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Brief paper

Convergence of max–min consensus algorithms[☆]

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ABSTRACT

In this paper, we propose a distributed max–min consensus algorithm for a discrete-time n-node system. Each node iteratively updates its state to a weighted average of its own state together with the minimum and maximum states of its neighbors. In order for carrying out this update, each node needs to know the positive direction of the state axis, as some additional information besides the relative states from the neighbors. Various necessary and/or sufficient conditions are established for the proposed max–min consensus algorithm under time-varying interaction graphs. These convergence conditions do not rely on the assumption on the positive lower bound of the arc weights.

1. Introduction

In the past decade, distributed consensus algorithms have been extensively studied in the literature, due to its wide applicability in engineering, computer science, and social science (DeGroot, 1974; Diekmann, Frommer, & Monien, 1999; Golub & Jackson, 2007; Jadbabaie, Lin, & Morse, 2003; Tsitsiklis, Bertsekas, & Athans, 1986). In many cases consensus algorithms seek to compute the average of the nodes' initials over the network (Jadbabaie et al., 2003; Tsitsiklis et al., 1986), and various efforts have been devoted to analyzing how the underlying communication graphs influence the convergence or the convergence rate for both continuous-time and discrete-time agent dynamics (Cao, Morse, & Anderson, 2008a,b; Morse, 2005; Nedic, Olshevsky, Ozdaglar, & Tsitsiklis, 2009; Olfati-Saber & Murray, 2004; Ren & Beard, 2005). Weighted average consensus algorithms, also draw attentions in which all nodes eventually reach an agreement as a weighted average of the initial values (Ren & Beard, 2005). Weighted average consensus is resulted from the missing of balance in the communication graph (Ren & Beard, 2005), and it has been shown that even a weighted agreement still leads to certain wisdom for the networks under quite general conditions (Golub & Jackson, 2007).

A great advantage in distributed consensus algorithms is that they do not rely on a centralized coordinate system. Each node can carry on the computation using only relative state information from its neighbors. A convenient way of modeling the switching node interactions is to assume that the communication graphs are defined by a sequence of time-dependent graphs over the node set. The connectivity of this sequence of graphs plays an important role for the network to reach consensus. Joint connectivity, i.e., connectivity of the union graph over time intervals, has been considered, and various convergence conditions have been established (Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005; Cao et al., 2008a,b; Jadbabaie et al., 2003; Morse, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Tsitsiklis et al., 1986).

On the other hand, it is however true that in most existing works, the convergence of consensus algorithms highly depends on some critical conditions on network information flow. Most asymptotic convergence results are based on the assumption that the arc weights always have a positive lower bound over time, and particularly it is commonly assumed that the underlying communication graphs always keep self-loops reflected as node self-confidence in the node state updates (Blondel et al., 2005; Cao et al., 2008a,b; Jadbabaie et al., 2003; Ren & Beard, 2005; Tsitsiklis et al., 1986).

In this paper, we propose a distributed max–min consensus algorithm for an n-node system. In the proposed algorithm, each node iteratively updates its state to a weighted average of its

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own state together with the minimum and maximum states of its neighbors. This dynamics provides a natural model for extreme-biased opinion evolution over social networks. In classical DeGroot's model (DeGroot, 1974), weights of exchanged opinions are put on different nodes during interactions, without identifying specific opinions. Variants to DeGroot's belief evolution taking into account biases in different opinions have been considered. Krause's model (Krause, 1997) introduced state-dependent interactions where nodes interact with neighbors within certain range of opinions and therefore put a zero weight to opinions outside this interaction range. Recent work (Dandekar, Goel, & Lee, 2013) proposes a biased social interaction model with greater interactions between like-minded individuals and shows that this biased model often leads to polarization of opinions. The proposed max–min consensus algorithm actually defines extreme-biased belief evolution in which nodes put weights only on the extreme (max and min) opinions in the neighborhood, right opposite to the homophily effects studied in Dandekar et al. (2013) and Krause (1997). We show that this extreme-biased dynamics leads to convergence to an agreement under more general conditions, compared to DeGroot type updates.

