Extremum seeking on submanifolds in the Euclidean space

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Abstract

Extremum seeking is a powerful control method to steer a dynamical system to an extremum of a partially unknown function. In this paper, we introduce extremum seeking systems on submanifolds in the Euclidean space. Using a trajectory approximation technique based on Lie brackets, we prove that uniform asymptotic stability of the so-called Lie bracket system on the manifold implies practical uniform asymptotic stability of the corresponding extremum seeking system on the manifold. We illustrate the approach with an example of extremum seeking on a torus.

1. Introduction

Extremum seeking control is used to steer the states of a dynamical system to an extremum of a function which depends on the system states. A characteristic feature of extremum seeking control is that neither a model of the plant nor the function which has to be optimized need not to be known. Only the values of the function must be available (measurable) at the actual system state. The idea of extremum seeking control is to inject periodic signals into the control loop in order to estimate an optimizing direction for the unknown function (Krstić & Ariyur, 2003; Tan, Moase, Manzie, Nesic, & Mareels, 2010). These properties render extremum seeking attractive for many application areas. For example, it has been applied in brake system control (Drakunov, Ozguner, Dix, & Ashrafi, 1995), in real-time optimization of bioprocesses (Guay, Dochain, & Perrier, 2004), in flow control (Becker, King, Petz, & Nitsche, 2007; King et al., 2006) and in multi-agent systems (Binetti, Ariyur, Krstić, & Bernelli, 2003; Stanković, Johansson, & Stipanović, 2012) to mention only a few. Classical extremum seeking can be interpreted as a method for unconstrained optimization problems. In recent years, several extensions have been established for constrained optimization problems. One can distinguish between three main approaches, the Lagrangian approach, the barrier function approach and the approach for optimization on manifolds. The former two were addressed in Coito, Lemos, and Alves (2005) and DeHaan and Guay (2005) in the context of extremum seeking. Furthermore, in e.g. Poveda and Quijano (2012), a special class of manifolds, defined using linear constraints, were considered and an extremum seeking system was constructed that leaves this subspace invariant. In this paper, we construct an extremum seeking system that solves optimization problems on submanifolds in the Euclidean space $\mathbb{R}^n$ equipped with the standard scalar product. These are optimization problems with constraints that describe a smooth submanifold. They have many interesting applications in areas such as eigenvalue computation, principle component analysis or consensus problems, see Absil, Mahony, and Sepulchre (2008) and Helmke and Moore (1994) for more details.

The contribution of this paper is threefold. First, we introduce an extremum seeking feedback that steers a dynamical system to the solution of an optimization problem with constraints given by a submanifold in the Euclidean space. Hereby, we do not require the knowledge of the gradient of the objective function. The class of submanifolds considered in this paper contains many important manifolds such as spheres, tori, matrix Lie groups and the isospectral manifold (see e.g. Absil et al., 2008 and Helmke & Moore, 1994) which appear in many applications. We formulate the dynamics of the extremum seeking feedback using the coordinates of the ambient space, which has the advantage that no local coordinate charts

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of the manifold are required and therefore singularities of these charts are avoided.

Second, we prove non-local stability properties of the extremum seeking feedback and calculate a system whose trajectories approximate those of the extremum seeking system. In particular, we generalize the Lie bracket approximation method for extremum seeking systems introduced in Dürre, Stankovic, Ebenbauer, and Johansson (2013). We show in this paper, that the results in Dürre et al. (2013) also hold for extremum seeking systems on manifolds.

Third, we show a novel example for extremum seeking on manifolds. The example is motivated by synchronization problems of oscillators based only on distance measurements. This leads to extremum seeking on the torus. In particular, we derive an extremum seeking feedback for two oscillators that yield synchronization using only distance measurements. We show that there is a close relationship between the extremum seeking system and the well-known Kuramoto model.

The remainder of this paper is structured as follows. In Section 2, we introduce the necessary mathematical preliminaries. In Section 3, we present the main idea and the stability results for extremum seeking systems on submanifolds in the Euclidian space. In Section 4, we present the numerical example. In Section 5, we summarize the results and give an outlook on future research topics.

