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Multi-agent Systems with Compasses: Cooperative and Cooperative-antagonistic networks

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Abstract: In this paper, we first study agreement protocols for coupled continuous-time nonlinear dynamics over cooperative multi-agent networks. To guarantee convergence for such systems, it is common in the literature to assume that the vector field of each agent is pointing inside the convex hull formed by the states of the agent and its neighbors given the relative states between each agent and its neighbors are available. This convexity condition is relaxed in this paper, as we show that it is enough that the vector field belongs to a strict tangent cone based on a local supporting hyperrectangle. The new condition has the natural physical interpretation of adding a compass for each agent in addition to the available local relative states, as each agent needs only to know in which orthant each of its neighbor is. It is proven that the multi-agent system achieves exponential state agreement if and only if the time-varying interaction graph is uniformly jointly quasi-strongly connected. Cooperative–antagonistic multi-agent networks are also considered. For these systems, the (cooperative–antagonistic) relation has a negative sign for arcs corresponding to antagonistic interactions. State agreement may not be achieved for cooperative–antagonistic multi-agent systems. Instead it is shown that asymptotic modulus agreement is achieved if the time-varying interaction graph is uniformly jointly strongly connected.

Key Words: State agreement, modulus agreement, nonlinear systems, cooperative-antagonistic network

1 Introduction

1.1 Motivation

In the last decade, coordinated control of multi-agent systems has attracted extensive attention due to its broad applications in engineering, physics, biology and social sciences [10, 18, 23]. A fundamental idea is that by carefully implementing distributed control protocols for each agent (node), collective tasks can be reached for a network system using only neighboring information exchange and interactions. Agreement protocols, where the goal is to drive the node states to a common value, serve as primary problems and canonical solutions to distributed controller design [17].

The idea of state agreement protocol and its fundamental convergence properties were established for linear systems in the classical work [22]. Various efforts have been made towards a clear understanding on how the underlying communication graph influences the convergence and convergence rate of linear agreement seeking, just to name a few [4, 5, 10, 19]. In the meantime, agreement protocols with nonlinear agent dynamics are also intriguing due to the nonlinear nature of many real-world network systems. In fact, classical Kuramoto model and Vicsek's model of coupled node dynamics are both of nonlinear form [11, 23]. Due to the challenge raised by the nonlinearity of node interactions, results on the agreement seeking of nonlinear multi-agent systems are quite limited in the literature, especially for time-varying communication graphs [3, 13, 16].

These existing linear or nonlinear agreement protocols are functioning all because of a fundamental convexity assumption on the node interactions. For discrete-time models, it is usually assumed that each agent updates its state as a convex combination of its neighbors' states [4, 14, 16]. For continuous-time models, the vector field for each agent must fall into the relative interior of a tangent cone formed by the convex hull of the relative state vectors in its neighborhood [13]. Another standing assumption in above works lies in that agents in the network must be cooperative, which is often not the case in reality. Recently, motivated from opinion dynamics evolving over social networks and security of network systems, state agreement problems over cooperative–antagonistic networks were studied in [1, 20]. In such networks, each arc is associated with a positive/negative sign indicating cooperative/antagonistic relations.

1.2 Contributions

In this paper, we intend to answer the following questions for nonlinear agreement protocols.

Q1. When and how the fundamental convexity assumption on node interactions can be relaxed?

Q2. Can we explicitly characterize the convergence rate of nonlinear multi-agent systems?

Q3. What is the fundamental difference in asymptotical state evolution between cooperative and cooperative-antagonistic networks?

We show that the convexity condition needed for agreement seeking of multi-agent systems, can be relaxed, at the cost of equipping each agent with a "compass" with the help of which the direction of each axis can be observed for a prescribed global coordinate system. We do not require that each agent has access to its own or its neighbors' states, but the information exchange is based on relative states of the agents as usual. Using the compass, each agent can derive a strict tangent cone from a local supporting hyperrectangle. This cone defines the feasible set of local control actions for the agent to guarantee convergence to state agreement. It is argued that the vector field of an agent can be outside of the convex hull formed by the states of the agent and its neighbors, and thus provides a relaxed condition for agreement.

We remark that a magnetic compass is naturally present in many biological systems. For instance, the European Robin bird can detect and navigate through the Earth's magnetic

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field, providing them with a compass in addition to their normal vision [21]. Engineering systems, such as multi-robot networks, can be equipped with an electronic compass at a rather low cost.

Under a precise and general definition to nonlinear multi-agent systems with compasses, we establish two main results:

- For cooperative networks, we show that the underlying communication graph associated with the nonlinear interactions being uniformly jointly quasi-strongly connected is necessary and sufficient for exponential agreement. The convergence rate is explicitly given.
- For cooperative-antagonistic networks, we propose a general model following the sign-flipping interpretation along an antagonistic arc introduced in [1]. We show that the underlying graph being uniformly jointly strongly connected, irrespective with the sign of the arcs, is sufficient for asymptotic modulus agreement in the sense that the absolute value of each agent state component reaches an agreement asymptotically.

We believe these results have largely extended the previous understandings on multi-agent systems with nonlinear node dynamics and with possible antagonistic interactions.

1.3 Paper Organization

The remainder of the paper is organized as follows. In Section 2, we give some mathematical preliminaries on convex sets, graph theory, and Dini derivatives. The nonlinear multi-agent dynamics, the interaction graph, the compass, and and the convergence definitions are presented in Section 3. The main result on agreement for cooperative multi-agent system is presented in Section 4. The main result on asymptotic modulus agreement for cooperative–antagonistic network is given in Section 5. A brief concluding remark is given in Section 6.

2 Preliminaries

In this section, we introduce some mathematical preliminaries on convex analysis [2], graph theory [9], and Dini derivatives [8].

2.1 Convex analysis

For any nonempty set $\mathcal{S} \subseteq \mathbb{R}^d$, $\|x\|_{\mathcal{S}} = \inf_{y\in\mathcal{S}} \|x-y\|$ is called the distance between $x \in \mathbb{R}^d$ and $\mathcal{S}$, where $\|\cdot\|$ denotes the Euclidean norm. A set $\mathcal{S} \subset \mathbb{R}^d$ is called convex if $(1-\zeta)x + \zeta y \in \mathcal{S}$ when $x \in \mathcal{S}$, $y \in \mathcal{S}$, and $0 \le \zeta \le 1$. A convex set $\mathcal{S} \subset \mathbb{R}^d$ is called a convex cone if $\zeta x \in \mathcal{S}$ when $x \in \mathcal{S}$ and $\zeta > 0$. The convex hull of $\mathcal{S} \subset \mathbb{R}^d$ is denoted $\mathrm{co}(\mathcal{S})$ and the convex hull of a finite set of points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ denoted $\mathrm{co}\{x_1, x_2, \dots, x_n\}$.