Compared to standard consensus algorithms, in the proposed algorithm each node needs to know the positive direction of the state axis, as some additional information besides the relative states from the neighbors. This piece of additional information is indeed centralized, but obviously it is not expensive in many practical applications. Various necessary and/or sufficient conditions are established for the proposed max–min consensus algorithm under time-dependent interaction graphs. These conditions are consistent with the infinite flow property and persistent connectivity conditions in the literature which are utilized to study consensus algorithms (Hendrickx & Tsitsiklis, 2013; Martin & Girard, 2013; Touri & Nedic, 2011, 2012). The derived convergence conditions for directed graphs do not rely on the condition on the positive lower bound of the arc weights, which usually show up for the study of standard consensus algorithms. In other words, this small amount of centralized information has brought nontrivial relaxation to the convergence requirements, which is consistent with the recent study on the role of centralized information in queueing systems (Tsitsiklis & Xu, 2011).

The rest of the paper is organized as follows. In Section 2 we introduce the considered network model and the proposed max–min consensus algorithm. Some impossibilities of finite-time or asymptotic consensus are established in Section 3. Then sufficient convergence conditions for asymptotic consensus are established for time-dependent graphs in Section 4.[2] Finally some concluding remarks are given in Section 5.

2. Problem definition

In this section, we introduce the network model, the considered algorithm, and define the problem of interest.

2.1. Network

We first recall some concepts and notation in graph theory (Godsil & Royle, 2001). A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An element $e = (i, j) \in \mathcal{E}$ is called an *arc* from node $i \in \mathcal{V}$ to $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ of nodes $v_i \in \mathcal{V}$ and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, k$, the sequence is called a (directed) *path* of *length* k. If there exists a path from node i to node j, then node j is said to be reachable from node i. Each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. A digraph $\mathcal{G}$ is said to be *strongly connected* if node i is reachable from j for any two nodes $i, j \in \mathcal{V}$; *quasi-strongly connected* if $\mathcal{G}$ has a center (Berge & Ghouila-Houri, 1965). The *distance* from i to j in a digraph $\mathcal{G}$, $d(i, j)$, is the length of a shortest simple path from i to j if j is reachable from i, and the *diameter* of $\mathcal{G}$ is $\mathrm{diam}(\mathcal{G}) = \max\{d(i, j) | i, j \in \mathcal{V},\ j \text{ is reachable from } i\}$. The union of two digraphs with the same node set $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$. A digraph $\mathcal{G}$ is said to be bidirectional if for every two nodes i and j, $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. A bidirectional graph $\mathcal{G}$ is said to be *connected* if there is a path between any two nodes. A bidirectional underlying graph of a directed graph $\mathcal{G}$ is obtained by replacing all directed edges of $\mathcal{G}$ with bidirectional edges.

Consider a network with node set $\mathcal{V} = \{1, 2, \dots, n\}$, $n \geq 3$. Time is slotted. Denote the state of node i at time $k \geq 0$ as $x_i(k) \in \mathbb{R}$. Then $x(k) = \big(x_1(k) \dots x_n(k)\big)^T$ represents the network state. The interactions among the nodes are determined by a given sequence of digraphs with node set $\mathcal{V}$, denoted as $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$, $k = 0, 1, \dots$.

Throughout this paper, we call node j a *neighbor* of node i if there is an arc from j to i in the graph. Each node is supposed to always be a neighbor of itself. Let $\mathcal{N}_i(k)$ represent the neighbor set of node i at time k.

2.2. Algorithm

In this paper, we propose the following max–min consensus algorithm for node i's update:

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + \big(1 - \eta_k - \alpha_k\big) \max_{j \in \mathcal{N}_i(k)} x_j(k), \tag{1}$$

where $\alpha_k, \eta_k \geq 0$ and $\alpha_k + \eta_k \leq 1$. We denote the set of all algorithms of the form (1) by $\mathcal{A}$, when the parameters (α_k, η_k) take values as $\eta_k \in [0, 1]$, $\alpha_k \in [0, 1 - \eta_k]$. We use $\mathcal{A}_1$ to denote the set of algorithms in the form of (1) with parameters $\eta_k \in (0, 1]$, $\alpha_k \in [0, 1 - \eta_k]$ and use $\mathcal{A}_2$ to denote the set of algorithms in the form of (1) with parameters $\eta_k = 0$ for $k \geq 0$ and $\alpha_k \in [0, 1]$.

Algorithm (1) provides a natural model for extreme-biased opinion dynamics in social networks, where the biased node only assigns weights to extreme opinions in its neighborhood.