2. Preliminaries

We use the following notation. \( \mathbb{N} \) denotes the set of natural numbers and \( \mathbb{Q} \) the (positive) rational numbers. We use \( s \in \mathbb{C} \) for the complex variable of the Laplace transformation. The Euclidian space, considered in this paper, is \( \mathbb{R}^n \) equipped with the standard scalar product (standard metric) \( \langle x_1, x_2 \rangle := x_1^t x_2 \), we denote it by \( \mathbb{R}^n \), \( 0 \) is the standard basis vectors in \( \mathbb{R}^n \). We identify the tangent space \( T_p \mathbb{R}^n \) of \( \mathbb{R}^n \) at \( x \) with \( \mathbb{R}^n \) and, with a slight abuse of notation, we also denote the constant vector field \( 1 \) by \( e_1 \). The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is denoted by \( \|x\| := \sqrt{x^t x} \). Let \( M \subseteq \mathbb{R}^n \) be a smooth submanifold. The tangent space of \( M \) at \( x \) is denoted by \( T_x M \). The \( n \)-neighborhood of a set \( A \subseteq \mathbb{R}^n \) is denoted by \( U_n(A) := \{ x \in \mathbb{R}^n : \| x - y \| < n \} \). We denote by \( C^0 \) the set of functions that are continuously differentiable. The Lie bracket \( [\cdot, \cdot] : T_p \mathbb{R}^n \times T_p \mathbb{R}^n \to T_p \mathbb{R}^n \) between two vector fields \( g_1, g_2 \in C^1 : M \to \mathbb{R}^n \) is defined as: \( [g_1, g_2](x) := \frac{d}{dt} g_1(x) - g_1(x) \frac{d}{dt} g_2(x) \). The vector field \( f(x) \) is again a vector field on \( M \) (see Corollary 8.28 in Lee, 2003). Let \( U \subseteq \mathbb{R}^n \) be open, \( M \subseteq \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \). Let \( f \) be \( f|_M : M \to \mathbb{R}^n \) the restriction of \( f \) to \( M \). The gradient vector field of \( f \) on the Riemannian manifold \( (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \) is denoted by \( \nabla f = [\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_n}{\partial x_n}]^t \) and the gradient vector field of \( f \) on the manifold \( M \) is denoted by \( \nabla f \big|_M \).

An optimization problem on a manifold is an optimization problem of the form

\[
\min f(x) \quad \text{s.t.} \quad x \in M
\]

where \( M \) is a manifold and \( f : M \to \mathbb{R} \) a continuously differentiable function. A necessary condition for a local minimum of (1) is that the gradient must necessarily vanish at that point, i.e., if \( x^* \) is a local minimum of (1) then \( \nabla f \big|_{M}(x^*) = 0 \) (see e.g., p. 284 in Bertsekas, 1995). Typically extremum seeking tries to solve (1) with \( M = \mathbb{R}^n \) by seeking points where the gradient vanishes. In this work, we consider the case where \( M \) is a submanifold in \( \mathbb{R}^n \).

In order to characterize the gradient on \( M \) in terms of \( \nabla f \), we need the following lemma (see e.g. p. 48 in Absil et al., 2008):

**Lemma 1.** Let \( U \subseteq \mathbb{R}^n \) be open, \( M \subseteq U \) and \( \nabla f \) be the gradient vector field of \( f : U \to \mathbb{R}^n \), defined by the standard scalar product \( \langle \cdot, \cdot \rangle \), in \( \mathbb{R}^n \). Then the induced gradient vector field \( \nabla f \big|_M : M \to T_xM \) is given by

\[
\nabla f \big|_M(x) = P(\nabla f(x)),
\]

where \( P(y) \) denotes the orthogonal projection of \( y \in \mathbb{R}^n \) onto \( T_xM \).

Next, we review some results from Dürre et al. (2013). As it is shown in Dürre et al. (2013), certain extremum seeking systems can be written as input-affine systems

\[
\dot{x} = b_0(x) + \sum_{i=1}^{m} b_i(x) \sqrt{c_i(x)} \omega_i(t)
\]

with \( x(t) = x_0 \in \mathbb{R}^n, \omega > 0 \) and \( m \in \mathbb{N} \cup \{0\} \) the number of vector fields.

We impose the following assumptions on \( b_i \) and \( u_i \):

A1 \( b_i \in C^2 : \mathbb{R}^n \to \mathbb{R}^n, i = 0, \ldots, m; \)

A2 \( u_i \in C^0 : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, m \) and for every \( i = 1, \ldots, m \) there exist constants \( M_i > 0 \) such that \( \sup_{u_0(\theta)} |u_i(\theta)| \leq M_i; \)

A3 \( u_i(\cdot) \) is \( T \)-periodic, i.e., \( u_i(\theta + T) = u_i(\theta) \) and has zero average, i.e., \( \frac{1}{T} \int_0^T u_i(\theta) d\theta = 0 \), with \( T > 0 \) for all \( \theta \in \mathbb{R}, i = 1, \ldots, m \).