Let $\mathcal{S}$ be a convex set. Then there is a unique element $P_{\mathcal{S}}(x) \in \mathcal{S}$, called the convex projection of x onto $\mathcal{S}$, satisfying $\|x - P_{\mathcal{S}}(x)\| = \|x\|_{\mathcal{S}}$ associated to any $x \in \mathbb{R}^d$. It is also known that $\|x\|^2_{\mathcal{S}}$ is continuously differentiable for all $x \in \mathbb{R}^d$, and its gradient can be explicitly computed [2]:

$$\nabla \|x\|^2_{\mathcal{S}} = 2(x - P_{\mathcal{S}}(x)). \tag{1}$$

Let $\mathcal{S} \subset \mathbb{R}^d$ be convex and closed. The interior and boundary of $\mathcal{S}$ is denoted by $\mathrm{int}(\mathcal{S})$ and $\partial\mathcal{S}$, respectively. If $\mathcal{S}$ contains the origin, the smallest subspace containing $\mathcal{S}$ is the carrier subspace denoted by $\mathrm{cs}(\mathcal{S})$. The relative interior of $\mathcal{S}$, denoted by $\mathrm{ri}(\mathcal{S})$, is the interior of $\mathcal{S}$ with respect to the subspace $\mathrm{cs}(\mathcal{S})$ and the relative topology used. If $\mathcal{S}$ does not contain the origin, $\mathrm{cs}(\mathcal{S})$ denotes the smallest subspace containing $\mathcal{S} - z$, where z is any point in $\mathcal{S}$. Then, $\mathrm{ri}(\mathcal{S})$ is the interior of $\mathcal{S}$ with respect to the subspace $z + \mathrm{cs}(\mathcal{S})$. Similarly, we can define the relative boundary $\mathrm{rb}(\mathcal{S})$.

Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed convex set and $x \in \mathcal{S}$. The tangent cone to $\mathcal{S}$ at x is defined as the set

$$\mathcal{T}(x,\mathcal{S}) = \Big\{ z \in \mathbb{R}^d : \liminf_{\zeta\to 0} \frac{\|x+\zeta z\|_{\mathcal{S}}}{\zeta} = 0 \Big\}.$$

Note that if $x \in \mathrm{int}(\mathcal{S})$, then $\mathcal{T}(x,\mathcal{S}) = \mathbb{R}^d$. Therefore, the definition of $\mathcal{T}(x,\mathcal{S})$ is essential only when $x \in \partial\mathcal{S}$.

2.2 Graph Theory

A directed graph $\mathcal{G}$ consists of a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, n\}$ is a finite, nonempty set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of ordered pairs of nodes denoted arcs. The set of neighbors of node i is denoted $\mathcal{N}_i := \{j : (j,i) \in \mathcal{E}\}$. A directed path in a directed graph is a sequence of arcs of the form $(i,j), (j,k), \dots$. If there exists a path from node i to j, then node j is said to be reachable from node i. If for node i, there exists a path from i to any other node, then i is called a root of $\mathcal{G}$. $\mathcal{G}$ is said to be strongly connected if each node is reachable from any other node. $\mathcal{G}$ is said to be quasi-strongly connected if $\mathcal{G}$ has a root.

2.3 Dini derivatives

Consider the differential equation

$$\dot{x} = f(t,x), \tag{2}$$

where $f : \mathbb{R} \times \mathcal{M} \to \mathbb{R}^d$ is continuous in $x \in \mathcal{M} \subset \mathbb{R}^d$ for fixed t and piecewise continuous in t for fixed x. Let $V(t,x) : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ be locally Lipschitz with respect to x and uniformly continuous with respect to t. Define

$$D_f^+ V(t,x) = \lim_{\tau\to 0^+} \sup \frac{V(t+\tau, x+\tau f(t,x)) - V(t,x)}{\tau}.$$

The function $D_f^+ V$ is called the upper Dini derivative of V along the trajectory of (2). Suppose that for an initial condition $x(t_0)$, (2) has a solution $x(t)$ defined on an interval $[0,\epsilon)$ and let $D^+V(t,x(t))$ be the upper Dini derivative of $V(t,x(t))$ with respect to t, i.e.,

$$D^+ V(t,x) = \lim_{\tau\to 0^+} \sup \frac{V(t+\tau, x(t+\tau)) - V(t,x(t))}{\tau}.$$

Let $t^* \in [0,\epsilon)$ and put $x(t^*) = x^*$. Then we have that

$$D^+V(t^*, x(t^*)) = D_f^+ V(t^*, x^*).$$

The following lemma can be found in [7].

Lemma 1. *Suppose for each $i \in \mathcal{V}$, $V_i : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ is continuously differentiable. Let $V(t,x) = \max_{i\in\mathcal{V}} V_i(t,x)$, and let $\widehat{\mathcal{V}}(t) = \{i \in \mathcal{V} : V_i(t,x(t)) = V(t,x(t))\}$ be the set of indices where the maximum is reached at time t. Then*

$$D^+V(t,x(t)) = \max_{i\in\widehat{\mathcal{V}}(t)} \dot{V}_i(t,x(t)).$$

3 Multi-agent System

In this section, we present the model of the considered multi-agent systems, introduce the corresponding interaction graph, and define some useful geometric concepts used in the control laws.

Consider a multi-agent system with agent set $\mathcal{V} = \{1,\dots,n\}$. Let $x_i \in \mathbb{R}^d$ denote the state of agent i. Let $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_n^{\mathrm{T}})^{\mathrm{T}}$ and denote $\mathcal{D} = \{1, 2, \dots, d\}$.

3.1 Nonlinear multi-agent dynamics

Let $\mathfrak{P}$ be a given (finite or infinite) set of indices. An element in $\mathfrak{P}$ is denoted by p. For any $p \in \mathfrak{P}$, we define a function $f_p(x_1, x_2, \dots, x_n) : \mathbb{R}^{dn} \to \mathbb{R}^{dn}$ associated with p, where

$$f_p(x_1, x_2, \dots, x_n) = \begin{pmatrix} f_p^1(x_1, x_2, \dots, x_n) \\ \vdots \\ f_p^n(x_1, x_2, \dots, x_n) \end{pmatrix}$$

with $f_p^i : \mathbb{R}^{dn} \to \mathbb{R}^d$, $i = 1, 2, \dots, n$.

Let $\sigma(t) : [t_0, \infty) \to \mathfrak{P}$ be a piecewise constant function, so, there exists a sequence of increasing time instances $\{t_l\}_0^\infty$ such that $\sigma(t)$ remains constant for $t \in [t_l, t_{l+1})$ and switches at $t = t_l$.

The dynamics of the multi-agent systems is described by the switched nonlinear system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \tag{3}$$

We place some mild assumptions on this system.

Assumption 1 (Dwell time). *There exists a lower bound $\tau_d > 0$, such that $\inf_l (t_{l+1} - t_l) \geq \tau_d$.*

Assumption 2 (Uniformly locally Lipschitz). *$f_p(x)$ is uniformly locally Lipschitz on $\mathbb{R}^{dn}$,* i.e., *for every $x \in \mathbb{R}^{dn}$, we can find a neighborhood $\mathcal{U}_\alpha(x) = \{y \in \mathbb{R}^{dn} : \|y - x\| \leq \alpha\}$ for some $\alpha > 0$ such that there exits a real number $L(x) > 0$ with $\|f_p(a) - f_p(b)\| \leq L(x)\|a - b\|$ for any $a, b \in \mathcal{U}_\alpha(x)$ and $p \in \mathfrak{P}$.*

Under Assumptions 1 and 2, the Caratheodory solutions of (3) exist for arbitrary initial conditions, and they are absolutely continuous functions for almost all t on the maximum interval of existence [6, 8]. All our further discussions will be on the Caratheodory solutions of (3) without specific mention.

3.2 Interaction graph

Having the dynamics defined for the considered multi-agent system, we introduce next its interaction graph.

