2018 IEEE Conference on Decision and Control (CDC)
Miami Beach, FL, USA, Dec. 17-19, 2018

Distributed Optimization with Dynamic Event-Triggered Mechanisms

Wen Du, Xinlei Yi, Jemin George, Karl H. Johansson, and Tao Yang

Abstract— **In this paper, we consider the distributed optimization problem, whose objective is to minimize the global objective function, which is the sum of local convex objective functions, by using local information exchange. To avoid continuous communication among the agents, we propose a distributed algorithm with a dynamic event-triggered communication mechanism. We show that the distributed algorithm with the dynamic event-triggered communication scheme converges to the global minimizer exponentially, if the underlying communication graph is undirected and connected. Moreover, we show that the event-triggered algorithm is free of Zeno behavior. For a particular case, we also explicitly characterize the lower bound for inter-event times. The theoretical results are illustrated by numerical simulations.**

I. Introduction

For a networked system of multiple agents, each of which has a local private convex objective function, the objective of the distributed optimization problem is to find the global minimizer that minimizes the global objective function, which is the sum of the objective functions of all agents, in a distributed manner. Distributed optimization has gained a growing interest over the last decade, due to its wide applications in machine learning, power systems, communication networks, and sensor networks [1].

To solve the distributed optimization problem, various distributed algorithms have been proposed. These algorithms can be generally divided into two categories depending on whether they are discrete-time or continuous-time.

Most distributed optimization algorithms are discrete-time and are based on the consensus and distributed (sub)gradient descent (DGD) method, see, e.g., [2]–[7]. Although the simple DGD algorithm and its variants are applicable to non-smooth convex functions, the convergence speed is usually rather slow due to the diminishing step-size. Thus, in order to reduce the communication overheads, recent works focus on speeding up the convergence process for more structured local convex objective functions, such as smooth strongly convex ones, see, e.g., [8]–[10]. The common approach in these studies is to use some sort of historical information to correct the error caused by the distributed gradient method with a fixed step-size. On the other hand, to accelerate the convergence process, various continuous-time distributed algorithms based on the proportional-plus integral control strategy have also been developed, see, e.g., [11]–[16].

This work was supported in part by the Ralph E. Powe Junior Faculty Enhancement Award for the Oak Ridge Associated Universities (ORAU), the Knut and Alice Wallenberg Foundation, and the Swedish Research Council.

Wen Du and Tao Yang are with the Department of Electrical Engineering, University of North Texas, Denton, TX 76203 USA (e-mail: WenDu@my.unt.edu, Tao.Yang@unt.edu).

Xinlei Yi and Karl H. Johansson are with the ACCESS Linnaeus Center, School of Electrical Engineering and Computer Science, Royal Institute of Technology, 100 44 Stockholm, Sweden (e-mail: {xinleiy,kallej}@kth.se).

Jemin George is with the U.S. Army Research Laboratory, Adelphi, MD 20783, USA (email: jemin.george.civ@mail.mil).

Note that all the aforementioned distributed algorithms require continuous information exchange among the agents, which may be impractical in physical applications. Moreover, distributed networks are usually resources-constrained and communication is energy-consuming. In order to avoid continuous communication and reduce communication overheads, the idea of event-triggered communication and control has been proposed. The early works focus on the single system [17]–[19] and have been extended to the multi-agent system setting [20], [21]. Event-triggered communication mechanisms for the consensus problem have been proposed in [22]–[25]

However, for the distributed optimization problem, it is more challenging since in addition to achieve consensus, it also requires that the consensus state is an optimal solution. There are a few works which propose distributed algorithms with event-triggered communication mechanisms for solving the distributed optimization over undirected graphs [26], [27]. In particular, the authors of [26] develop an event-triggered communication scheme which is free of Zeno behavior [28], i.e., an infinite number of triggered events in a finite period of time, and establish its convergence to a neighborhood of the global minimizer. Motivated by the zero-gradient-sum (ZGS) algorithm proposed in [12], the authors of [27] propose a ZGS algorithm with a periodical time-triggered communication mechanism.

Statement of Contributions: In this paper, we develop a distributed ZGS algorithm with a novel class of event-triggered communication mechanisms that use an additional internal dynamic variable, which is why we named dynamic event-triggered mechanism. We show that the ZGS algorithm with the dynamic event-triggered communication scheme exponentially converges to the global minimizer if the underlying graph is undirected and connected. Moreover, we show that the proposed event-triggered distributed algorithm is free of Zeno behavior.

Compared to the time-triggered communication scheme proposed in [27], our event-triggered mechanism is more energy efficient. Compared to the distributed algorithm with an event-triggered communication scheme proposed in [26], which only converges to the neighborhood of the global minimizer, our proposed algorithm with the dynamic event-triggered mechanism converges to the global minimizer.