One main result in Dürre et al. (2013) is to approximate the trajectories of (3) by the trajectories of a so-called Lie bracket system

\[
\dot{z} = b_0(z) + \sum_{j=1}^{m} b_j(z) |y_j|
\]

where \( y_j = \frac{1}{T} \int_0^T u_j(\theta) \frac{d}{dt} u_j(\theta) d\theta \). Furthermore, we introduce a set \( B \) of initial conditions for (4) which have bounded solutions, i.e., there exists an \( A > 0 \) such that

\[
\|z(0) - B \Rightarrow z(t) \in U_{B_0}, \quad t \geq 0.
\]

Theorem 2. Let Assumptions A1–A3 be satisfied. Then for every bounded set \( K \subseteq B \) in \( (5) \), for every \( B > 0 \) and for every \( T_0 \) there exists an \( \alpha > 0 \) such that for every \( \omega > \alpha \), for every \( t_0 \in K \) and every \( x_0 \in X \) there exist solutions \( x, z : \mathbb{R} \to \mathbb{R}^n \) of (3) and (4) through \( x(t_0) = z(t_0) = x_0 \) which satisfy \( \|x(t) - z(t)\| < \delta, t_0 \leq t \leq t_0 + T_0 \).

For systems like (3) we need a notion of stability which is closely related to Lyapunov stability and applies to systems depending on a parameter.

**Definition 1.** A compact set \( E \subseteq M \) is said to be practically uniformly stable for (3) if for every \( \epsilon > 0 \) there exist a \( \delta > 0 \) and an \( \alpha > 0 \) such that for all \( t_0 \in \mathbb{R} \) and for all \( \omega > \alpha \) \( \omega x(t_0) \in U_{\delta} \Rightarrow x(t) \in U_{\epsilon}, t \geq t_0 \).

**Definition 2.** A compact set \( E \subseteq M \) is said to be practically uniformly attractive for (3) if there exists a \( \delta > 0 \) such that for every \( \epsilon > 0 \) there exist a \( t_0 \in \mathbb{R} \) and an \( \alpha > 0 \) such that for all \( t_0 \in \mathbb{R} \) and all \( \omega > \alpha \) \( \omega x(t_0) \in U_{\delta} \Rightarrow x(t) \in U_{\epsilon}, t \geq t_0 + t_f \).

**Definition 3.** A compact set \( E \subseteq M \) is said to be practically asymptotically stable for (3) if it is practically uniformly stable and it is practically uniformly attractive.

Using Lemma 2 it is shown in Dürre et al. (2013) that the stability properties of systems (3) and (4) are linked. This is captured in the next lemma, where practical stability is shown for (3).
Lemma 3. Let Assumptions A1–A3 be satisfied and suppose that a compact set \( E \subseteq M \) is asymptotically stable for (4). Then \( E \subseteq M \) is practically uniformly asymptotically stable for (3).

For systems like (4) which are independent of “\( \omega \)” and “\( t \)” we drop the terms “practically” and “uniformly” in the definitions above.

3. Main results

This section consists of two parts. First, we recall extremum seeking in \( \mathbb{R}^n \). Second, we propose extremum seeking systems on submanifolds in the Euclidean space.

3.1. Extremum seeking in \( \mathbb{R}^n \)

Consider an extremum seeking system as e.g. in Dürr et al. (2013) and Krstić and Ariyur (2003), given by the differential equations

\[
x_1 = cf(x)\sqrt{\omega_1} \cos(\omega_1 t) + \alpha \sqrt{\omega_1} \sin(\omega_1 t)
\]

\[
h_i = cf(x)\sqrt{\omega_i} \cos(\omega_i t) + \alpha \sqrt{\omega_i} \sin(\omega_i t),
\]

where \( f \in C^2 : \mathbb{R}^n \rightarrow \mathbb{R}, x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) and \( \alpha, \gamma > 0 \) and \( \omega_i > 0, i \neq j, i, j \in \{1, \ldots, n\} \). Identifying \( u_i^1(\omega_i t) = \sqrt{\omega_i} \cos(\omega_i t) \) and \( u_i^2(\omega_i t) = \sqrt{\omega_i} \sin(\omega_i t) \) as inputs and calculating the Lie bracket system associated to (6), we obtain

\[
\dot{z} = -\frac{\alpha \gamma}{2} \nabla f(z).
\]