Definition 1 (Interaction graph). *The graph $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}_p)$ associated with f_p is the directed graph on node set $\mathcal{V} = \{1, 2, \dots, n\}$ and arc set $\mathcal{E}_p$ such that $(j, i) \in \mathcal{E}_p$ if and only if f_p^i depends on x_j,* i.e., *there exist $x_j, \overline{x}_j \in \mathbb{R}^d$ such that*

$$f_p^i(x_1, \dots, x_j, \dots, x_n) \neq f_p^i(x_1, \dots, \overline{x}_j, \dots, x_n).$$

The set of neighbors of node i in $\mathcal{G}_p$ is denoted by $\mathcal{N}_i(p)$. The dynamic interaction graph associated with system (3) is denoted by $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$. The joint graph of $\mathcal{G}_{\sigma(t)}$ during time interval $[t_1, t_2)$ is defined by $\mathcal{G}_{\sigma(t)}([t_1, t_2)) = \bigcup_{t \in [t_1, t_2)} \mathcal{G}(t) = (\mathcal{V}, \bigcup_{t \in [t_1, t_2)} \mathcal{E}_{\sigma(t)})$. We impose the following definition on the connectivity of $\mathcal{G}_{\sigma(t)}$.

Definition 2 (Joint connectivity). *(i) $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is quasi-strongly connected for any $t \geq t_0$.*

(ii) $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is strongly connected for any $t \geq t_0$.

For each $p \in \mathfrak{P}$, the node relation along an interaction arc $(i, j) \in \mathcal{E}_p$ may be cooperative, or antagonistic. These different types of arcs are modeled by signed graphs. We assume that there is a sign, "+1" or "-1", associated with each $(i, j) \in \mathcal{E}_p$, denoted by sgn_p^{ij}. To be precise, if j is cooperative to i, $\mathrm{sgn}_p^{ij} = +1$, and if j is antagonistic to i, $\mathrm{sgn}_p^{ij} = -1$.

Definition 3 (Cooperative and cooperative-antagonistic networks). *If $\mathrm{sgn}_p^{ij} = +1$, for all $(j, i) \in \mathcal{E}_p$ and all $p \in \mathfrak{P}$, the considered multi-agent network is called a cooperative network. Otherwise, it is called a cooperative-antagonistic network.*

3.3 Compass, hyperrectangle, and strict tangent cone

We assume that each agent has access to a compass corresponding to a common Cartesian coordinate system. We use $(\overrightarrow{r_1}, \overrightarrow{r_2}, \dots, \overrightarrow{r_d})$ to represent the basis of the $\mathbb{R}^d$ Cartesian coordinate system. Here $\overrightarrow{r_k} = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the unit vector in the direction of axis $k \in \mathcal{D}$. Obviously, a point in $\mathbb{R}^d$ can be described by $z = z_1\overrightarrow{r_1} + z_2\overrightarrow{r_2} + \dots + z_d\overrightarrow{r_d}$, where z_k is a real number for all $k \in \mathcal{D}$.

A hyperrectangle is the generalization of a rectangle to higher dimensions. An axis-aligned hyperrectangle is a hyperrectangle subject to the constraint that the edges of the hyperrectangle are parallel to the Cartesian coordinate axes.

Definition 4 (Supporting hyperrectangle). *Let $\mathcal{C} \subset \mathbb{R}^d$ be a bounded set. The supporting hyperrectangle $\mathcal{H}(\mathcal{C})$ is the axis-aligned hyperrectangle*

$$\mathcal{H}(\mathcal{C}) = [\min(\mathcal{C})_1, \max(\mathcal{C})_1] \times [\min(\mathcal{C})_2, \max(\mathcal{C})_2] \times \dots \\ \times [\min(\mathcal{C})_d, \max(\mathcal{C})_d],$$

where $\min(\mathcal{C})_k := \min\{y_k : y_k \text{ is the } k\text{th entry of } y \in \mathcal{C}\}$, *and* $\max(\mathcal{C})_k := \max\{y_k : y_k \text{ is the } k\text{th entry of } y \in \mathcal{C}\}$.

In other words, a supporting hyperrectangle of a bounded set $\mathcal{C}$ is an axis-aligned minimum bounding hyperrectangle.

Definition 5 (Strict tangent cone). *Let $\mathcal{A} \subset \mathbb{R}^d$ be an axis-aligned hyperrectangle and $\gamma > 0$ a constant. The γ-strict tangent cone to $\mathcal{A}$ at $x \in \mathbb{R}^d$ is the set*

$$\mathcal{T}_\gamma(x, \mathcal{A}) = \begin{cases} \mathrm{cs}(\mathcal{A}); & \textit{if } x \in \mathrm{ri}(\mathcal{A}) \\ \mathcal{T}(x, \mathcal{A}) \bigcap \{z \in \mathbb{R}^d : |\langle z, \overrightarrow{r_k} \rangle| \geq \gamma D_k(\mathcal{A})\}; & \\ & \textit{if } x \in \mathrm{rb}_k(\mathcal{A}), \end{cases}$$

where $\mathrm{rb}_k(\mathcal{A})$ denotes one of the two facets of $\mathcal{A}$ perpendicular to the axis $\overrightarrow{r_k}$, and $D_k(\mathcal{A}) = |\max(\mathcal{A})_k - \min(\mathcal{A})_k|$ denotes the side length parallel to the axis $\overrightarrow{r_k}$.

3.4 Uniformly asymptotic agreement, exponential agreement, and asymptotic modulus agreement

Definition 6 (Uniformly asymptotic agreement)**.** *The switched coupled system* (3) *is said to achieve uniformly asymptotic agreement on* $\mathcal{S}_0 \subseteq \mathbb{R}^d$ *if*

1) *point-wise uniform agreement can be achieved, i.e., for all* $\eta \in \mathcal{J}$*,* $\forall \varepsilon > 0$*, there exists a positive constant* $\delta(\varepsilon)$ *such that* $\forall t_0 \geq 0$ *and* $x(t_0) \in \mathcal{S}_0^n$*, and*

$$\|x(t_0) - \eta\| < \delta \quad \Rightarrow \quad \|x(t) - \eta\| < \varepsilon, \ \forall t \geq t_0,$$

where agreement manifold is defined as $\mathcal{J} = \{x \in \mathcal{S}_0^n : x_1 = x_2 = \cdots = x_n\}$*;*

2) *uniform agreement attraction can be achieved, i.e.,* $\forall \varepsilon > 0$*, there exist a* $\eta \in \mathcal{J}$ *and a positive constant* $T(\varepsilon)$ *such that for all* $t_0 \geq 0$ *and* $x(t_0) \in \mathcal{S}_0^n$*,*

$$\|x(t) - \eta\| < \varepsilon, \ \forall t \geq t_0 + T.$$

Definition 7 (Exponential agreement)**.** *The switched coupled system* (3) *is said to achieve exponential state agreement on* $\mathcal{S}_0 \subseteq \mathbb{R}^d$ *if*

1) *point-wise uniform agreement can be achieved; and*
2) *exponential agreement attraction can be achieved, i.e., there exist a* $\eta \in \mathcal{J}$ *and positive constants* $k(\mathcal{S}_0)$*,* $\lambda(\mathcal{S}_0)$*, such that for all* $t_0 \geq 0$ *and* $x(t_0) \in \mathcal{S}_0^n$*,*

$$\|x(t) - \eta\| \leq ke^{-\lambda(t-t_0)}\|x(t_0) - \eta\|.$$

Asymptotic modulus agreement of system (3) is defined as follows.

Definition 8 (Asymptotic modulus agreement)**.** *System* (3) *achieves asymptotic modulus agreement for initial time* $t_0 \geq 0$ *and initial state* $x(t_0) \in \mathbb{R}^{nd}$ *if there exist a* $\eta \in \overrightarrow{\mathcal{J}}$ *such that*

$$\lim_{t\to\infty} \||x(t)|_* - \eta\| = 0,$$

where $\overline{\mathcal{J}} = \{x \in \mathbb{R}^{dn} : |x_1|_* = |x_2|_* = \cdots = |x_n|_*\}$*, and the componentwise absolute value* $|\cdot|_*$ *is defined as* $|z|_* = [|z_1|, |z_2|, \ldots, |z_d|]^{\mathrm{T}}$ *for the vector* $z = [z_1, z_2, \ldots, z_d]^{\mathrm{T}}$.