The remainder of the paper is organized as follows. In Section II, some preliminaries are introduced. In Section III, we first formulate the distributed optimization problem, and then motivate our study. In Section IV, we develop a distributed optimization algorithm with a dynamic event-triggered communication scheme, and establish its exponential convergence to the global minimizer for undirected connected graphs. Moreover, we show that the proposed distributed event-triggered algorithm is free of Zeno behavior. For a particular case, we also explicitly characterize the lower bound for the inter-event times. Section V presents simulation examples. Finally, concluding remarks are offered in Section VI.

II. Preliminaries

In this section, we provide some basic concepts of graph theory and convex analysis.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote an undirected weighted graph with the set of nodes (agents) $\mathcal{V} = \{1, \dots, n\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. In this paper, we also assume that there is no self-loops, i.e., $a_{ii} = 0$ for all $i \in \mathcal{V}$. The neighbor set of agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} \,|\, a_{ij} > 0\}$. A path from node i_1 to node i_k is a sequence of nodes $\{i_1, \dots, i_k\}$, such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$ in the undirected graph. An undirected graph is said to be connected if there exists a path between any pair of distinct nodes.

For an undirected weighted graph $\mathcal{G}$, the weighted Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N\times N}$ is defined as $L_{ii} = \sum_{j=1}^{N} a_{ij}$ and $L_{ij} = -a_{ij}$ for $j \neq i$. It is well known that the Laplacian matrix has the property that all the row sums are zero. If the undirected weighted graph $\mathcal{G}$ is connected, then the Laplacian matrix L has a simple eigenvalue at zero with corresponding right eigenvector $\mathbf{1}$, and all other eigenvalues are strictly positive.

A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is locally strongly convex if for any convex and compact set $D \subset \mathbb{R}^n$, there exists a constant $\theta > 0$ such that the following equivalent conditions hold:

$$f(y) - f(x) - \nabla f(x)^{\top}(y-x) \geq \frac{\theta}{2}\|y-x\|^2, \quad \forall x, y \in D \quad (1)$$

$$(\nabla f(y) - \nabla f(x))^{\top}(y-x) \geq \theta\|y-x\|^2, \quad \forall x, y \in D \quad (2)$$

$$\nabla^2 f(x) \geq \theta I_n, \quad \forall x \in D \quad (3)$$

where $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is the gradient of f, $\nabla^2 : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the Hessian of f. The function f is strongly convex if there exists a constant $\theta > 0$ such that the above equivalent conditions hold for $D = \mathbb{R}^n$, in which case θ is called the convexity parameter of f. For a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, any convex set $D \subset \mathbb{R}^n$, and any constant $\Theta > 0$, the following equivalent conditions are equivalent:

$$f(y) - f(x) - \nabla f(x)^{\top}(y-x) \leq \frac{\Theta}{2}\|y-x\|^2, \forall x, y \in D \quad (4)$$

$$(\nabla f(y) - \nabla f(x))^{\top}(y-x) \leq \Theta\|y-x\|^2, \quad \forall x, y \in D \quad (5)$$

$$\nabla^2 f(x) \leq \Theta I_n, \forall x \in D. \quad (6)$$

III. Problem Formulation and Motivation

Consider a network of N agents, each of which has a local private convex objective function $f_i : \mathbb{R}^n \to \mathbb{R}$. The global objective function of the network is $f(x) = \sum_{i=1}^{N} f_i(x)$. All the agents aim to cooperatively solve the following optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) = \sum_{i=1}^{N} f_i(x), \quad (7)$$

in a distributed manner using only local communication, which is described by an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the agent set, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise.

In the literature, various algorithms have been developed to solve the optimization problem (7) in a distributed manner, see, e.g., the recent survey papers [1] and references therein. However, most existing distributed algorithms require continuous information exchange among the agents, which results in high energy consumption. Moreover, it is impractical in physical applications and not desirable in the multi-agent systems since each agent is usually equipped with a limited energy resource.

The goal of this paper is to overcome these problems by developing a distributed algorithm with event-triggered communication schemes. For this purpose, we make the following assumption about local objective functions.

Assumption 1. *For each $i \in \mathcal{V}$, the objective function $f_i : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, strongly convex with convexity parameter $m_i > 0$, and has a locally Lipschitz Hessian $\nabla^2 f_i$.*

Under Assumption 1, it follows from [29] that the optimization problem (7) has a unique global minimizer, which is denoted by $x^* \in \mathbb{R}^n$. Moreover, the necessary and sufficient optimality condition is $\nabla f(x^*) = \sum_{i=1}^{N} \nabla f_i(x^*) = 0$.

IV. Distributed Algorithm with a Dynamic Event Triggering Mechanism

In this section, we first propose a distributed algorithm with a dynamic event-triggered communication scheme and analyze its convergence.