2.3. Problem

Let $\big\{x(k; x^0) = \big(x_1(k; x^0) \dots x_n(k; x^0)\big)^T\big\}_{k_0}^{\infty}$ be the sequence generated by (1) for initial time k_0 and initial value $x^0 = x(k_0) = \big(x_1(k_0) \dots x_n(k_0)\big)^T \in \mathbb{R}^n$. We will identify $x(k; x^0)$ as $x(k)$ in the following discussions. We introduce the following definition on the convergence of the considered algorithm.

Definition 1. (i) Asymptotic consensus is achieved for Algorithm (1) for initial condition $x(k_0) = x^0 \in \mathbb{R}^n$ if there exists $z_*(x^0) \in \mathbb{R}$ such that $\lim_{k\to\infty} x_i(k) = z_*$, $i = 1, \dots, n$. Global asymptotic consensus is achieved if asymptotic consensus is achieved for all $k_0 \geq 0$ and $x^0 \in \mathbb{R}^n$.

(ii) Finite-time consensus is achieved for Algorithm (1) for initial condition $x(k_0) = x^0 \in \mathbb{R}^n$ if there exist $z_*(x^0) \in \mathbb{R}$ and an

[2] The proposed max–min consensus algorithms can also be studied for a type of state-dependent graphs as shown in Shi and Johansson (2013b).

integer $T_*(x^0) > 0$ such that $x_i(T_*) = z_*,\ i = 1, \ldots, n$. Global finite-time consensus is achieved if finite-time consensus is achieved for all $k_0 \geq 0$ and $x^0 \in \mathbb{R}^n$.

Algorithm (1) reaching consensus is equivalent to that $x(k)$ converging to a point on the manifold

$$C = \Big\{x = (x_1 \ldots x_n)^T : x_1 = \cdots = x_n\Big\}.$$

We call C the consensus manifold. Its dimension is one.

In the following, we focus on the impossibilities and possibilities of asymptotic or finite-time consensus.

3. Convergence impossibilities

In this section, we discuss the impossibilities of asymptotic or finite-time convergence for the consensus algorithms in $\mathcal{A}_1$. First we present the following conclusion.

Proposition 1. *For every consensus algorithm in $\mathcal{A}_1$, finite-time consensus fails for all initial values in $\mathbb{R}^n$ except for initial values on the consensus manifold.*

Proof. We define

$$h(k) = \min_{i\in\mathcal{V}} x_i(k); \qquad H(k) = \max_{i\in\mathcal{V}} x_i(k).$$

Introduce $\Phi(k) = H(k) - h(k)$. Then clearly asymptotic consensus is achieved if and only if $\lim_{k\to\infty} \Phi(k) = 0$.

Take a node i satisfying $x_i(k) = h(k)$. We have

$$\begin{aligned} x_i(k+1) &= \eta_k x_i(k) + \alpha_k \min_{j\in\mathcal{N}_i(k)} x_j(k) \\ &\quad + \big(1-\eta_k-\alpha_k\big) \max_{j\in\mathcal{N}_i(k)} x_j(k) \\ &\leq (\alpha_k+\eta_k)h(k) + (1-\eta_k-\alpha_k)H(k). \end{aligned} \tag{2}$$

Similarly, taking another node m satisfying $x_m(k) = H(k)$, we obtain

$$\begin{aligned} x_m(k+1) &= \eta_k x_m(k) + \alpha_k \min_{j\in\mathcal{N}_m(k)} x_j(k) \\ &\quad + \big(1-\eta_k-\alpha_k\big) \max_{j\in\mathcal{N}_m(k)} x_j(k) \\ &\geq \alpha_k h(k) + (1-\alpha_k)H(k). \end{aligned} \tag{3}$$

With (2) and (3), we conclude that

$$\begin{aligned} \Phi(k+1) &= \max_{i\in\mathcal{V}} x_i(k+1) - \min_{i\in\mathcal{V}} x_i(k+1) \\ &\geq x_m(k+1) - x_i(k+1) \geq \eta_k \Phi(k). \end{aligned} \tag{4}$$

Therefore, since (4) holds for all k, we immediately obtain that for every algorithm in the set $\mathcal{A}_1$,

$$\Phi(K) \geq \Phi(k_0) \prod_{k=k_0}^{K-1} \eta_k > 0 \tag{5}$$

for all $K \geq k_0$ as long as $\Phi(k_0) > 0$. Noticing that the initial values satisfying $\Phi(k_0) = 0$ are on the consensus manifold, the desired conclusion follows. □

Since the consensus manifold is a one-dimensional manifold in $\mathbb{R}^n$, Proposition 1 indicates that finite-time convergence is impossible for almost all initial values for algorithms in $\mathcal{A}_1$.