Remark 1. *The concept of "agreement" is just the same as that of "consensus", e.g., [17]. "Modulus agreement" means that the absolute values of the node states eventually reach the same value. In this case, it is possible that the agents converge to the orgin, a non-zero state, or two different states. We call the case of converging to the origin a "trivial" agreement as the agents do not agree on anything that is a function of their states. Agreement [16] and bipartite agreement [1] can be considered as special cases of modulus agreement.*

4 Cooperative Multi-agent Systems: Exponential Agreement

In this section, we study the convergence property of the nonlinear switched system (3) over a cooperative network defined by an interaction graph. Introduce $\mathcal{C}_p^i(x) = \mathrm{co}\{x_i, x_j : j \in \mathcal{N}_i(p)\}$. We impose the following assumption.

Assumption 3 (Vector field)**.** *For all* $i \in \mathcal{V}$*,* $p \in \mathfrak{P}$*, and* $x \in \mathbb{R}^{dn}$*, it holds that* $f_p^i(x) \in \mathcal{T}_\gamma(x_i, \mathcal{H}(\mathcal{C}_p^i(x)))$.

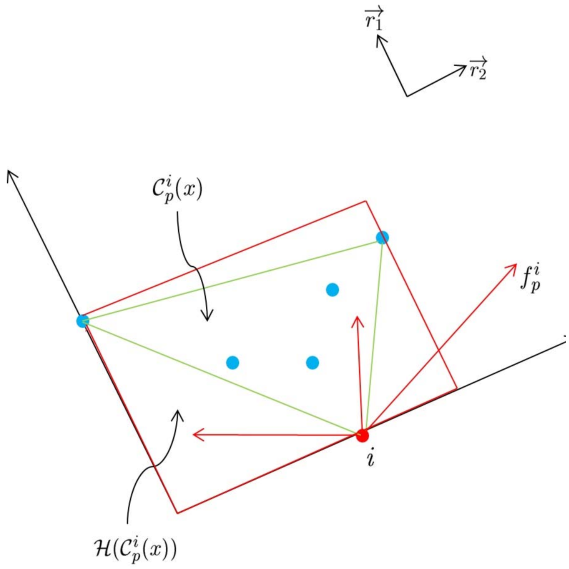

Figure 1: Convex hull, supporting hyperrectangle, and feasible vectors f_p^i satisfying Assumption 3

Remark 2. *In Assumption 3, the vector field* f_p^i *can be chosen freely from the set* $\mathcal{T}_\gamma(x_i, \mathcal{H}(\mathcal{C}_p^i(x)))$*. Hence, the assumption specifies constraints on the feasible controls for the considered multi-agent system. Here* $\mathcal{C}_p^i(x)$ *denotes the convex hull formed by agent* i *and its neighbors,* $\mathcal{H}(\mathcal{C}_p^i(x))$ *(defined in Section 3.3) denotes the supporting hyperrectangle of the set* $\mathcal{C}_p^i(x)$*, and* $\mathcal{T}_\gamma(x_i, \mathcal{H}(\mathcal{C}_p^i(x)))$ *(also defined in Section 3.3) denotes the* γ*-strict tangent cone to* $\mathcal{H}(\mathcal{C}_p^i(x))$ *at* x_i*. Figure 1 gives an example of the convex hull and the supporting hyperrectangle formed by agent* i *and its' neighbors. Three feasible vectors* f_p^i *are presented.*

Remark 3. *It is essential to capture what information exchange is required in a multi-agent system implementing a control law fulfilling Assumption 3. Each agent uses its own coordinate system to locate in which orthant each of its neighbor is. Then the agent constructs the supporting hyperrectangle based on the relative states between itself and its neighbors, similarly to conventional agreement protocols for multi-agent systems. When the agent is inside its supporting hyperrectangle, the vector field for the agent can be chosen arbitrary. When the agent is on the boundary of its supporting hyperrectangle, the feasible control is just any direction pointing inside the halfspace of its supporting hyperrectangle. Note that the absolute state of the agents is not needed, but each agent needs to identify* $d-1$ *absolute directions such that it can identify the direction of its neighbors with respect to itself. For example, for the planar case* $d = 2$*, each agent just needs to be equipped with a compass (providing direction information) to implement this protocol. The compass provides the quadrant location information of its neighbors. For* $d > 2$*, the (generalized) compass gives information on which orthant the neighbors belong to.*

The main result for the agreement seeking of the nonlinear multi-agent dynamics over cooperative networks is given as follows.

Theorem 1. *Suppose $\mathcal{S}_0$ is compact and that Assumptions 1–3 hold. The cooperative multi-agent system* (3) *achieves exponential agreement on $\mathcal{S}_0$ if and only if its interaction graph $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected.*

In order to highlight the improvement of the proposed "supporting hyperrectangle condition" with respect to the usual convex hull condition [13, 16], we next present a uniformly asymptotic agreement result based on the relative interior condition of a tangent cone formed by the supporting hyperrectangle.

Assumption 4 (Vector field)**.** *For all $i \in \mathcal{V}$, $p \in \mathfrak{P}$, and $x \in \mathbb{R}^{dn}$, it holds that $f_p^i(x) \in \mathrm{ri}\left(\mathcal{T}(x_i, \mathcal{H}(\mathcal{C}_p^i(x)))\right)$.*

Proposition 1. *Suppose $\mathcal{S}_0$ is compact and that Assumptions 1, 2, and 4 hold. The cooperative multi-agent system* (3) *achieves uniformly asymptotic agreement on $\mathcal{S}_0$ if and only if its interaction graph $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected.*

Remark 4. *Theorem 1 and Proposition 1 are consistent with the main results in [13, 14, 16]. Our analysis relies on some techniques developed in [12]. Proposition 1 allows that the vector field belongs to a larger set compared with the convex hull condition proposed in [13, 14, 16]. In addition, we allow the agent dynamics to switch over a possibly infinite set and we show exponential agreement and derive in the proof the explicit exponential rate for the convergence in Theorem 1.*

Due the space limitations, we next prove Theorem 1 by analyzing a contraction property of (3) and omit the proof of Proposition 1. Before moving on, we first present several important lemmas without proofs. Detailed proofs of these lemmas can be found in [15].

4.1 Technical lemmas

Definition 9 (Invariant set)**.** *A set $\mathcal{M} \subset \mathbb{R}^{dn}$ is an invariant set for the system* (3) *if for all $t_0 \geq 0$,*

$$x(t_0) \in \mathcal{M} \quad \Longrightarrow \quad x(t) \in \mathcal{M},\ \forall t \geq t_0.$$

For all $k \in \mathcal{D}$, define

$$M_k(x(t)) = \max_{i \in \mathcal{V}}\{x_{ik}(t)\},\ \ m_k(x(t)) = \min_{i \in \mathcal{V}}\{x_{ik}(t)\},$$

where x_{ik} denotes kth entry of x_i. In addition, define the supporting hyperrectangle by the initial states of all agents as $\mathcal{H}_0 := \mathcal{H}(\mathcal{C}(x(t_0)))$, where $\mathcal{C}(x) = \mathrm{co}\{x_1, x_2, \ldots, x_n\}$.

In the following lemma, we show that the supporting hyperrectangle formed by the initial states of all agents is nonexpanding over time.

Lemma 2. *Let Assumptions 1–3 hold. Then, $\mathcal{H}_0^n$ is an invariant set,* i.e., $x_i(t) \in \mathcal{H}_0$, $\forall i \in \mathcal{V}$, $\forall t \geq t_0$.