Consider the following distributed algorithm with an event-triggered communication scheme:

$$\dot{x}_i(t) = \gamma\Big(\nabla^2 f_i(x_i(t))\Big)^{-1} \sum_{j\in\mathcal{N}_i} a_{ij}\Big(x_j(t^j_{k_j(t)}) - x_i(t^i_k)\Big),$$
$$t \in [t^i_k, t^i_{k+1}), \quad (8a)$$

$$x_i(0) = x_i^*, \quad i \in \mathcal{V}, \quad (8b)$$

where $x_i(t) \in \mathbb{R}^n$ is agent i's estimate of the unique global minimizer x^*, $\gamma > 0$ is the gain parameter, x_i^* is the minimizer of the local objective function $f_i(x)$, and the increasing sequence $\{t^j_k\}_{k=1}^{\infty}$, $\forall j \in \mathcal{V}$ to be determined later is the triggering times and $t^j_{k_j(t)} = \max\{t^j_k : t^j_k \leq t\}$.

We assume $t_1^j = 0,\ \forall j \in \mathcal{V}$. For ease of presentation, let $\hat{x}_j(t) = x_j(t^j_{k_j(t)})$, and $e_j(t) = \hat{x}_j(t) - x_j(t)$ for any $j \in \mathcal{V}$.

Remark 1. *Note that the distributed algorithm (8) without an event-triggered communication scheme is the zero-gradient-sum (ZGS) algorithm proposed in [12]. However, in order to avoid continuous communication, we equip the ZGS algorithm with an event-triggered communication scheme. Note that it follows from (8) that* $\frac{d}{dt}\sum_{i\in\mathcal{V}}\nabla f_i(x_i(t)) = \sum_{i\in\mathcal{V}}\nabla^2 f_i(x_i(t))\dot{x}_i(t) = \gamma\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} a_{ij}\big(\hat{x}_j(t) - \hat{x}_i(t)\big) = 0$ *for undirected connected graphs. It then follows from* $\nabla f_i(x_i^*) = 0$ *that*

$$\sum_{i\in\mathcal{V}}\nabla f_i(x_i(t)) = 0, \quad \forall t \geq 0. \tag{9}$$

Therefore, the zero-gradient-sum property is still satisfied for the event-triggered algorithm (8).

In order to determine the triggering times for agent $i \in \mathcal{V}$, we design a novel class of triggering mechanisms that use an additional internal dynamic variable $\chi_i(t)$ satisfying the following equation:

$$\dot{\chi}_i(t) = -\beta_i\chi_i(t) - \delta_i(L_{ii}\|e_i(t)\|^2 - \frac{\sigma_i}{2}\hat{q}_i(t)),\ i \in \mathcal{V}, \tag{10}$$

where $\chi_i(0) > 0,\ \beta_i > 0,\ \delta_i \in [0,1]$, and $\sigma_i \in (0,1)$ are design parameters, and

$$\hat{q}_i(t) = -\frac{1}{2}\sum_{j\in\mathcal{N}_i} L_{ij}\|\hat{x}_j(t) - \hat{x}_i(t)\|^2 \geq 0. \tag{11}$$

In the following theorem, we propose a dynamic event-triggered law to determine the triggering times and establish the exponential convergence of the event-triggered algorithm.

Theorem 1. *Assume that Assumption 1 is satisfied, and that the undirected graph $\mathcal{G}$ is connected. Given $\theta_i > \frac{1-\delta_i}{\beta_i}$ and the first triggering time $t_1^i = 0$, each agent $i \in \mathcal{V}$ determines the triggering times $\{t_k^i\}_{k=2}^{\infty}$ by*

$$t^i_{k+1} = \min\{t : \theta_i(L_{ii}\|e_i(t)\|^2 - \frac{\sigma_i}{2}\hat{q}_i(t)) \geq \chi_i(t),\ t \geq t^i_k\}, \quad k = 1, 2, \ldots. \tag{12}$$

with $\hat{q}_i(t)$ and $\chi_i(t)$ defined in (11) and (10), respectively. Then, the distributed algorithm (8) with the dynamic event-triggered mechanism (12) solves the distributed optimization problem (7) exponentially, i.e., $x_i(t) \to x^*$ exponentially fast as $t \to \infty$ for any $i \in \mathcal{V}$.

Proof: We first note that it follows from the way we determine the triggering times by (12) that

$$\theta_i(L_{ii}\|e_i(t)\|^2 - \frac{\sigma_i}{2}\hat{q}_i(t)) \leq \chi_i(t),\ \forall t \geq 0. \tag{13}$$

This together with (10) implies that

$$\dot{\chi}_i(t) \geq -\beta_i\chi_i(t) - \frac{\delta_i}{\theta_i}\chi_i(t),\ \forall t \geq 0.$$

Therefore,

$$\chi_i(t) \geq \chi_i(0)e^{-(\beta_i + \frac{\delta_i}{\theta_i})t} > 0,\ \forall t \geq 0. \tag{14}$$

Next, consider the following function

$$V(x(t)) = \sum_{i=1}^{N}\big(f_i(x^*) - f_i(x_i(t)) - \nabla f_i(x_i(t))^{\mathrm{T}}(x^* - x_i(t))\big), \tag{15}$$

where $x(t) = [x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Nn}$.