We see that the Lie bracket system (7) reveals the optimizing behavior of the extremum seeking system (6). With Lemma 2 the trajectories of the Lie bracket system (7) can be uniformly approximated by trajectories of the extremum seeking system (8) for sufficiently large \( \omega \) and for pairwise distinct \( \omega_i, \omega_j \). Moreover, due to Lemma 3, stability properties of (7) turn into practical stability properties of (8), see Dürr et al. (2013) for details.

3.2. Extremum seeking on submanifolds in \( \mathbb{R}^n \)

The main idea is based on the following observations. First, note that (6) can be more compactly written as follows:

\[
\dot{x} = \sum_{i=1}^{n} cf(x)e_i\sqrt{\omega_i} \cos(\omega_i t) + \alpha e_i \sqrt{\omega_i} \sin(\omega_i t).
\]

Using the identity

\[
[f(z)e_i, e_j] = -\langle \nabla f(z), e_i \rangle e_j,
\]

we see that the Lie bracket system associated to (8) can be written as

\[
\dot{z} = \frac{\alpha \gamma}{2} \sum_{i=1}^{n} [f(z)e_i, e_i] = -\frac{\alpha \gamma}{2} \nabla f(z).
\]

Second, we recall that the gradient vector field \( \nabla f|_M \) of a submanifold \( M \) in \( \mathbb{R}^n \), which is induced by the standard scalar product in \( \mathbb{R}^n \), is the orthogonal projection of \( \nabla f \) onto the tangent space of \( M \) (see Lemma 1). Note that (9) also holds if we replace \( e_i \) by (tangent) vector fields \( g_i = g_i(x) \) on the submanifold \( M \), i.e.,

\[
[f(z)g_i(z), g_j(z)] = -\langle \nabla f(z), g_i(z) \rangle g_j(z).
\]

The idea of this paper is now as follows. Note that (11) can be interpreted as a projection of \( \nabla f \) onto the vector field \( g_i \). If the vector fields \( g_i \) form an orthonormal basis of the tangent space \( T_x M \) of \( M \), we can see that

\[
\nabla f|_M (z) = \sum_{i=1}^{n} (\nabla f(z), g_i(z))g_i(z).
\]

Consequently using (9) and by replacing the \( e_i \)'s in (8) and (10) by \( g_i \)'s, the Lie bracket system (10) turns into a gradient flow on the manifold \( M \). Based on this idea we propose an approach for extremum seeking on submanifolds in the Euclidean space. Consider the following assumption:

Assumption 1. (1) \( M \subseteq \mathbb{R}^n \) is a smooth, \( m \)-dimensional Riemannian submanifold without boundary. The metric \( \langle \cdot, \cdot \rangle_M : T_x M \times T_x M \rightarrow \mathbb{R} \) on \( M \) is the metric \( \langle \cdot, \cdot \rangle \) induced by the inner product of the ambient space \( \mathbb{R}^n \), i.e., \( \langle x_1, x_2 \rangle_M := \langle x_1, x_2 \rangle \) with \( x_1, x_2 \in T_x M \);

(2) there are \( p \geq m \) vector fields \( g_i \in C^2 : \mathbb{R}^n \rightarrow T_x M, i = 1, \ldots, p \) on \( M \) such that \( \text{span} [g_1(x), \ldots, g_p(x)] = T_x M \) for all \( x \in M \),

\[
\text{i.e., for each point } x \text{ on } M \text{, the tangent vectors } g_i(x), i = 1, \ldots, m \text{, span the tangent space } T_x M \text{ and for } p = m \text{ the tangent vectors } g_i(x) \text{ form a basis of } T_x M;
\]

(3) let \( U \subseteq \mathbb{R}^n \) be open, \( M \subseteq U \) and \( f \in C^2 : U \rightarrow \mathbb{R} \). The set of local minima \( E \) of \( f |_M \) is nonempty and we denote with \( E_s \subseteq E \) a compact connected component of \( E \).