Lemma 3. *Let Assumptions 1–3 hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. Then, for any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^n$, any $\varepsilon > 0$, and any $T^* > 0$, if $x_{ik}(t_2) \leq M_k(x(t_1)) - \varepsilon$ at some $t_2 \geq t_1$ for some $k \in \mathcal{D}$, then $x_{ik}(t) \leq M_k(x(t_1)) - \delta$, where $\delta = e^{-L_1^* T^*}\varepsilon$ for all $t \in [t_2, t_2 + T^*]$, and L_1^* is a positive constant related to $\mathcal{H}_0$.*

Lemma 4. *Let Assumptions 1–3 hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^n$, any $\varepsilon > 0$, and any $T^* > 0$, if $x_{ik}(t_2) \geq m_k(x(t_1)) + \varepsilon$ at some $t_2 \geq t_1$ for some $k \in \mathcal{D}$, then $x_{ik}(t_2) \geq m_k(x(t_1)) + \delta$, where $\delta = e^{-L_2^* T^*}\varepsilon$ for all $t \in [t_2, t_2 + T^*]$, and L_2^* is a positive constant related to $\mathcal{H}_0$.*

Lemma 5. *Let Assumptions 1–3 hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^n$, any $\delta_1 > 0$ and any $T^* > 0$, if there is an arc (j, i) and a time $t_2 \geq t_1$ such that $j \in \mathcal{N}_i(\sigma(t))$, and $x_{jk}(t) \leq M_k(x(t_1)) - \delta_1$ for all $t \in [t_2, t_2 + \tau_d]$, then there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t) \leq M_k(x(t_1)) - \delta_2$, for all $t \in [t_3, t_3 + T^*]$, where $\delta_2 = \frac{\gamma\tau_d}{L_1^+\tau_d + \gamma\tau_d + 1} e^{-L_1^* T^*}\delta_1$ for some constants L_1^* and L_1^+ related to $\mathcal{H}_0$.*

Lemma 6. *Let Assumptions 1–3 hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^n$, any $\delta_1 > 0$ and any $T^* > 0$, if there is an arc (j, i) and a time $t_2 \geq t_1$ such that $j \in \mathcal{N}_i(\sigma(t))$, and $x_{jk}(t) \geq m_k(x(t_1)) + \delta_1$, then there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t) \geq m_k(x(t_1)) + \delta_2$, for all $t \in [t_3, t_3 + T^*]$, where $\delta_2 = \frac{\gamma\tau_d}{L_2^+\tau_d + \gamma\tau_d + 1} e^{-L_2^* T^*}\delta_1$ for some constants L_2^* and L_2^+ related to $\mathcal{H}_0$.*

4.2 Proof of Theorem 1

The necessity proof follows a similar argument as the proof of Theorem 3.8 of [13]. It is therefore omitted. We prove the sufficiency.

We first prove point-wise uniform agreement. Choose any $\eta \in \mathcal{J}$ and any $\varepsilon > 0$, where $\mathcal{J} = \{x \in \mathcal{S}_0^n : x_1 = x_2 = \cdots = x_n\}$. We define $\mathcal{A}_a(\eta) = \{x \in \mathcal{S}_0^n : \|x - \eta\|_\infty \leq a\}$. It is obvious from Lemma 2 that $\mathcal{A}_a(\eta)$ is a invariant set since a hypercube is a special case of a hyperrectangle. Therefore, by setting $\delta = \frac{\varepsilon}{\sqrt{n}}$, we know that

$$\|x(t_0) - \eta\| \leq \delta \quad \Rightarrow \quad \|x(t) - \eta\| \leq \varepsilon,\ \ \forall t \geq t_0.$$

This shows that point-wise uniform agreement is achieved on $\mathcal{S}_0$.

We next focus on the analysis of agreement attraction. Define

$$V(x) = \rho(\mathcal{H}(\mathcal{C}(x))),$$

where $\rho(\mathcal{H}(\mathcal{C}(x)))$ denotes the diameter of the hyperrectangle $\mathcal{H}(\mathcal{C}(x))$. Clearly, it follows from Lemma 2 that $V(x)$ is nonincreasing along (3) and $x_i(t) \in \mathcal{H}_0$, $\forall i \in \mathcal{V}$, $\forall t \geq t_0$. We prove this theorem by showing that $V(x)$ is strictly shrinking over the time.

Since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected, there is a $T > 0$ such that the union graph $\mathcal{G}([t_0, t_0 + T])$ is quasi-strongly connected. Define $T_1 = T + 2\tau_d$, where τ_d is the dwell time. Denote $\kappa_1 = t_0 + \tau_d$, $\kappa_2 = t_0 + T_1 + \tau_d$, $\ldots$, $\kappa_{n^2} = t_0 + (n^2 - 1)T_1 + \tau_d$. Thus, there exists a node $i_0 \in \mathcal{V}$ such that i_0 has a path to every other nodes jointly on time interval $[\kappa_{l_i}, \kappa_{l_i} + T]$, where $i = 1, 2, \ldots, n$ and $1 \leq l_1 \leq l_2 \leq \cdots \leq l_n \leq n^2$. Denote $\overline{T} = n^2 T_1$.

We divide the rest of the proof into three steps.

(Step I). Consider the time interval $[t_0, t_0 + \overline{T}]$ and $k = 1$. In this step, we show that an agent that does not belong to the interior set will become an interior agent due to the attraction of interior agent i_0.

More specifically, define $\varepsilon_1 = \frac{M_1(x(t_0)) - m_1(x(t_0))}{2}$. It is trivial to show that $M_1(x(t)) = m_1(x(t))$, $\forall t \geq t_0$ when

$M_1(x(t_0)) = m_1(x(t_0))$ based on Definition 5. Therefore, we assume that $M_1(x(t_0)) \neq m_1(x(t_0))$ without loss of generality. Split the node set into two disjoint subsets $\mathcal{V}_1 = \{j | \, x_{j1}(t_0) \leq M_1(x(t_0)) - \varepsilon_1\}$ and $\overline{\mathcal{V}}_1 = \{j | j \notin \mathcal{V}_1\}$.

Assume that $i_0 \in \mathcal{V}_1$. This implies that $x_{i_0 1}(t_0) \leq M_1(x(t_0)) - \varepsilon_1$. It follows from Lemma 3 that $x_{i_0 1}(t) \leq M_1(x(t_0)) - \delta_1$, $\forall t \in [t_0, t_0 + \overline{T}]$, where $\delta_1 = e^{-L_1^* \overline{T}} \varepsilon_1$. Considering the time interval $[\kappa_{l_1}, \kappa_{l_1} + T]$, we can show that there is an arc $(i_1, j_1) \in \mathcal{V}_1 \times \overline{\mathcal{V}}_1$ such that i_1 is a neighbor of j_1 because otherwise there is no arc (i_1, j_1) for any $i_1 \in \mathcal{V}_1$ and $j_1 \in \overline{\mathcal{V}}_1$ (this contradicts the fact $i_1 \in \mathcal{V}_1$ has a path to every other nodes jointly on time interval $[\kappa_{l_1}, \kappa_{l_1} + T]$). Therefore, there exists a time $\tau \in [\kappa_{l_1}, \kappa_{l_1} + T] = [t_0 + (l_1 - 1)T + \tau_d, t_0 + l_1 T - \tau_d]$ such that $j_1 \in \mathcal{N}_i(\sigma(\tau))$. Based on Assumption 1, it follows that there is time interval $[\overline{\tau}_1, \overline{\tau}_1 + \tau_d] \subset [t_0 + (l_1 - 1)T, t_0 + l_1 T]$ such that $j_1 \in \mathcal{N}_i(\sigma(\tau))$, for all $t \in [\overline{\tau}_1, \overline{\tau}_1 + \tau_d]$.