Since Assumption 1 is satisfied, the first-order strong convexity condition implies that

$$V(x) \geq \sum_{i=1}^{N}\frac{m_i}{2}\|x^* - x_i\|^2, \quad \forall x \in \mathbb{R}^{Nn}. \tag{16}$$

The Lie derivative of $V(x(t))$ along (8) is

$$\begin{aligned}
&\dot{V}(x(t))\\
&= \sum_{i=1}^{N}(x_i(t) - x^*)^{\mathrm{T}}\nabla^2 f_i(x_i(t))\dot{x}_i(t)\\
&= -\gamma\sum_{i=1}^{N}(x_i(t) - x^*)^{\mathrm{T}}\sum_{j=1}^{N}L_{ij}\big(\hat{x}_j(t) - \hat{x}_i(t)\big)\\
&= -\gamma\sum_{i=1}^{N}x_i^{\mathrm{T}}(t)\sum_{j=1}^{N}L_{ij}\hat{x}_j(t)\\
&= -\gamma\sum_{i=1}^{N}\big(\hat{x}_i(t) - e_i(t)\big)^{\mathrm{T}}\sum_{j=1}^{N}L_{ij}\hat{x}_j(t)\\
&\overset{*}{=} -\gamma\sum_{i=1}^{N}\hat{q}_i(t) + \gamma\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^{\mathrm{T}}(t)L_{ij}\hat{x}_j(t)\\
&= -\gamma\sum_{i=1}^{N}\hat{q}_i(t) + \gamma\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}e_i^{\mathrm{T}}(t)L_{ij}\big(\hat{x}_j(t) - \hat{x}_i(t)\big)\\
&\leq -\gamma\sum_{i=1}^{N}\hat{q}_i(t) - \gamma\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}L_{ij}\|e_i(t)\|^2\\
&\quad - \gamma\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{1}{4}L_{ij}\|\hat{x}_j(t) - \hat{x}_i(t)\|^2\\
&= -\gamma\sum_{i=1}^{N}\hat{q}_i(t) + \gamma\sum_{i=1}^{N}L_{ii}\|e_i(t)\|^2\\
&\quad - \gamma\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4}L_{ij}\|\hat{x}_j(t) - \hat{x}_i(t)\|^2\\
&\overset{*}{=} -\frac{\gamma}{2}\sum_{i=1}^{N}\hat{q}_i(t) + \gamma\sum_{i=1}^{N}L_{ii}\|e_i(t)\|^2,
\end{aligned} \tag{17}$$

where the third equality holds due to the fact that $L = L^{\mathrm{T}}$ and $L\mathbf{1} = \mathbf{0}$, the equalities denoted by $\overset{*}{=}$ hold since it follows from (11) that

$$\sum_{i=1}^{N}\hat{q}_i(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{x}_i^{\mathrm{T}}(t)L_{ij}\hat{x}_j(t) = \hat{x}^{\mathrm{T}}(t)(L \otimes I_n)\hat{x}(t),$$

where $\hat{x}(t) = [\hat{x}_1^{\mathrm{T}}(t), \ldots, \hat{x}_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Nn}$, and the inequality holds since $-a^{\mathrm{T}}b \leq \|a\|\|b\| \leq \|a\|^2 + \frac{1}{4}\|b\|^2$ for all $a, b \in \mathbb{R}^n$.

Next, consider the following Lyapunov candidate

$$W(x(t), \chi(t)) = V(x(t)) + \gamma \sum_{i=1}^{N} \chi_i(t), \tag{18}$$

where $\chi(t) = [\chi_1(t), \ldots, \chi_N(t)]^{\mathrm{T}}$. The Lie derivative of $W(x(t), \chi(t))$ along (8) and (10) is

$$\begin{aligned}
&\dot{W}(x(t), \chi(t)) \\
&= \dot{V}(x(t)) + \gamma \sum_{i=1}^{N} \dot{\chi}_i(t) \\
&\leq \gamma\Big\{ -\sum_{i=1}^{N} \frac{1}{2}(1-\sigma_i)\hat{q}_i(t) - \sum_{i=1}^{N} \beta_i \chi_i(t) \\
&\quad + \sum_{i=1}^{N} (\delta_i - 1)\big(\frac{\sigma_i}{2}\hat{q}_i(t) - L_{ii}\|e_i(t)\|^2\big)\Big\} \\
&\leq \gamma\Big\{ -\frac{1}{2}(1-\sigma_{\max})\hat{x}^{\mathrm{T}}(t)(L \otimes I_n)\hat{x}(t) - k_d \sum_{i=1}^{N} \chi_i(t)\Big\},
\end{aligned} \tag{19}$$

where $\sigma_{\max} = \max_i \sigma_i < 1$, $k_d = \min_i\{\beta_i - \frac{1-\delta_i}{\theta_i}\} > 0$, the first inequality holds due to (17), and the second inequality holds due to (13).