We introduce the extremum seeking system on the manifold \( M \) as follows (which differs from (8) by replacing the unit vectors \( e_i \) by the tangent vector fields \( g_i \)):

\[
\dot{x} = \sum_{i=1}^{p} cf(x)g_i(x)\sqrt{\omega_i} \cos(\omega_i t) + \alpha g_i(x)\sqrt{\omega_i} \sin(\omega_i t)
\]

\[
\dot{z} = -\frac{1}{2} \sum_{i=1}^{p} \alpha \gamma i \langle \nabla f(z), g_i(z) \rangle g_i(z).
\]

Remark 1. Compared to the extremum seeking setup in Krstić and Ariyur (2003) and Tan et al. (2010), where the dynamics of the extremum seeking feedback and the dynamic plant are separated, we consider a slightly different setup. In this paper, the dynamics in (14) can be interpreted in two ways. First, the dynamics of the plant are assumed to be in quasi-steady state and thus appears static to the dynamics of the extremum seeking feedback, i.e., the dynamics in (14) contain only the extremum seeking dynamics. Second, the dynamics in (14) may include the dynamics of a plant in terms of a drift term \( f_0 \), i.e.,

\[
\dot{x} = f_0(x) + \sum_{i=1}^{p} cf(x)g_i(x)\sqrt{\omega_i} \cos(\omega_i t)
\]

\[
+ \alpha g_i(x)\sqrt{\omega_i} \sin(\omega_i t),
\]

where the extremum seeking feedback is static and consists only of the sinusoidal perturbations. Note that, the results herein can also be extended to extremum seeking systems of the form (16).

Using Lemmas 2 and 3, i.e., the methodology developed in Dürr et al. (2013), we can establish the main results.
Theorem 1. Consider the Lie bracket system (15) and let Assumption 1 be satisfied. Let $W \subseteq M$ be an open set and let $E_c$ be a compact connected set of minima of $f |_{\tilde{M}}$ which is contained in $W$. Assume that the gradient of $f |_{\tilde{M}}$ vanishes in $W$ only at points in $E_c$, i.e., $\nabla f |_{\tilde{M}} = 0$ if and only if $z \in E_c$. Then the set $E_c$ of equilibria is asymptotically stable. Moreover, $E_c$ is practically uniformly asymptotically stable with respect to the extremum seeking system (14). 

In other words, the extremum seeking system locally converges arbitrary close to the set of local minima of $f |_{\tilde{M}}$ for sufficiently large $\alpha_t, i = 1, \ldots, p$. The existence of a set $W$ around a compact connected component of local minima usually exists except in certain pathological cases, see for example Whitney (1935).

Proof. First, define the Lyapunov function candidate $V(x) = f(x)$. Since $\nabla f |_{\tilde{M}} = 0$ if and only if $x \in E_c$ for all $x \in W$ and since $E_c$ is a compact connected component of (local) minima of $f$ on $M$, we know that $V(x) > 0$. Thus $V$ is a suitable Lyapunov function. The Lie derivative along the Lie bracket system (15) is negative semidefinite: $\mathcal{L}_{f} V = -\frac{1}{2} \sum_{i=1}^{n} \alpha_i c_i (V^2)(z, g_i z)^2 \leq 0$. From Assumption 1 span$\{g_1, \ldots, g_n\} = T_M$ for all $z \in M$ so that $\mathcal{L}_{f} V = 0$ and $\mathcal{L}_{f} V = 0$ is orthogonal to $T_M$. By Lemma 1, this is equivalent to $\nabla f |_{\tilde{M}} = 0$ if and only if $x \in E_c$. Thus, we conclude that the set $E_c$ is asymptotically stable and $E_c$ is a set of equilibria. Second, the extremum seeking system (14) fulfills Assumptions A1–A3. Thus, all assumptions of Lemma 3 are satisfied and we conclude that the set $E_c$ of minima of $f |_{\tilde{M}}$ is practically uniformly asymptotically stable for the extremum seeking system (14).

Theorem 1 provides a local stability result of the extremum seeking system based on the Lie bracket system. The next theorem provides a nonlocal result. This requires the introduction of a modified version of practical stability, that also captures the region of attraction.

Definition 4. Let $S \subseteq M$. A compact set $E \subseteq M$ is said to be $S$-practically uniformly asymptotically stable for (3) if it is practically uniformly asymptotically stable and for every $\delta, \epsilon > 0$ there exist a $t_0 \geq 0$, $a > 0$ and $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega > \omega_0$ $x(t_0) \in S \cap U_{\delta}^{\epsilon} \Rightarrow x(t) \in U_{t-t_0}^{\epsilon}$ and $x(t) \in U_{t-t_0}^{\epsilon}$ for $t \geq t_0$.