Next, we discuss the impossibility of asymptotic consensus. The following lemma is well-known.

Lemma 1 (Knopp, 1990). *Let $\{b_k\}_0^\infty$ be a sequence of real numbers with $b_k \in [0, 1)$ for all k. Then $\sum_{k=0}^\infty b_k = \infty$ if and only if $\prod_{k=0}^\infty (1-b_k) = 0$.*

The following result on asymptotic convergence holds.

Proposition 2. *For every consensus algorithm in $\mathcal{A}_1$, asymptotic consensus fails for all initial values in $\mathbb{R}^n$ except for initial values on the consensus manifold, if $\sum_{k=0}^\infty \big(1-\eta_k\big) < \infty$.*

Proof. In light of Lemma 1 and (5), we see that for every algorithm in the set $\mathcal{A}_1$,

$$\lim_{K\to\infty} \Phi(K) \geq \Phi(k_0) \prod_{k=k_0}^{\infty} \eta_k > 0$$

if $\sum_{k=0}^\infty \big(1-\eta_k\big) < \infty$ for all initial values satisfying $\Phi(k_0) > 0$. The desired conclusion thus follows. □

Proposition 2 indicates that $\sum_{k=0}^\infty \big(1-\eta_k\big) = \infty$ is a necessary condition for algorithms in $\mathcal{A}_1$ to reach asymptotic consensus. Note that η_k measures node self-confidence. Thus, the condition $\sum_{k=0}^\infty \big(1-\eta_k\big) = \infty$ characterizes the maximal self-confidence that nodes can hold and still reach consensus.

It is worth pointing out that Propositions 1 and 2 hold for any communication graph. In the following discussions, we turn to the possibilities for consensus. Then, however, the communication graph plays an important role.

4. Convergence to consensus

In this section, we establish a series of conditions on asymptotic convergence of the considered algorithm under switching directed communication graphs.

Denote the union graph of $\mathcal{G}_k$ over time interval $[k_1, k_2]$ as $\mathcal{G}\big([k_1, k_2]\big) = (\mathcal{V}, \cup_{k\in[k_1,k_2]} \mathcal{E}(k))$, where $0 \leq k_1 \leq k_2 \leq +\infty$. We introduce the following definitions on the joint connectivity of time-varying graphs.

Definition 2. (i) $\mathcal{G}_k$ is *uniformly jointly quasi-strongly connected* (strongly connected) if there exists an integer $B \geq 1$ such that $\mathcal{G}\big([k, k+B-1]\big)$ is quasi-strongly connected (strongly connected) for all $k \geq 0$.

(ii) $\mathcal{G}_k$ is *infinitely jointly strongly connected* if $\mathcal{G}\big([k, \infty)\big)$ is strongly connected for all $k \geq 0$.

(iii) Suppose $\mathcal{G}_k$ is bidirectional for all k. Then $\mathcal{G}_k$ is *infinitely jointly connected* if $\mathcal{G}\big([k, \infty)\big)$ is connected for all $k \geq 0$.

The following conclusion holds for uniformly jointly quasi-strongly connected graphs.

Theorem 1. *Suppose $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if either*

$$\sum_{s=0}^{\infty}\Bigg[\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1} \alpha_k\Bigg] = \infty \tag{6}$$

or

$$\sum_{s=0}^{\infty}\Bigg[\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1} \big(1-\alpha_k-\eta_k\big)\Bigg] = \infty. \tag{7}$$

Theorem 1 hence states that divergence of certain products of the algorithm parameters guarantees global asymptotic consensus.

The following corollary follows from Theorem 1 straightforwardly.

Corollary 1. *Suppose $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected.*

(i) *Assume that $\alpha_k \geq \alpha_{k+1}$ for all k. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty} \alpha_k^{(n-1)^2 B} = \infty$.*
(ii) *Assume that $\alpha_k + \eta_k \leq \alpha_{k+1} + \eta_{k+1}$ for all k. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty} \big(1 - \alpha_k - \eta_k\big)^{(n-1)^2 B} = \infty$.*

For uniformly jointly strongly connected graphs, it turns out that consensus can be achieved under weaker conditions on (α_k, η_k).