Note that

$$\begin{aligned}
&x^{\mathrm{T}}(t)(L \otimes I_n)x(t) \\
=&\big(\hat{x}(t) - e(t)\big)^{\mathrm{T}}(L \otimes I_n)\big(\hat{x}(t) - e(t)\big) \\
\leq& 2\hat{x}^{\mathrm{T}}(t)(L \otimes I_n)\hat{x}(t) + 2e^{\mathrm{T}}(t)(L \otimes I_n)e(t) \\
\leq& 2\hat{x}^{\mathrm{T}}(t)(L \otimes I_n)\hat{x}(t) + 2\|L\|\|e(t)\|^2 \\
\leq& \Big(2 + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\Big)\hat{x}^{\mathrm{T}}(t)(L \otimes I_n)\hat{x}(t) + \frac{2\|L\|}{\min_i\{\theta_i L_{ii}\}} \sum_{i=1}^{N} \chi_i(t) \\
\leq& k_x \hat{x}^{\mathrm{T}}(t) L \hat{x}(t) + \frac{2\|L\|}{\min_i\{\theta_i L_{ii}\}} \sum_{i=1}^{N} \chi_i(t),
\end{aligned} \tag{20}$$

where

$$k_x = \max\left\{2 + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}, \ \frac{2(1-\sigma_{\max})\|L\|}{k_d \min_i\{\theta_i L_{ii}\}}\right\}, \tag{21}$$

the first inequality holds since the Laplacian matrix L is positive semi-definite and that $-a^{\mathrm{T}}(L \otimes I_n)b \leq \frac{1}{2}a^{\mathrm{T}}(L \otimes I_n)a + \frac{1}{2}b^{\mathrm{T}}(L \otimes I_n)b$ for all $a, b \in \mathbb{R}^{Nn}$, the second inequality holds since $a^{\mathrm{T}}(L \otimes I_n)a \leq \|L\|\|a\|^2$ for all $a \in \mathbb{R}^{Nn}$, and the third inequality holds due to (13).

It then follows from (20) and (21) that

$$\begin{aligned}
&-\frac{1}{2}(1-\sigma_{\max})\hat{x}^{\mathrm{T}}(t)(L \otimes I_n)\hat{x}(t) \\
&\leq -\frac{1}{2k_x}(1-\sigma_{\max})x^{\mathrm{T}}(t)(L \otimes I_n)x(t) + \frac{k_d}{2}\sum_{i=1}^{N} \chi_i(t).
\end{aligned}$$

This together with (19) implies that

$$\begin{aligned}
\dot{W}(x(t), \chi(t)) \leq& \gamma\Big\{ -\frac{1}{2k_x}(1-\sigma_{\max})x^{\mathrm{T}}(t)(L \otimes I_n)x(t) \\
&- \frac{k_d}{2}\sum_{i=1}^{N} \chi_i(t)\Big\}.
\end{aligned} \tag{22}$$

In order to establish the exponential convergence, we will upper bound the right-hand side of (22) in terms of the Lyapunov function $W(x(t), \chi(t))$ defined in (18). To begin with, we first define the set

$$\begin{aligned}
\mathcal{C}_i = \Big\{x \in \mathbb{R}^n : \ & f_i(x^*) - f_i(x) - \nabla f_i(x)^{\mathrm{T}}(x^* - x) \\
& \leq W(x(0), \chi(0))\Big\},
\end{aligned}$$

where the initial condition $x(0) = [x_1^{*\mathrm{T}}, x_2^{*\mathrm{T}}, \ldots, x_N^{*\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}$ and $\chi(0) \in \mathbb{R}^N$. Note that it follows from (15), (18) and (22) that the set $\mathcal{C}_i$ is nonempty and invariant. Moreover from Assumption 1, we know that $\mathcal{C}_i$ is compact.

Next define $\mathcal{C} = \mathrm{conv} \cup_{i \in \mathcal{V}} \mathcal{C}_i$, where conv denotes the convex hull. Note that the set $\mathcal{C}$ is compact and $x_i(t) \in \mathcal{C}$, $\forall t \geq 0$, $\forall i \in \mathcal{V}$. Then, again from Assumption 1, we know that there exists a constant $\Theta_i \geq m_i$ such that

$$\nabla^2 f_i(x) \leq \Theta_i I_n, \ \forall x \in \mathcal{C}. \tag{23}$$

Let $\eta(t) = \frac{1}{N}\sum_{i \in \mathcal{V}} x_i(t)$, then $\eta(t) \in \mathcal{C}$ since $\mathcal{C}$ is convex. Since x^* is the unique solution to the optimization problem (7), we know that $\sum_{i \in \mathcal{V}} f_i(x^*) \leq \sum_{i \in \mathcal{V}} f_i(\eta(t))$. Thus, it follows from (9) and (15) that

$$\begin{aligned}
&V(x(t)) \\
&\leq \sum_{i \in \mathcal{V}} f_i(\eta(t)) - f_i(x_i(t)) - \nabla f_i(x_i(t))^{\mathrm{T}}(\eta(t) - x_i(t)).
\end{aligned}$$