As before, we drop the term “practically” and “uniformly” in the definition above for systems like (4). If (4) is $S$-asymptotically stable then $E$ is asymptotically stable and $S$ belongs to the region of attraction.

Theorem 2. Consider the Lie bracket system (15) and let Assumption 1 be satisfied. Let $S \subseteq M$ and assume a compact connected set $E_c$ of local minima of $f |_{\tilde{M}}$ is $S$-asymptotically stable. Then, $E_c$ is $S$-practically uniformly asymptotically stable with respect to the extremum seeking system (14). In other words, this theorem states if $S$ is a subset of the region of attraction of $E_c$ for the Lie bracket system, then $S$ is also a subset of the ‘practical’ region of attraction of $E_c$ for the extremum seeking system. This notion is similar as to the notion of semi-global practical stability (e.g. Dürre et al., 2013 and Moreau & Aeyels, 2000).

Proof. By assumption the set $E_c$ is asymptotically stable for the Lie bracket system (15). Since the extremum seeking system (14) satisfies all assumptions of Lemma 3 we conclude that $E_c$ is practically uniformly asymptotically stable.

Now we prove the second claim of the theorem, i.e., for every $\delta, \epsilon > 0$ there exist a $t_0 \geq 0$, $a > 0$ and $\omega_0 > 0$ such that for all $\omega > \omega_0$ and all $t_0 \in \mathbb{R}$ $x(t_0) \in S \cap U_{\delta}^{\epsilon} \Rightarrow x(t) \in U_{t-t_0}^{\epsilon}$, $t \geq t_0 + \tau$ and $x(t) \in U_{t-t_0}^{\epsilon}$, $t \geq t_0$. The following proof consists of two steps. First, we exploit the result above, i.e., trajectories initialized in the vicinity of $E_c$ are practically asymptotically stable. Second, we establish practical convergence of the extremum seeking system for all initial conditions in $S$ to $E_c$. Combining these facts, we obtain that trajectories of the extremum seeking system initialized in $S$ converge to a vicinity of $E_c$ and by practical asymptotic stability they do not leave the vicinity of $E_c$ and practically converge to $E_c$. First step: given any $\delta, \epsilon > 0$. Note that by practical uniform asymptotic stability, there exist a $\delta_1 > 0$ and an $\omega_1 > 0$ such that for all $\omega > \omega_1$ and all $t_0 \in \mathbb{R}$ we have that $x(t_0) \in U_{\delta_1}^{\omega_1} \Rightarrow x(t) \in U_{t-t_0}^{\epsilon}$, $t \geq t_0$.

Note that, this also implies that $\delta_1 \leq \epsilon$. Second step: let $0 < \epsilon_2 < \delta_1$. Since the set $E_c$ is $S$-asymptotically stable for the Lie bracket system (15), there exist a $\tau > 0$ and a $\epsilon_2 > 0$ such that $z(t) \in U_{t-t_0}^{\epsilon_2} \Rightarrow x(t) \in U_{t-t_0}^{\epsilon}$, $t \geq t_0$ and $z(t) \in U_{t-t_0}^{\epsilon_2}$, $t \geq 0$.

Now, we apply Lemma 2. Let $B := K = S \cap U_{\delta_1}^{\omega_1}, D = \delta_1 - \epsilon_2$ and $\tau$ as above and note that due to (19) $B$ satisfies (5). Thus, there exists an $\omega > 0$ such that for every $\omega > \omega_1, t_0 \in \mathbb{R}$ and $x_0 \in K$ we have for $x(t_0) = z(t_0)$ that $||x(t) - z(t)|| < D, t_0 \leq t \leq t_0 + \tau$. Observe that this together with (19) leads for all $\omega > \omega_0 := \max\{\alpha_1, \omega_2\}$ and all $t_0 \in \mathbb{R}$ to $x(t_0) \in S \cap U_{\delta_1}^{\omega_1} \Rightarrow x(t) \in U_{t-t_0}^{\epsilon_2}$ and $x(t) \in U_{t-t_0}^{\epsilon_2}$, $t_0 \leq t \leq t_0 + \tau$.

