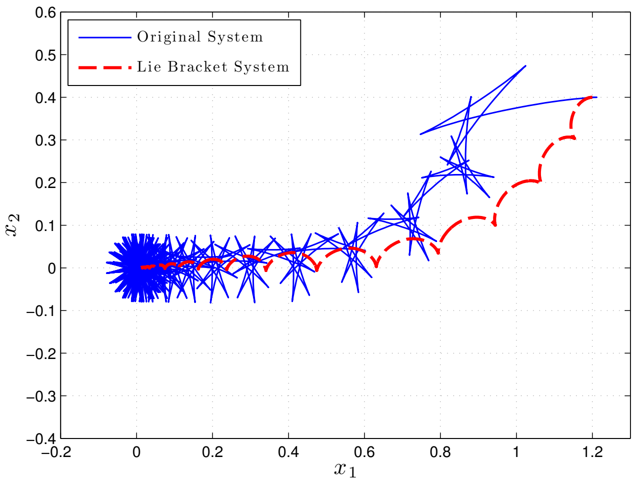
Fig. 3. Single Integrator Extremum Seeking compared to Lie Bracket Approximation

which demonstrates that the Lie bracket system can be used to understand the qualitative behavior of the original one.

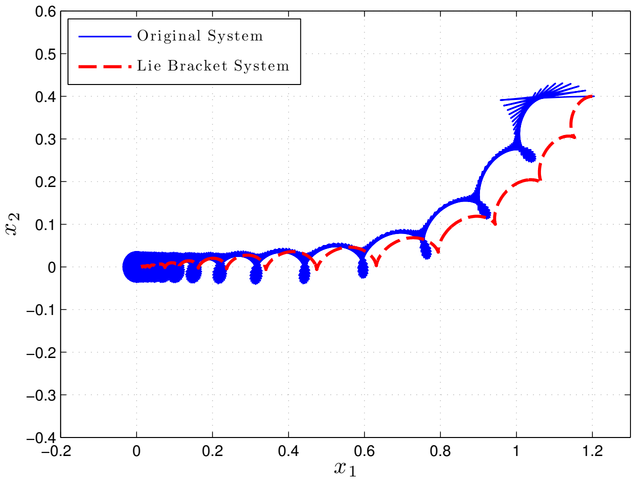
Furthermore, higher order terms and couplings in the nonlinear map as well as the dynamics of the vehicle, influence not only the trajectory of the original system, but also the trajectory of the corresponding Lie bracket system. Regarding the equations for the Lie bracket systems, where one can see very well that the extremum seeking does not follow only the gradient of the map, but also admits bias terms that do not vanish even for higher values of ω. This is the crucial result in the presented analysis and extends the understanding of the extremum seeking feedbacks compared to the results of the authors Zhang et al. (2007a) and Zhang et al. (2007b) who neglected higher order terms in their approximating system using pure averaging techniques.

5. CONCLUSIONS AND FUTURE WORK

The presented results showed that the extremum seeking can be interpreted as the Lie bracket motion because of the periodic excitation that leads to an unaccessible direction of the system. The trajectories of the approximated system



(a) $\omega = 10$



(b) $\omega = 50$

Fig. 4. Unicycle Extremum Seeking compared to Lie Bracket Approximation

are the limit of the trajectories of the original one, for $\omega \to \infty$. By examining the properties of the approximate system, one can analyze and understand the extremum seeking from a completely new viewpoint. We used this method to prove global practical stability of different two-dimensional schemes.

We claim that this methodology can be used to analyze the behavior of more complex systems involving extremum seeking loops. For example, it is possible to extend the schemes to multi-dimensional or multi-agent systems, where, by choosing different frequencies for each agent, it is possible to decouple the dynamics and treat the stability in a similar way. Furthermore, we would like to extend the proposed algorithms in such a way that asymptotic stability in the sense of Lyapunov, instead of practical stability, is achieved. The Lie bracket analysis, especially Lemma 1 allows to introduce time-varying, vanishing gains. For this purpose it is necessary to find suitable conditions in order to be able to prove asymptotic stability. The approximation of the extremum seeking by a Lie bracket system can be extended to analyze other properties such as obstacle avoidance or the construction of piecewise constant inputs instead of sinusoids, which might be more convenient for certain applications.

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