This together with (23), (4) and (6) implies that for all $t \geq 0$,

$$V(x(t) \leq \sum_{i \in \mathcal{V}} \frac{\Theta_i}{2}\|\eta(t) - x_i(t)\|^2 = x^{\mathrm{T}}(t)(P \otimes I_n)x(t),$$

where $P = [P_{ij}] \in \mathbb{R}^{N \times N}$ is a positive seim-definite matrix given by

$$P_{ij} = \begin{cases} (\frac{1}{2} - \frac{1}{N})\Theta_i + \frac{1}{2N^2}\sum_{\ell \in \mathcal{V}} \Theta_\ell, & \text{if } i = j, \\ -\frac{\Theta_i + \Theta_j}{2N} + \frac{1}{2N^2}\sum_{\ell \in \mathcal{V}} \Theta_\ell, & \text{otherwise.} \end{cases} \tag{24}$$

It is straightforward to check that $P\mathbf{1} = \mathbf{0}$. Then, by using a similar analysis as the proof of eq. (5) in [30], for an undirected and connected graph, we have

$$P \leq \frac{\rho(P)}{\lambda_2(L)} L, \tag{25}$$

where $\lambda_2(L)$ is the second smallest eigenvalue of the Laplacian matrix L, and $\rho(P)$ is the spectral radius of matrix P. It then follows from (22) and (25) that

$$\begin{aligned}
&\dot{W}(x(t), \chi(t)) \\
\leq& \gamma\Big\{ -\frac{1}{2k_x}(1-\sigma_{\max})x^{\mathrm{T}}(t)(L \otimes I_n)x(t) - \frac{k_d}{2}\sum_{i=1}^{N} \chi_i(t)\Big\}
\end{aligned}$$

$$\leq \gamma \Big\{ -\frac{\lambda_2(L)}{2k_x\rho(P)}(1-\sigma_{\max})V(x(t)) - \frac{k_d}{2}\sum_{i=1}^{N}\chi_i(t)\Big\}$$
$$\leq -k_W W(x(t),\chi(t)),$$

where

$$k_W = \min\Big\{\frac{\lambda_2(L)}{2k_x\rho(P)}(1-\sigma_{\max})\gamma,\ \frac{k_d}{2}\Big\}. \tag{26}$$

Hence,

$$W(x(t),\chi(t)) \leq W(x(0),\chi(0))e^{-k_W t},\ \forall t\geq 0.$$

This together with (18), (16), and the fact that $\chi_i(t) > 0$ given in (14), implies that

$$\sum_{i=1}^{N}\frac{m_i}{2}\|x_i(t)-x^*\|^2 \leq W(x(t),\chi(t)) - \gamma\sum_{i=1}^{N}\chi_i(t)$$
$$\leq e^{-k_W t}W(x(0),\chi(0)).$$

Therefore,

$$\|x(t)-\mathbf{1}_N\otimes x^*\| \leq ce^{-\frac{k_W}{2}t}, \tag{27}$$

where

$$c = \sqrt{\frac{2}{m}W(x(0),\chi(0))}, \tag{28}$$

where $m = \min_{i\in\mathcal{V}}\{m_i\}$. This implies that the algorithm (8) with the dynamic event-triggering mechanism (12) exponentially converges to the global minimizer with the rate at least equal to $\frac{k_W}{2}$. ■

If the parameter θ_i goes to ∞ in the dynamic triggering law (12), then it would become the following static triggering law:

$$t^i_{k+1} = \min\{t : L_{ii}\|e_i(t)\|^2 - \frac{\sigma_i}{2}\hat{q}_i(t)\geq 0,\ t\geq t^i_k\},$$
$$k = 1,2,\dots. \tag{29}$$

The following corollary shows that the algorithm (8) with the static triggering law (29) also exponentially converges to the global minimizer. The proof is very similar to that of Theorem 1 and thus omitted.

Corollary 1. *Under the same assumptions as Theorem 1, the distributed algorithm (8) with the static event-triggered mechanism (29) solves the distributed optimization problem (7) exponentially, i.e., $x_i(t)\to x^*$ exponentially fast as $t\to\infty$ for any $i\in\mathcal{V}$.*

The main purpose of using event-triggered communication mechanisms is to reduce the overall need of continuous communication among agents, so it is essential to exclude Zeno behavior. However, as stated in [25], Zeno behavior may not be excluded under the static triggering law (29). On the other hand, Zeno behavior is excluded under the dynamic triggering law (12) as shown in the next theorem.