Theorem 2. *Suppose $\mathcal{G}_k$ is uniformly jointly strongly connected. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if either*

$$\sum_{s=0}^{\infty} \Bigg[\prod_{k=s(n-1)B}^{(s+1)(n-1)B-1} \alpha_k \Bigg] = \infty \tag{8}$$

or

$$\sum_{s=0}^{\infty} \Bigg[\prod_{k=s(n-1)B}^{(s+1)(n-1)B-1} \big(1 - \alpha_k - \eta_k\big) \Bigg] = \infty. \tag{9}$$

Similarly, Theorem 2 leads to the following corollary.

Corollary 2. *Suppose $\mathcal{G}_k$ is uniformly jointly strongly connected.*

(i) *Assume that $\alpha_k \geq \alpha_{k+1}$ for all k. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty} \alpha_k^{(n-1)B} = \infty$.*
(ii) *Assume that $\alpha_k + \eta_k \leq \alpha_{k+1} + \eta_{k+1}$ for all k. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty} \big(1 - \alpha_k - \eta_k\big)^{(n-1)B} = \infty$.*

These results are consistent with the notion of *infinite flow* which has been proposed in Touri and Nedic (2011, 2012) to study the consensus and ergodicity of a chain of stochastic matrices as well as the concept of *persistent graph* proposed in Shi and Johansson (2013c).

We continue to show that consensus can still be reached as long as the α_k are varying sufficiently *slow* if $\eta_k = 0$ for all k. In this case, asymptotic convergence of Algorithm (1) is guaranteed under general conditions no longer relying on the merits of the information flow property. We note that the assumption that $\eta_k = 0$ for all k is adopted for the ease of presentation, and it is clear from its proof that more general conditions with η_k involved can be easily derived using similar argument.

Theorem 3. *Suppose $\eta_k = 0$ for all k.*

(i) *Suppose $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected with respect to $B \geq 1$. Algorithms in the set $\mathcal{A}_2$ achieve global asymptotic consensus if there exists $0 < \delta < 1$ such that $\big|\alpha_{k+1} - \alpha_k\big| \leq \frac{\delta}{(n-1)^2 B}$, $k \geq 0$.*
(ii) *Suppose $\mathcal{G}_k$ is uniformly jointly strongly connected with respect to $B \geq 1$. Algorithms in the set $\mathcal{A}_2$ achieve global asymptotic consensus if there exists $0 < \delta < 1$ such that $\big|\alpha_{k+1} - \alpha_k\big| \leq \frac{\delta}{(n-1)B}$, $k \geq 0$.*

For bidirectional graphs, the conditions are much simpler to state. We present the following result which in contrast to Theorems 1 and 2 imposes a lower bound on the parameter in Algorithm (1).

Theorem 4. *Suppose $\mathcal{G}_k$ is bidirectional for all k and $\mathcal{G}_k$ is infinitely jointly connected. Algorithms in the set $\mathcal{A}_1$ achieve global asymptotic consensus if there exists a constant $\alpha_* \in (0, 1)$ such that either $\alpha_k \geq \alpha_*$ or $1 - \alpha_k - \eta_k \geq \alpha_*$ for all k.*

4.1. Proof of Theorem 1

We continue to use the following notation:

$$h(k) = \min_{i \in \mathcal{V}} x_i(k), \qquad H(k) = \max_{i \in \mathcal{V}} x_i(k),$$

and $\Phi(k) = H(k) - h(k)$. Following any solution of (1), it is obvious to see that $h(k)$ is non-decreasing and $H(k)$ is non-increasing.

Note that if (6) guarantees asymptotic consensus of algorithm (1), replacing the node state $x_i(k)$ with $-x_i(k)$ leads to that (7) guarantees asymptotic consensus of algorithm (1) for $-x_i(k)$, $i = 1, \dots, n$. Since consensus for $x_i(k)$, $i = 1, \dots, n$ is equivalent to consensus for $-x_i(k)$, $i = 1, \dots, n$, (6) and (7) are equivalent in terms of consensus convergence. Thus, we just need to show that (6) is a sufficient condition for asymptotic consensus.

Take $k_* \geq 0$ as any moment in the iterative algorithm. Take $(n-1)^2$ intervals $[k_*, k_*+B-1], [k_*+B, k_*+2B-1], \dots, [k_*+(n^2-2n)B, k_*+(n-1)^2B-1]$. Since $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected, the union graph on each of these intervals has at least one center node. Consequently, there must be a node $v_0 \in \mathcal{V}$ and $n-1$ integers $0 \leq b_1 < b_2 < \cdots < b_{n-1} \leq n^2 - 2n$ such that v_0 is a center of $\mathcal{G}\big([k_* + b_i B, k_* + (b_i + 1)B - 1]\big)$, $i = 1, \dots, n-1$. Assume that $x_{v_0}(k_*) \leq \frac{1}{2}h(k_*) + \frac{1}{2}H(k_*)$.