With this result and since (18) holds uniformly in $t_0$ we choose $t_0 = t_0 + \tau$ and obtain $x(t_0) \in S \cap U_{\delta_1}^{\omega_1} \Rightarrow x(t) \in U_{t-t_0}^{\epsilon}$, $t \geq t_0 + \tau$. Finally, with (20) and $c = \max\{c + D, \epsilon\}$ we obtain the desired result in (17). We conclude that the set $E_c$ is $S$-practically uniformly asymptotically stable for the extremum seeking system (14).

Remark 2. For the sake of simplicity we removed the washout filter (see, e.g., Dürre et al., 2013 and Krstić & Arriyud, 2003) which is usually present in extremum seeking. The role of the filter is to improve the transient behavior and to remove constant offsets in the nonlinear function. Note that, the stability definitions explicitly include the existence of a lower bound $\omega_0$ for the parameter $\omega$ in (14). The washout filter does not influence the existence of such a lower bound, but it may change its value.

The proofs of Theorems 1 and 2 go along similar lines as in Dürre et al. (2013). Therein, local practical uniform asymptotic stability and semi-global practical uniform asymptotic stability were shown using the methodology developed in Moreau and Aeyels (2000). This methodology also allows to approximate the trajectories of the extremum seeking system by the trajectories of the Lie bracket system. This property directly translates to the case of extremum seeking on submanifolds in $\mathbb{R}^n$.

Note that instead of using sinusoids for the perturbations of the extremum seeking feedbacks, the results herein also cover a larger class of periodic perturbation signals (see Dürre et al., 2013).

4. Example

In this section, we illustrate the results on a numerical example of extremum seeking on submanifolds of $\mathbb{R}^n$. It is motivated by synchronization problems in the spirit of Kuramoto’s gradient flow model, e.g. Sarlette, Tuna, Blondel, and Sepulchre (2008). Consider
two controlled harmonic oscillators given by
\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 
\end{bmatrix} = (1 + u_i) \begin{bmatrix}
    x_{i2} \\
    -x_{i1} 
\end{bmatrix},
\]
where we denote the state of the oscillators by \( x_i = [x_{i1}, x_{i2}]^T \), \( i = 1, 2 \), and the overall state by \( x = [x_1^T, x_2^T]^T \). The goal is to find inputs \( u_1, u_2 \) such that the oscillators synchronize by using only relative distance measurements \( f(x) = \|x_1 - x_2\|^2 = (x_1 - x_2, x_1 - x_2) \). Each of the oscillators evolves on the circle \( S^1 \). The problem of synchronizing the oscillators can be formulated as the minimization problem
\[
\min \frac{1}{2} \|x_1 - x_2\|^2 \quad \text{s.t.} \quad x_i' = 1, \quad i = 1, 2.
\]

The state space of the overall system is a torus because the system evolves on the product of two unit circles \( T^2 = S^1 \times S^1 = \{x_1, x_2 \in \mathbb{R}^2 : x_1^2 x_1 = 1, x_2^2 x_2 = 1\} \). The tangent space of \( T^2 \) is given by \( T_T^2 = \{(u^T, v^T) \in \mathbb{R}^4 : x_1^2 u = 0, x_2^2 v = 0\} \). We choose \( g_1(x_1) = [x_2, -x_1, 0, 0]^T \) and \( g_2(x_2) = [0, 0, x_2, -x_1]^T \). One can again easily verify that these vector fields span the tangent space \( T_T^2 \) for all \( x \in T^2 \). The extremum seeking system is given by (21) with \( u_i = c f(x) \sqrt{\omega_i} \cos(\omega_i t) + \alpha_i \sqrt{\omega_i} \sin(\omega_i t) \). The overall extremum seeking system can be written as
\[
\dot{x} = \sum_{i=1}^2 g_i(x_i) + c f(x) g_i(x_i) \sqrt{\omega_i} \cos(\omega_i t)
+ \alpha_i g_i(x_i) \sqrt{\omega_i} \sin(\omega_i t).
\]
Notice that, the system has a drift term \( g_1(x_1) + g_2(x_2) \) and thus it is not of the form (14). However, the results in Section 3 can be easily extended to extremum seeking systems with drifts, which directly follow from the results established in Dürr et al. (2013). In contrast to the first example and in view of Remark 1, the drift vector field can be interpreted as the dynamics of the plants, which are, in this case, the oscillators.