Theorem 2. *Under the same assumptions as Theorem 1, the distributed algorithm (8) with the dynamic event triggering law (12) does not exhibit Zeno behavior.*

Proof: The proof is based on a contradiction argument that

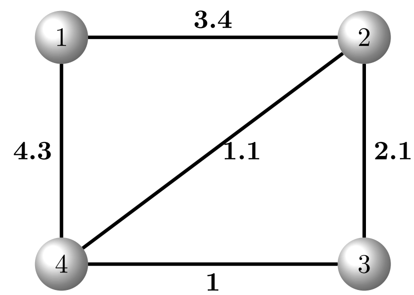

Fig. 1. Network of four agents.

is similar to the proof of [25, Theorem 3.1]. Due to the space limitation, we have omitted the detailed proof. ■

Theorem 2 shows that the algorithm (8) with the dynamic triggering law (12) is free of Zeno behavior by a contradiction argument. However, the inter-event times are not clear. The next theorem explicitly characterizes the lower-bound for the inter-event times for a particular case. Due to the space limitation, we have omitted the proof.

Theorem 3. *Assume that all the assumptions of Theorem 1 are satisfied. In addition, if $\chi_i(0) > 0$, $\delta_i = 0$, $\sigma_i\in(0,\frac{1}{1+\sigma})$ where σ is any positive constant, for all $i\in\mathcal{V}$, and $\beta_1 = \beta_2 = \cdots = \beta_N = \beta$ with $0<\beta\leq k_W$, then for any $i\in\mathcal{V}$, there exists a positive constant τ_i such that $t^i_{k+1} - t^i_k \geq \tau_i$ for all $k = 1,2,\dots$.*

V. Simulations

In this section, we illustrate and validate the proposed distributed algorithm (8) with the dynamic event-triggered communication mechanism (12) by considering a numerical example, which is adopted from [27] for comparison purpose. In particular, we consider an undirected network with $N = 4$ agents whose communication topology is given by Fig. 1. Note that the undirected graph is connected. The objective functions are $f_i(x) = \frac{1}{2}(x-y_i)^2$, where $[y_1,y_2,y_3,y_4]^\top = [1.12, 2.04, 2.98, 3.82]^\top$. For this case, the minimizer for $f_i(x)$ is $x_i^* = y_i$.

The simulation results for the algorithm (8) under the dynamic triggered mechanism (12) with $\gamma = 1$, $\chi_i(0) = 10$, $\beta_i = 1$, $\delta_i = 1$, $\sigma_i = 0.5$, and $\theta_i = 1$ for all $i\in\mathcal{V}$, are given in Fig. 2. The evolution of the states of the four agents is plotted in Fig. 2a, where we see that all agents converge to the global minimizer $x^* = 2.49$, which agrees with [27]. This confirms the result of Theorem 1. Moreover, the corresponding triggering times for each agent are plotted in Fig. 2b, which clearly shows that the dynamic event-triggered scheme (12) has less triggering times compared to the periodical time-triggered mechanism proposed in [27].

VI. Conclusions

In this paper, we studied the distributed optimization problem where the local objective functions are twice continuously differentiable, strongly convex, and have locally Lipschitz Hessians. To avoid continuous communication among the agents, we proposed a distributed algorithm

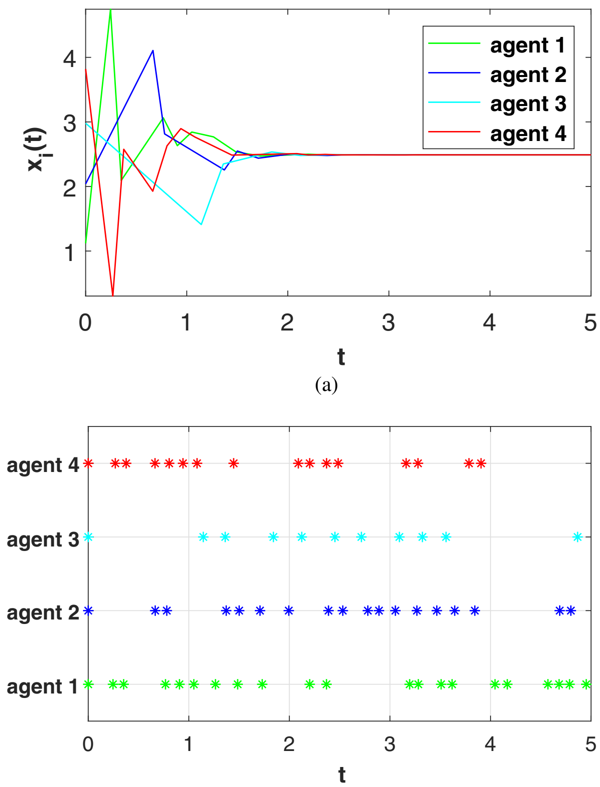

Fig. 2. Simulation Results

with a dynamic event-triggered communication mechanism. We showed that the proposed distributed event-triggered algorithm exponentially converges to the global minimizer if the undirected graph is connected. Moreover, we showed that the proposed distributed event-triggered algorithm is free of Zeno behavior. For a particular case, we also explicitly characterized the lower bound for the inter-event times. The future direction is to extend the proposed event-triggered algorithm to directed graphs.