We first bound $x_{v_0}(k)$ for $k \in [k_*, k_* + (n-1)^2 B]$. It is not hard to see that

$$\begin{aligned} x_{v_0}(k_* + 1) &= \eta_{k_*} x_{v_0}(k_*) + \alpha_{k_*} \min_{j \in \mathcal{N}_{v_0}(k_*)} x_j(k_*) \\ &\quad + \big(1 - \alpha_{k_*} - \eta_{k_*}\big) \max_{j \in \mathcal{N}_{v_0}(k_*)} x_j(k_*) \\ &\leq \big(\alpha_{k_*} + \eta_{k_*}\big)\Big(\frac{1}{2}h(k_*) + \frac{1}{2}H(k_*)\Big) \\ &\quad + \big(1 - \alpha_{k_*} - \eta_{k_*}\big)H(k_*) \\ &\leq \alpha_{k_*}\Big(\frac{1}{2}h(k_*) + \frac{1}{2}H(k_*)\Big) + \big(1 - \alpha_{k_*}\big)H(k_*) \\ &= \frac{\alpha_{k_*}}{2}h(k_*) + \Big(1 - \frac{\alpha_{k_*}}{2}\Big)H(k_*). \end{aligned}$$

Proceeding, we obtain

$$x_{v_0}(k_* + m) \leq \frac{\prod_{k=k_*}^{k_*+m-1} \alpha_k}{2} h(k_*) + \left(1 - \frac{\prod_{k=k_*}^{k_*+m-1} \alpha_k}{2}\right) H(k_*), \quad m = 0, 1, \dots. \tag{10}$$

Since v_0 is a center of $\mathcal{G}\big([k_* + b_1 B, k_* + (b_1 + 1)B - 1]\big)$, there exists another node v_1 such that v_0 is a neighbor of v_1 for some $k_1 \in [k_* + b_1 B, k_* + (b_1 + 1)B - 1]$. As a result, based on (10), we have

$$\begin{aligned} &x_{v_1}(k_1 + 1) \\ &= \eta_{k_1} x_{v_1}(k_1) + \alpha_{k_1} \min_{j \in \mathcal{N}_{v_1}(k_1)} x_j(k_1) \\ &+ \big(1 - \alpha_{k_1} - \eta_{k_1}\big) \max_{j \in \mathcal{N}_{v_1}(k_1)} x_j(k_1) \\ &\leq \alpha_{k_1} x_{v_0}(k_1) + \big(1 - \alpha_{k_1}\big)H(k_1) \end{aligned}$$

$$\leq \alpha_{k_1}\left(\frac{\prod_{k=k_*}^{k_1-1}\alpha_k}{2}h(k_*)+\left(1-\frac{\prod_{k=k_*}^{k_1-1}\alpha_k}{2}\right)H(k_*)\right)+\left(1-\alpha_{k_1}\right)H(k_*)$$

$$=\frac{\prod_{k=k_*}^{k_1}\alpha_k}{2}h(k_*)+\left(1-\frac{\prod_{k=k_*}^{k_1}\alpha_k}{2}\right)H(k_*).$$

Proceeding, we have

$$x_{v_1}(k_*+m)\leq\frac{\prod_{k=k_*}^{k_*+m-1}\alpha_k}{2}h(k_*)+\left(1-\frac{\prod_{k=k_*}^{k_*+m-1}\alpha_k}{2}\right)H(k_*),$$

for $m=(b_1+1)B,\ldots$.

Continuing the analysis on time intervals $[k_*+b_iB, k_*+(b_i+1)B-1]$ for $i=2,\ldots,n-1$ and nodes $v_2,v_3,\ldots,v_{n-1}$, similar upper bounds for each node can be obtained:

$$x_{v_i}(k_*+m)\leq\frac{\prod_{k=k_*}^{k_*+m-1}\alpha_k}{2}h(k_*)+\left(1-\frac{\prod_{k=k_*}^{k_*+m-1}\alpha_k}{2}\right)H(k_*),$$

for $m=(b_i+1)B,\ldots$. This immediately leads to

$$x_{v_i}(k_*+(n-1)^2B)\leq\frac{\prod_{k=k_*}^{k_*+(n-1)^2B-1}\alpha_k}{2}h(k_*)+\left(1-\frac{\prod_{k=k_*}^{k_*+(n-1)^2B-1}\alpha_k}{2}\right)H(k_*),$$

for $i=0,1,\ldots,n-1$, which implies

$$H\big(k_*+(n-1)^2B\big)\leq\frac{\prod_{k=k_*}^{k_*+(n-1)^2B-1}\alpha_k}{2}h(k_*)+\left(1-\frac{\prod_{k=k_*}^{k_*+(n-1)^2B-1}\alpha_k}{2}\right)H(k_*).$$