Also note that $i_1 \in \mathcal{V}_1$ implies that $x_{i_1 1}(t_0) \leq M_1(x(t_0)) - \varepsilon_1$. This further shows that $x_{i_1 1}(t) \leq M_1(x(t_0)) - \delta_1$, $\forall t \in [t_0, t_0 + \overline{T}]$ based on Lemma 3. Therefore, it follows from Lemma 5 that there exists a $t_2 \in [t_0, \overline{\tau}_1 + \tau_d]$ such that $x_{j_1 1}(t_2) \leq M_1(x(t_0)) - \varepsilon_2$ and $x_{j_1 1}(t) \leq M_1(x(t_0)) - \delta_2$, $\forall t \in [t_2, t_2 + \overline{T}]$, where $\varepsilon_2 = \frac{\gamma\tau_d}{L_1^+ \tau_d + \gamma\tau_d + 1} e^{-L_1^* \overline{T}} \varepsilon_1$ and $\delta_2 = \frac{\gamma\tau_d}{L_1^+ \tau_d + \gamma\tau_d + 1} e^{-L_1^* \overline{T}} \delta_1$. To this end, we have shown that at least two agents are not on the upper boundary at $t_0 + l_1 T$.

(Step II). In this step, we show that the side length of the hyperrectangle $\mathcal{H}(\mathcal{C}(x))$ parallel to the kth axis $\overrightarrow{r_k}$ at $t_0 + \overline{T}$ is strictly less than that at t_0.

We can now redefine two disjoint subsets $\mathcal{V}_2 = \{j | \, x_j(t_0) \leq M_1(x(t_0)) - \varepsilon_2\}$ and $\overline{\mathcal{V}}_2 = \{j | j \notin \mathcal{V}_2\}$. It then follows that $\mathcal{V}_2$ has at least two nodes. By repeating the above analysis, we can show that $x_i(t) \leq M_1(x(t_0)) - \delta_n$, $\forall i \in \mathcal{V}$, $\forall t \in [t_n, t_n + \overline{T}]$ by noting that $\delta_n = \min_{i \in \mathcal{V}}\{\delta_i\}$, where $t_n \in [t_0, \overline{\tau}_n + \tau_d] \subseteq [t_0 + (l_n - 1)T_1, t_0 + l_n T_1]$ and $\delta_n = e^{-nL_1^* \overline{T}} \frac{(\gamma\tau_d)^{n-1}}{(L_1^+ \tau_d + \gamma\tau_d + 1)^{n-1}} \varepsilon_1$.

Instead, if $i_1 \in \overline{\mathcal{V}}_1$, or what is equivalent, $x_{i_1 1}(t_0) \geq m_1(x(t_0)) + \varepsilon_1$, we can similarly show that $x_i(t) \geq m_1(x(t_0)) - \overline{\delta}_n$, $\forall i \in \mathcal{V}$, $\forall t \in [t_n, t_n + \overline{T}]$, where $t_n \in [t_0, \overline{\tau}_n + \tau_d] \subseteq [t_0 + (l_n - 1)T_1, t_0 + l_n T_1]$ and $\overline{\delta}_n = e^{-nL_2^* \overline{T}} \frac{(\gamma\tau_d)^{n-1}}{(L_2^+ \tau_d + \gamma\tau_d + 1)^{n-1}} \varepsilon_1$ using Lemmas 4 and 6.

Therefore, it follows that $D_1(\mathcal{H}(x(t_0 + \overline{T}))) \leq D_1(\mathcal{H}(x(t_0))) - \beta_1 D_1(\mathcal{H}(x(t_0)))$, where $\beta_1 = e^{-nL^* \overline{T}} \frac{(\gamma\tau_d)^{n-1}}{2(L^+ \tau_d + \gamma\tau_d + 1)^{n-1}}$ and $L^* = \max\{L_1^*, L_2^*\}$ and $L^+ = \max\{L_1^+, L_2^+\}$.

(Step III). In this step, we show that $\rho(\mathcal{H}(\mathcal{C}(x)))$ at $t_0 + d\overline{T}$ is strictly less than that at t_0 and thus prove the theorem by showing that V is strictly shrinking.

We consider the time interval $[t_0 + \overline{T}, t_0 + 2\overline{T}]$ and $k = 2$. Following similar analysis as of Step I and Step II, we can show that $D_2(\mathcal{H}(x(t_0 + 2\overline{T}))) \leq D_2(\mathcal{H}(x(t_0))) - \beta_2 D_2(\mathcal{H}(x(t_0)))$, where $\beta_2 = e^{-nL^* \overline{T}} \frac{(\gamma\tau_d)^{n-1}}{2(L^+ \tau_d + \gamma\tau_d + 1)^{n-1}}$.

By repeating the above analysis, it follows that

$$V(x(t_0 + d\overline{T})) - V(x(t_0)) \leq -\beta V(x(t_0)),$$

where $\beta = e^{-nL^* \overline{T}} \frac{(\gamma\tau_d)^{n-1}}{2(L^+ \tau_d + \gamma\tau_d + 1)^{n-1}}$.

Then, let N be the smallest positive integer such that $t \leq t_0 + Nd\overline{T}$. It then follows that

$$\begin{aligned} V(x(t)) &\leq (1 - \beta)^{N-1} V(x(t_0)) \\ &\leq \frac{1}{1 - \beta} (1 - \beta)^{\frac{t - t_0}{d\overline{T}}} V(x(t_0)) \\ &= \frac{1}{1 - \beta} e^{-\beta^*(t - t_0)} V(x(t_0)), \end{aligned}$$

where $\beta^* = \frac{1}{d\overline{T}} \ln \frac{1}{1 - \beta}$. Denote $\mathcal{H}(\mathcal{S}_0)$ as the supporting hyperrectangle of $\mathcal{S}_0$. Since $x(t_0) \in \mathcal{H}_0^n \subseteq \mathcal{H}^n(\mathcal{S}_0)$, it follows that the above inequality holds for any $x(t_0) \in \mathcal{H}^n(\mathcal{S}_0)$ or any $x(t_0) \in \mathcal{S}_0^n$. By choosing $k = \frac{1}{1 - \beta}$ and $\lambda = \beta^*$, we have that exponential attraction is achieved on $\mathcal{S}_0$. This proves the desired theorem.

5 Cooperative–antagonistic Multi-agent Systems: Asymptotic Modulus Agreement

In this section, we study state agreement over cooperative–antagonistic networks. Define $\widehat{\mathcal{C}}_p^i(x) := \mathrm{co}\{x_i, x_j \mathrm{sgn}_p^{ij} : j \in \mathcal{N}_i(p)\}$. For cooperative–antagonistic networks, we impose the following assumption, instead of Assumption 3.

Assumption 5 (Vector field)**.** *For all $i \in \mathcal{V}$, $p \in \mathfrak{P}$ and $x \in \mathbb{R}^{dn}$, it holds that $f_p^i(x) \in \mathcal{T}_\gamma(x_i, \mathcal{H}(\widehat{\mathcal{C}}_p^i(x)))$.*

Assumption 5 follows the model for antagonistic interactions introduced in [1]. Simple examples (see, e.g., [1]) can be found that state agreement cannot be achieved for cooperative–antagonistic networks. Instead, it is possible that different agents hold different values with opposite signs, which is known as bipartite consensus [1]. Therefore, we are interested in the modulus agreement in this part. We present the following result on modulus agreement of cooperative–antagonistic networks.