One can show that the Lie bracket system associated to (23) is a gradient flow for (22), therefore it coincides with a classical Kuramoto model, e.g., Sarlette et al. (2008). This allows us to interpret the proposed extremum seeking system as a coupled Kuramoto model that requires only distance measurements and which leads to practical synchronization with similar properties as the classical Kuramoto model. This observation is in the following, discussed in more detail. Since \( x_i \in S^1 \), i.e., \( x_1^2 + x_2^2 = 1 \), we may introduce the coordinate transformation \( x_1 = \cos(\eta_1), x_2 = \sin(\eta_1) \), \( \eta_1 = \arctan(\frac{x_2}{x_1}) + \eta_2 \) and with \( \eta = [\eta_1, \eta_2]^T \) we obtain for \( f(x_1, x_2) = \frac{1}{2} \|x_1 - x_2\|^2 \) in (22) \( f(\eta) = \frac{1}{2} \|\cos(\eta_1) - \cos(\eta_2)\|^2 + \frac{1}{2} \|\sin(\eta_1) - \sin(\eta_2)\|^2 \). Furthermore, \( \dot{\eta}_1 = \frac{x_1 x_2 - x_2 x_1}{x_1^2 + x_2^2} = x_1 \dot{x}_2 - x_2 \dot{x}_1 \), and with (21), this yields
\[
\dot{\eta}_1 = 1 + f(\eta) \sqrt{\omega_i} \cos(\omega_i t) + \sqrt{\omega_i} \sin(\omega_i t),
\]
\( i = 1, 2 \). Note that this system is in the form (8) and we can now calculate the corresponding Lie bracket system, which yields
\[
\begin{align*}
\dot{\theta}_1 &= 1 - \nabla_\theta f(\theta) = 1 - \sin(\theta_1 - \theta_2), \\
\dot{\theta}_2 &= 1 - \nabla_\theta f(\theta) = 1 - \sin(\theta_2 - \theta_1).
\end{align*}
\]
Note that, (25) is the well-known Kuramoto model for two oscillators, e.g., Kuramoto (1975) and Sarlette et al. (2008). The synchronization seeking system can be generalized to multiple oscillators, for further details we refer to Dürr, Stanković, Johansson, and Ebenbauer (2013). In that paper, we also present an example for synchronization of oscillators on the special orthogonal group \( SO(3) \). In Montenbruck, Dürr, Ebenbauer and Allgower (2014), we show how extremum seeking on manifolds can be used for obstacle avoidance problems on \( SO(3) \).

Next, we show a numerical simulation with parameters \( c_1 = c_2 = 0.5, \alpha_1 = \alpha_2 = 0.1, \omega_1 = \omega \) and \( \omega_2 = 1.1 \omega \) with \( \omega = 20 \).

The phase system associated to the coupled oscillators can be visualized as a system evolving on a torus depicted in Fig. 1. The phase of the first oscillator evolves in poloidal direction (small radius) and the phase of the second oscillator evolves on the toroidal direction (large radius) on the torus.

We see that the oscillators synchronize, i.e., \( x_1 = x_2 \) which is a solution of (22). Thus the overall system practically converges according to the results of the previous section.

### 5. Summary and outlook

In this paper, we introduced extremum seeking systems on submanifolds of the real Euclidian space. By exploiting the Lie bracket approximation analysis introduced in Dürr et al. (2013), we proved practical uniform asymptotic stability of the set of minima of an unknown function on a manifold when only the function values are measurable. We illustrated the results with a numerical example for extremum seeking on submanifolds of the Euclidian space.

Several interesting extensions of this setup are possible. First, an interesting extension is to consider non-integrable distributions, i.e., the vector fields in (13) do not span the tangent space but the Lie algebra generated by these vector fields do span the tangent space, i.e., extremum seeking for non-holonomic systems.

Second, the introduced tangent vector fields \( g \) can either be imposed by the proposed feedback or they can be part of the dynamics of the plant, i.e., in the above setup the dynamics evolve exclusively on the manifold and therefore cannot leave the manifold. For the former case, a future research direction is to consider the robustness of the system with respect to the possible presence of disturbances which drive the states away from the manifold. Third, it is also attractive to consider the minimization of dynamic maps, as it was done in the classical extremum seeking (see e.g. Krstić & Ariyur, 2003). This requires a combination of Lie bracket approximation method of extremum seeking systems with a singular perturbation analysis, which allows to consider the dynamics of the map as a quasi-steady map.

Finally, the introduced approach potentially allows to extend many results in the synchronization literature to the case when only distance measurements are available.

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