Thus, we further conclude

$$\Phi\big(k_*+(n-1)^2B\big)\leq\left(1-\frac{\prod_{k=k_*}^{k_*+(n-1)^2B-1}\alpha_k}{2}\right)\Phi(k_*).\tag{11}$$

From a symmetric analysis by bounding $x(k_*+(n-1)^2B)$ from below, we know that (11) also holds for the other condition with $x_{v_0}(k_*)\geq\frac{1}{2}h(k_*)+\frac{1}{2}H(k_*)$. Therefore, since k_* is selected arbitrarily, we can assume the initial time is $k_0=0$, without loss of generality, and then conclude that

$$\Phi\big(K(n-1)^2B\big)\leq\Phi(0)\prod_{s=0}^{K-1}\Bigg(1-\frac{1}{2}\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1}\alpha_k\Bigg).$$

The desired conclusion follows immediately from Lemma 1.

4.2. Proof of Theorem 2

Notice that in a strongly connected graph, every node is a center node. Therefore, when $\mathcal{G}_k$ is uniformly jointly strongly connected, taking $k_*\geq 0$ as any moment in the iteration and $n-1$ intervals $[k_*,k_*+B-1],[k_*+B,k_*+2B-1],\ldots,[k_*+(n-2)B,k_*+(n-1)B-1]$, any node $i\in\mathcal{V}$ is a center node for the union graph over each of these intervals. As a result, the desired conclusion follows repeating the analysis used in the proof of Theorem 1.

4.3. Proof of Theorem 3

We only present the detailed proof of (i), and (ii) can be obtained using a similar argument to that in the proof of Theorem 2.

We introduce

$$\underline{\alpha}=\liminf_{k\to\infty}\alpha_k.$$

Note that their existence is guaranteed by the boundedness of $\{\alpha_k\}_{k=0}^{\infty}$.

First of all we assume that $\underline{\alpha}>0$. Pick a constant $\epsilon>0$ such that $\underline{\alpha}-\epsilon>0$. By the definition of $\underline{\alpha}$, there exists an integer $k_*\geq 0$ such that for all $k\geq k_*$, $\alpha_k\geq\underline{\alpha}-\epsilon>0$. Applying the recursive analysis that we used to derive (11) in the proof of Theorem 1, we similarly obtain

$$\Phi\big(k_*+(n-1)^2B\big)\leq\Phi\big(k_*\big)\bigg(1-\big(\underline{\alpha}-\epsilon\big)^{(n-1)^2B}/2\bigg),$$

from which we can show that $\lim_{k\to\infty}\Phi\big(k\big)=0$ and the desired conclusion follows.

Suppose that $\underline{\alpha}=0$. By the definition of $\underline{\alpha}$, there exists an infinite sequence $k_1<k_2<\cdots$ with $k_{m+1}-k_m>(n-1)^2B$ such that

$$\alpha_{k_m}\leq\frac{\delta}{(n-1)^2B},$$

where $0<\delta<1$. Fix an $m\geq 1$ and consider α_{k_m}. Since

$$\big|\alpha_{k+1}-\alpha_k\big|\leq\frac{\delta}{(n-1)^2B},\quad k\geq 0,$$

it holds

$$\alpha_{k_m+s}\leq\delta<1,$$

for all $s=0,1,\ldots,(n-1)^2B-1$, which implies that

$$1-\alpha_{k_m+s}\geq 1-\delta>0,$$

for all $s=0,1,\ldots,(n-1)^2B-1$. Consider the evolution of $y(k)=(y_1(k)\ldots y_n(k))^T$ with $y_i(k)=-x_i(k)$, $i=1,\ldots,n$. Note that

$$\Phi(k)=\max_{i\in\mathcal{V}}x_i(k)-\min_{i\in\mathcal{V}}x_i(k)=\max_{i\in\mathcal{V}}y_i(k)-\min_{i\in\mathcal{V}}y_i(k).$$