References

[1] A. Nedić, "Convergence rate of distributed averaging dynamics and optimization in networks," *Foundations and Trends in Systems and Control*, vol. 2, no. 1, pp. 1–100, 2015.

[2] B. Johansson, T. Keviczky, M. Johansson, and K. H. Johansson, "Subgradient methods and consensus algorithms for solving convex optimization problems," in *Proceedings of the IEEE Conference on Decision and Control*, 2008, pp. 4185–4190.

[3] A. Nedić and A. Ozdaglar, "Distributed subgradient methods for multi-agent optimization," *IEEE Transactions on Automatic Control*, vol. 54, no. 1, pp. 48–61, 2009.

[4] M. Zhu and S. Martínez, "On distributed convex optimization under inequality and equality constraints," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 151–164, 2012.

[5] K. I. Tsianos, S. Lawlor, and M. G. Rabbat, "Push-sum distributed dual averaging for convex optimization," in *Proceedings of the IEEE Conference on Decision and Control*, 2012, pp. 5453–5458.

[6] A. Nedić and A. Olshevsky, "Distributed optimization over time-varying directed graphs," *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 601–615, 2015.

[7] T. Yang, J. Lu, D. Wu, J. Wu, G. Shi, Z. Meng, and K. H. Johansson, "A distributed algorithm for economic dispatch over time-varying directed networks with delays," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 6, pp. 5095–5106, 2017.

[8] W. Shi, Q. Ling, G. Wu, and W. Yin, "EXTRA: An exact first-order algorithm for decentralized consensus optimization," *SIAM Journal on Optimization*, vol. 25, no. 2, pp. 944–966, 2015.

[9] G. Qu and N. Li, "Harnessing smoothness to accelerate distributed optimization," *IEEE Transactions on Control of Network Systems*, 2017, to appear.

[10] J. Xu, S. Zhu, Y. C. Soh, and L. Xie, "Augmented distributed gradient methods for multi-agent optimization under uncoordinated constant stepsizes," in *Proceedings of the IEEE Conference on Decision and Control*, 2015, pp. 2055–2060.

[11] J. Wang and N. Elia, "Control approach to distributed optimization," in *Proceedings of the Annual Allerton Conference on Communication, Control, and Computing*, 2010, pp. 557–561.

[12] J. Lu and C. Y. Tang, "Zero-gradient-sum algorithms for distributed convex optimization: The continuous-time case," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2348–2354, 2012.

[13] B. Gharesifard and J. Cortés, "Distributed continuous-time convex optimization on weight-balanced digraphs," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 781–786, 2014.

[14] X. Zeng, P. Yi, and Y. Hong, "Distributed continuous-time algorithm for constrained convex optimizations via nonsmooth analysis approach," *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 5227–5233, 2017.

[15] Y. Xie and Z. Lin, "Global optimal consensus of multi-agent systems with bounded controls," *Systems & Control Letters*, vol. 102, pp. 104–111, 2017.

[16] Z. Li, Z. Ding, J. Sun, and Z. Li, "Distributed adaptive convex optimization on directed graphs via continuous-time algorithms," *IEEE Transactions on Automatic Control*, vol. 63, no. 5, pp. 1434–1441, 2017.

[17] K. J. Aström and B. M. Bernhardsson, "Comparison of riemann and lebesgue sampling for first order stochastic systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2002, pp. 2011–2016.

[18] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, 2007.

[19] A. Girard, "Dynamic triggering mechanisms for event-triggered control," *IEEE Transactions on Automatic Control*, vol. 60, no. 7, pp. 1992–1997, 2015.

[20] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 586–601, 2011.

[21] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Proceedings of the IEEE Conference on Decision and Control*, 2012, pp. 3270–3285.

[22] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2012.

[23] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, no. 1, pp. 245–252, 2013.

[24] X. Meng, L. Xie, Y. C. Soh, C. Nowzari, and G. J. Pappas, "Periodic event-triggered average consensus over directed graphs," in *Proceedings of the IEEE Conference on Decision and Control*, 2015, pp. 2055–2060.

[25] X. Yi, "Resource-constrained multi-agent control systems: Dynamic event-triggering, input saturation, and connectivity preservation," Licentiate Thesis, Royal Institute of Technology, Sweden, 2017.

[26] S. S. Kia, J. Cortés, and S. Martínez, "Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication," *Automatica*, vol. 55, pp. 254–264, 2015.

[27] W. Chen and W. Ren, "Event-triggered zero-gradient-sum distributed consensus optimization over directed networks," *Automatica*, vol. 65, pp. 90–97, 2016.

[28] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry, "On the regularization of zeno hybrid automata," *Systems & Control Letters*, vol. 38, no. 3, pp. 141 – 150, 1999.

[29] D. P. Bertsekas, *Nonlinear Programming*. Belmont, MA: Athena Scientific, 1999.

[30] X. Yi, W. Lu, and T. Chen, "Pull-based distributed event-triggered consensus for multiagent systems with directed topologies," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 1, pp. 71–79, 2017.