Theorem 2. *Let Assumptions 1, 2 and 5 hold. Then cooperative-antagonistic multi-agent system* (3) *achieves asymptotic modulus agreement for all initial time $t_0 \geq 0$ and all initial state $x(t_0) \in \mathbb{R}^{nd}$ if the interaction graph $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected.*

Remark 5. *The state agreement result in Theorem 1 relies on uniformly jointly quasi-strong connectivity, while the modulus agreement result in Theorem 2 needs uniformly jointly strong connectivity. In fact, we conjecture that strong connectivity is essential for modulus agreement in the sense that uniformly jointly quasi-strong connectivity might not be enough. The reason is that although Lemmas 3 and 5 can be rebuilt for the upper bound of the node absolute values for cooperative–antagonistic networks, the corresponding Lemmas 4 and 6 no longer hold.*

Remark 6. *Compared to the results given in [1], Theorem 2 requires no conditions on the structural balance properties. In other words, Theorem 2 shows that every positive or negative arc contributes to the convergence of the absolute values of the nodes' states, even for general nonlinear multi-agent dynamics.*

The proof of Theorem 2 will be given using a contradiction arguments, with the help of a series of preliminary lemmas. We omit the proofs of these lemmas here due to space limitation and the detailed proofs can be found in [15].

5.1 Technical Lemmas

We first construct an invariant set for the dynamics under the cooperative–antagonistic networks. For all $k \in \mathcal{D}$, define

$$M_k^{\dagger}(x(t)) = \max_{i \in \mathcal{V}} |x_{ik}(t)|.$$

In addition, define an origin-symmetric supporting hyperrectangle $\mathcal{H}(\widehat{\mathcal{C}}(x)) \subset \mathbb{R}^d$ as

$$\mathcal{H}(\widehat{\mathcal{C}}(x)) := [-M_1^{\dagger}(x), M_1^{\dagger}(x)] \times \cdots \times [-M_d^{\dagger}(x), M_d^{\dagger}(x)].$$

The origin-symmetric supporting hyperrectangle formed by the initial states of all agents $\widehat{\mathcal{H}}_0$ is given by

$$\big[-\max_{i \in \mathcal{V}} |x_{i1}(t_0)|, \max_{i \in \mathcal{V}} |x_{i1}(t_0)|\big] \times \dots \\ \times \big[-\max_{i \in \mathcal{V}} |x_{id}(t_0)|, \max_{i \in \mathcal{V}} |x_{id}(t_0)|\big].$$

Introduce the state transformation

$$y_{ik} = x_{ik}^2, \quad \forall i \in \mathcal{V}, \quad \forall k \in \mathcal{D}.$$

The analysis will be carried out on y_{ik}, instead of x_{ik} to avoid non-smoothness.

Lemma 7. *Let Assumptions 1, 2 and 5 hold. Then, for system* (3), $\widehat{\mathcal{H}}_0^n$ *is an invariant set,* i.e., $x_i(t) \in \widehat{\mathcal{H}}_0$, $\forall i \in \mathcal{V}$, $\forall t \geq t_0$.

Remark 7. *In Figures 2-3, we highlight the different invariant sets for cooperative and cooperative–antagonistic networks. The supporting hyperrectangle $\mathcal{H}(\mathcal{C}(x))$ given in Lemma 2 is illustrated in Figure 2 and the origin-symmetric supporting hyperrectangle $\mathcal{H}(\widehat{\mathcal{C}}(x))$ given in Lemma 7 is illustrated in Figure 3.*

Lemma 8. *Let Assumptions 1, 2, and 5 hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times \widehat{\mathcal{H}}_0^n$, any $\varepsilon > 0$ and any $T^* > 0$, if $y_{ik}(t_2) \leq y^* - \varepsilon$ at some $t_2 \geq t_1$ for some $k \in \mathcal{D}$, where $y^* \geq y_k(x(t_1))$ is a constant. Then $y_{ik}(t) \leq y^* - \delta$ for all $t \in [t_2, t_2 + T^*]$, where $\delta = e^{-L^* T^*}\varepsilon$, and L^* is a positive constant related to $\widehat{\mathcal{H}}_0$.*

Lemma 9. *Let Assumptions 1, 2, and 5 hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times \widehat{\mathcal{H}}_0^n$ and any $\delta > 0$, if there is an arc (j, i) and a time $t_2 \geq t_1$ such that $j \in \mathcal{N}_i(\sigma(t))$, and $y_{jk}(t) \leq y^* - \delta$ for all $t \in [t_2, t_2 + \tau_d]$ for some $k \in \mathcal{D}$, where $y^* \geq y_k(x(t_1))$ is a constant. Then there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $y_{ik}(t_3) \leq y^* - \varepsilon$, where $\varepsilon = \frac{\gamma \tau_d \delta}{2(L^+ \tau_d + \gamma \tau_d + 1)}$ and L^+ is a constant related to $\widehat{\mathcal{H}}_0$.*

5.2 Proof of Theorem 2

The theorem is proved using a contradiction argument.

Lemma 7 implies that for any initial time t_0 and initial value $x(t_0)$, there exist y_k^*, $k \in \mathcal{D}$ such that

$$\lim_{t \to \infty} y_k(t) = y_k^*, \quad k \in \mathcal{D}.$$

Define $\hbar_{ik} = \lim_{t\to\infty} \sup y_{ik}(t)$ and $\ell_{ik} = \lim_{t\to\infty} \inf y_{ik}(t)$, $\forall i \in \mathcal{V}$, $\forall k \in \mathcal{D}$. Clearly,

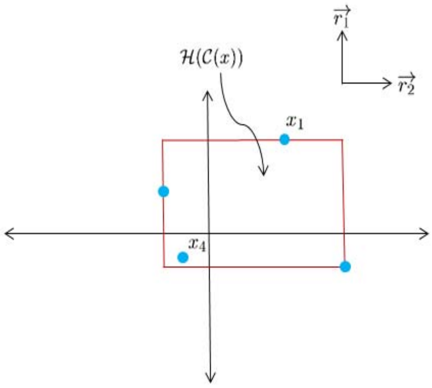

Figure 2: An example of the supporting hyperrectangle of $\mathcal{H}(\mathcal{C}(x))$.

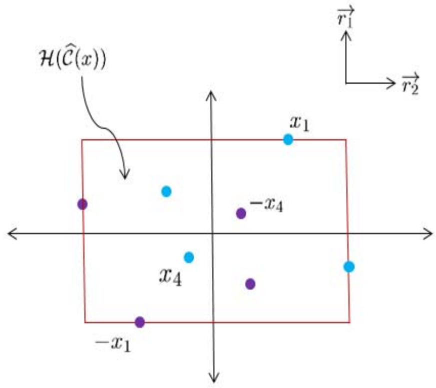

Figure 3: An example of the origin-symmetric supporting hyperrectangle $\mathcal{H}(\widehat{\mathcal{C}}(x))$.

$0 \leq \ell_{ik} \leq \hbar_{ik} \leq y_k^*$. Based on Definition 8, asymptotic modulus agreement is achieved if and only if $\hbar_{ik} = \ell_{ik} = y_k^*$, $\forall i \in \mathcal{V}, \forall k \in \mathcal{D}$. The desired conclusion holds trivially if $y_k^* = 0, k \in \mathcal{D}$. Therefore, we assume that $y_k^* > 0$ for some $k \in \mathcal{D}$ without loss of generality.