Applying the recursive analysis to the y-system that we used to derive (11) in the proof of Theorem 1, one has

$$\Phi\big(k_m + (n-1)^2 B\big) \le \Phi\big(k_m\big)\bigg(1 - \big(1-\delta\big)^{(n-1)^2 B}/2\bigg),$$

which in turn leads to

$$\Phi\big(k_{m+1}\big) \le \Phi\big(k_m\big)\bigg(1 - \big(1-\delta\big)^{(n-1)^2 B}/2\bigg),$$

noticing that $\Phi(k)$ is non-increasing. This proves $\lim_{k\to\infty}\Phi\big(k\big) = 0$ and therefore consensus is achieved. This completes the proof.

4.4. Proof of Theorem 4

Similar to the proof of Theorem 1, we only need to show that the existence of a constant $\alpha_* \in (0, 1)$ such that $\alpha_k \ge \alpha_*$ is sufficient for asymptotic consensus.

Take $k_1^* \ge 0$ as an arbitrary moment in the iterative algorithm. Take a node $u_0 \in \mathcal{V}$ satisfying $x_{u_0}(k_1^*) = h(k_1^*)$. We define

$$k_1 = \inf\big\{k \ge k_1^* : \text{ there exists another node} \\ \text{connecting } u_0 \text{ at time } k\big\}$$

and then

$$\mathcal{V}_1 = \big\{i \in \mathcal{V} : i \text{ is connected to } u_0 \text{ at time } k_1\big\}.$$

Thus, we have

$$x_{u_0}(k_1+1) \le \alpha_* h(k_1^*) + \big(1-\alpha_*\big)H(k_1^*)$$

and

$$x_i(k_1+1) \le \alpha_* h(k_1^*) + \big(1-\alpha_*\big)H(k_1^*)$$

for all $i \in \mathcal{V}_1$.

Note that if nodes in $\{u_0\} \cup \mathcal{V}_1$ are not connected with other nodes in $\mathcal{V}\backslash(\{u_0\}\cup\mathcal{V}_1)$ during $[k_1+1, k_1+s]$, $s \ge 1$, we have that for all $i \in \{u_0\}\cup\mathcal{V}_1$ and $m = 1, \dots, s+1$,

$$x_i(k_1+m) \le \alpha_* h(k_1^*) + \big(1-\alpha_*\big)H(k_1^*).$$

Continuing the estimate, $k_2, \dots, k_d$ and $\mathcal{V}_2, \dots, \mathcal{V}_d$ can be defined correspondingly until $\mathcal{V} = \{u_0\} \cup (\cup_{i=1}^d \mathcal{V}_i)$. Eventually we have

$$x_i(k_d+1) \le \alpha_*^d h(k_1^*) + \big(1-\alpha_*^d\big)H(k_1^*), \quad i = 1, \dots, n,$$

which implies

$$H(k_d+1) \le \alpha_*^d h(k_1^*) + \big(1-\alpha_*^d\big)H(k_1^*). \tag{12}$$

We denote $k_2^* = k_d + 1$. Because it holds that $d \le n-1$, we see from (12) that

$$\Phi(k_2^*) \le \big(1-\alpha_*^{n-1}\big)\Phi(k_1^*).$$

Since $\mathcal{G}_k$ is infinitely jointly connected, this process can be carried on for an infinite sequence $k_1^* < k_2^* < \cdots$. Thus, asymptotic consensus is achieved for all initial conditions. This completes the proof.

5. Conclusions

We have proposed and analyzed a max–min consensus algorithm for an n-node network. In the considered algorithm, each node iteratively updated its state to a weighted average of its own state together with the minimum and maximum states of its neighbors. Some necessary and/or sufficient conditions have been established for the proposed max–min consensus algorithm under time-dependent graphs. These convergence conditions do not rely on the assumption on the positive lower bound of the arc weights. The relaxation was exactly gained from the knowledge of the positive direction of the network state axis.

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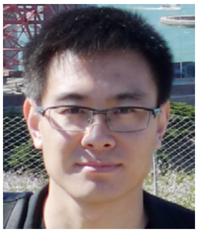

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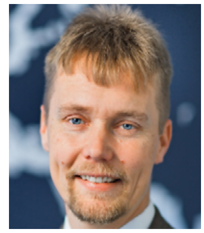

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