Suppose that there exists a node $i_1 \in \mathcal{V}$ such that $0 \leq \ell_{i_1 k} < \hbar_{i_1 k} \leq y_k^*$. Based on the fact that $\lim_{t\to\infty} y_k(t) = y_k^*$, it follows that for any $\varepsilon > 0$, there exists a $\widehat{t}(\varepsilon) > t_0$ such that

$$y_k^* - \varepsilon \leq y_k(t) \leq y_k^* + \varepsilon, \quad t \geq \widehat{t}(\varepsilon).$$

Take $\alpha_{1k} = \sqrt{\frac{1}{2}(\ell_{i_1 k} + \hbar_{i_1 k})}$. Therefore, there exists a time $t_1 \geq \widehat{t}(\varepsilon)$ such that $|x_{i_1 k}(t_1)| = \alpha_{1k}$. This shows that

$$\begin{aligned} x_{i_1 k}^2(t_1) &= \hbar_{i_1 k} - (\hbar_{i_1 k} - \alpha_{1k}^2) \\ &\leq y_k^* + \varepsilon - (\hbar_{i_1 k} - \alpha_{1k}^2) \\ &= y_k^* + \varepsilon - \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 = \hbar_{i_1 k} - \alpha_{1k}^2 > 0$ and the first inequality is based on the definition of $\hbar_{i_1 k}$.

Since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected, there is a $T > 0$ such that the union graph $\mathcal{G}([t_1, t_1 + T])$ is jointly strongly connected. Define $T_1 = T + 2\tau_d$, where τ_d is the dwell time. Denote $\kappa_1 = t_1 + \tau_d$, $\kappa_2 = t_1 + T_1 + \tau_d, \dots,$ $\kappa_n = t_1 + (n-1)T_1 + \tau_d$. For each node $i \in \mathcal{V}$, i has a path to every other nodes jointly on time interval $[\kappa_l, \kappa_l + T]$, where $l = 1, 2, \dots, n$. Denote $\overline{T} = nT_1$.

Consider time interval $[t_1, t_1 + \overline{T}]$. Based on the fact that $y_k(x(t_1)) \le y_k^* + \varepsilon$ and considering $y_k^* + \varepsilon$ as the role of y^* in Lemma 8, it follows that $y_{ik}(t) \le y_k^* + \varepsilon - \delta_1$, $\forall t \in [t_1, t_1 + \overline{T}]$, where $\delta_1 = e^{-L^* \overline{T}} \varepsilon_1$.

Since for each node $i \in \mathcal{V}$, i has a path to every other nodes jointly on time interval $[\kappa_l, \kappa_l + T]$, where $l = 1, 2, \dots, n$, there exists $i_2 \in \mathcal{V}$ such that i_1 is a neighbor of i_2 during the time interval $[\kappa_1, \kappa_1 + T]$. Based on Lemma 9, it follows that there exists $t_2 \in [t_1, \overline{\tau}_1 + \tau_d] \subset [t_1 + T, t_1 + 2T]$ such that $x_{i_2 k}^2(t_2) \le y_k^* + \varepsilon - \varepsilon_2$, where $\varepsilon_2 = \frac{\gamma \tau_d}{2(L^+ \tau_d + \gamma \tau_d + 1)} \delta_1$. This further implies that $x_{i_2 k}^2(t) \le y_k^* + \varepsilon - \delta_2$, $\forall t \in [t_2, t_1 + \overline{T}]$, where $\delta_2 = e^{-L^* \overline{T}} \varepsilon_2$. By repeating the above analysis, we can show that $y_{ik}(t) \le y_k^* + \varepsilon - \delta_n$, $\forall t \in [t_n, t_1 + \overline{T}]$, $\forall i \in \mathcal{V}$, where $t_n \in [t_1, \overline{\tau}_n + \tau_d] \subset [t_1 + (n-1)T, t_1 + nT]$, and δ_n can be iteratively obtained as $\delta_n = e^{-nL^* \overline{T}} \frac{\gamma^{n-1} \tau_d^{n-1}}{2^{n-1}(L^+ \tau_d + \gamma \tau_d + 1)^{n-1}}$. This is indeed true because $\delta_i \le \delta_{i-1}$, $\forall i = 2, 3, \dots, n$.

This shows that $y_k(t_1 + \overline{T}) = \max_{i \in \mathcal{V}} y_{ik} \le y_k^* + \varepsilon - \delta_n$, which indicates a contradiction for sufficiently small ε satisfying $\varepsilon < \delta_n / 2$. Therefore, $\hbar_{ik} = \ell_{ik} = y_k^*$, $\forall i \in \mathcal{V}$, $\forall k \in \mathcal{D}$. This proves asymptotic modulus agreement and the theorem holds.

6 Conclusions

Agreement protocols for nonlinear multi-agent dynamics over cooperative or cooperative–antagonistic networks were investigated. A class of nonlinear control laws were introduced based on a relaxed convexity condition. The price was that each agent must get access to the orientation of a common coordinate system, similar to a compass. Each agent specified a local supporting hyperrectangle with the help of the compass, and then a strict tangent cone was determined based on which local control can be found. Under mild conditions on the nonlinear dynamics and the interaction graph, we proved that for cooperative networks, exponential state agreement is achieved if and only if the communication topology is uniformly jointly quasi-strongly connected. For cooperative–antagonistic networks, modulus agreement is achieved asymptomatically if the time-varying communication topology is uniformly jointly strongly connected. The results generalized the existing studies on agreement seeking of multi-agent systems. Future works include higher-order agent dynamics and other dynamics on antagonistic arcs.

References

[1] C. Altafini. Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 58(4):935–946, 2013.

[2] J.-P. Aubin. *Viability Theory*. Birkhauser Boton, Boston, 1991.

[3] D. Bauso, L. Giarre, and R. Pesenti. Non-linear protocols for optimal distributed consensus in networks of dynamic agents. *Systems & Control Letters*, 55(11):918–928, 2006.

[4] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *44th IEEE Conference on Decision and Control*, pages 2996–3000, 2005.

[5] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: convergence rates, measurement delays, and asynchronous events. *SIAM Journal on Control and Optimization*, 47(2):601–623, 2008.

[6] J. Cortés. Discontinuous dynamical systems. a tutorial on solutions, nonsmooth analysis, and stability. *IEEE Control Systems Magazine*, 28(3):36–73, 2008.

[7] J. M. Danskin. The theory of max-min, with applications. *SIAM Journal on Applied Mathematics*, 14(4):641–664, 1966.

[8] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Norwell, MA: Kluwer, 1988.

[9] C. Godsil and G. Royle. *Algebraic Graph Theory*. New York: Springer-Verlag, 2001.

[10] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[11] Y. Kuramoto. In: H. Arakai (ed.), International Symposium on Mathematical Problems in Theoretical Physics. *Lecture Notes in Physics, Springer, New York*, 39:420, 1975.

[12] Z. Lin. *Coupled dynamic systems: From structure towards stability and stabilizability*. Ph. D. dissertation, University of Toronto, Toronto, 2006.

[13] Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM Journal on Control and Optimization*, 46(1):288–307, 2007.

[14] I. Matei and J. S. Baras. The asymptotic consensus problem on convex metric spaces. *to appear in IEEE Transactions on Automatic Control*.

[15] Z. Meng, G. Shi, and K. H. Johansson. Multi-agent systems with compasses. http://arxiv.org/abs/1308.0813.

[16] L. Moreau. Stability of multi-agent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

[17] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[18] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[19] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[20] G. Shi, M. Johansson, and K. H. Johansson. How agreement and disagreement evolve over random dynamic networks. *IEEE Journal on Selected Areas in Communications*, 31(6):1061–1071, 2013.

[21] A. M. Stoneham, E. M. Gauger, K. Porfyrakis, S. C. Benjamin, and B. W. Lovett. A new type of radical-pair-based model for magnetoreception. *Biophysical Journal*, 102(5):961–968, 2012.

[22] J. N. Tsitsiklis, D. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31(9):803–812, 1986.

[23] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Schochet. Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.