An Improved Hybrid Sensor Schedule for Remote State Estimation under Limited Communication Resources

Junfeng Wu, Karl Henrik Johansson and Ling Shi

***Abstract*— In this paper, we consider remote state estimation. A sensor locally processes its measurement data and sends its local estimate to a remote estimator for further processing. Due to the limited communication resources, the sensor can only communicate with the estimator for a pre-specified number within a given horizon. We propose a hybrid sensor data schedule which introduces an event-triggering mechanism on top of an optimal offline sensor schedule. This hybrid schedule, having a small implementation cost, leads to a smaller estimation error at the remote estimator when compared with the optimal offline sensor schedule.**

I. INTRODUCTION

Advances in modern control, communication and networking technologies enable a new generation of networked control systems (NCSs) [1]. The last decade has witnessed a wealth of NCS applications in smart grid, intelligent transportation systems, health care, environmental monitoring, etc. In many of these applications, remote state estimation is a key component, where sensor data is sent to a remote state estimator over a network. The communication resources for remote estimation, in terms of communication energy and bandwidth, are often scarce. Thus it is of significant importance to understand and obtain a desired tradeoff between the limited communication resources and the remote estimation quality. Such a tradeoff is sometimes possible to achieve via a good sensor scheduling scheme. Most existing sensor schedulers are either offline or online. The recent literature is briefly reviewed below.

Offline Schedulers: Mo et al. [2] proposed a network lifetime maximization policy under an estimation quality constraint. Ambrosino et al. [3] considered remote estimation, where sensors transmit measurements over a shared network to a central base station. Gupta et al. [4] proposed a stochastic sensor selection strategy which minimizes the expected error covariance. Similar approaches can also be found in [5]. Shi et al. [6] considered sensor data scheduling over packet-dropping networks. Due to its limited energy, a sensor has to decide whether to send its local estimate data to the remote estimator at low or high power level at each instance. They showed that the average estimation error is minimized when the transmission times at high power mode are separated as much as possible.

Online Schedulers: Åström and Bernhardsson [7] considered a simple first-order stochastic system. They showed that at the same average sampling rate of a periodic sampler, the event-based sampler leads to smaller state variance. Imer and Başar [8] considered an estimation problem over a scalar linear system with a limited number of observations. Upon observing the process, the observer needs to make a decision whether to send some observation information to the estimator. Li et al. [9] extended the results of [8] to vector linear systems. A sub-optimal event-trigger is given to minimize the mean square estimation error through a computationally efficient way. Cogill [10] considered the problem of controlling a system with limited actuation and sampling rate, where a control action is only applied when a certain event occurs. An event-based control policy was proposed which minimizes the upper bound on control performance using a quadratic approximate value function.

Offline schedules are often easier to compute then online schedules, but at the same time, may have worse performance. To make the best use of each approach, Shi et al. [11] proposed a novel hybrid sensor schedule which introduced an event-triggering mechanism on top of an optimal offline schedule. They considered a scenario when a sensor can only communicate with a remote state estimator m times within a time-horizon $T \gg m$. By selecting the event-triggering threshold $\delta > 0$ appropriately, the hybrid sensor schedule was shown to have better performance than the optimal offline one. However, the results of [11] have the following limitations:

1) When an event occurs, the error covariance matrix is not explicitly given, but only an upper bound.
2) The critical threshold $\delta_{\max}$ below which the hybrid schedule outperforms the optimal offline one is not explicitly given, but only a lower bound.
3) Selection of the optimal threshold δ relies on Monte Carlo simulations.

This paper presents an improved hybrid sensor schedule based on the results of [11]. The main contributions are summarized as follows.

1) A modified event-triggering mechanism is given under which a closed-form expression on the estimation error covariance when an event happens is derived.
2) A closed-form expression on the critical threshold $\delta_{\max}$ is given.
3) The optimal threshold δ is given analytically for first-

J. Wu and L. Shi are with the Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email: jfwu@ust.hk, eesling@ust.hk.

K. H. Johansson is with the ACCESS Linnaeus Center, School of Electrical Engineering, Royal Institute of Technology, Stockholm, Sweden. Email: kallej@ee.kth.se.

The work by J. Wu and L. Shi is supported by a HK RGC CRF grant HKUST11/CRF/10 and a HK RGC GRF grant GRF618612. The work by K. H. Johansson is supported by the Knut and Alice Wallenberg Foundation.

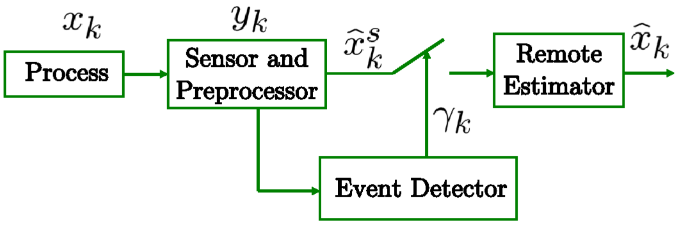


Fig. 1. Hybrid scheduling architecture for remote estimation.

order systems and can be solved efficiently for higher-order systems.

The remainder of the paper is organized as follows. Mathematical models of the system are given in Section II. Section III reviews some key results of [11]. A hybrid sensor schedule is introduced in Section IV and analyzed in Section V. Section VI demonstrates the main theoretical results. Some concluding remarks are given in the end.

Notations: $\mathbb{Z}$ is the set of non-negative integers. $\mathbb{N}$ is the set of positive integers. $k \in \mathbb{Z}$ is the time index. $\mathbb{R}^n$ is the n dimensional Euclidean space. $\mathbb{S}_+^n$ is the set of n by n positive semi-definite matrices. When $X \in \mathbb{S}_+^n$, it is written as $X \geq 0$. $X \geq Y$ if $X - Y \in \mathbb{S}_+^n$. $\mathbb{E}[\cdot]$ is the expectation of a random variable and $\mathbb{E}[\cdot|\cdot]$ is the conditional expectation. $\Pr(\cdot)$ is the probability of a random event. $\mathrm{Tr}(\cdot)$ is the trace of a matrix and $\|\cdot\|_\infty$ is the infinity norm of a vector. For functions $f, f_1, f_2 : \mathbb{S}_+^n \to \mathbb{S}_+^n$, $f_1 \circ f_2$ is defined as $f_1 \circ f_2(X) \triangleq f_1\big(f_2(X)\big)$ and f^t is defined as $f^t(X) \triangleq \underbrace{f \circ f \circ \cdots \circ f}_{t \text{ times}}(X)$.

II. Problem Setup

Consider the following system (Fig. 1)

$$x_{k+1} = Ax_k + w_k, \tag{1}$$
$$y_k = Cx_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the process state vector, $y_k \in \mathbb{R}^m$ is the observation vector, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero-mean Gaussian random vectors with $\mathbb{E}[w_k w_j{}'] = \delta_{kj} Q$ ($Q \geq 0$), $\mathbb{E}[v_k v_j{}'] = \delta_{kj} R$ ($R > 0$), $\mathbb{E}[w_k v_j{}'] = 0$ $\forall j, k$. The initial state x_0 is a zero-mean Gaussian random vector that is uncorrelated with w_k and v_k and has covariance $\Pi_0 \geq 0$. The pair (A, C) is assumed to be observable and $(A, \sqrt{Q})$ is controllable.

Assume the sensor runs a Kalman filter to compute

$$\hat{x}_k^s = \mathbb{E}[x_k | y_0, \ldots, y_k],$$

the local MMSE estimate of x_k in (1). The estimation error e_k^s and error covariance matrix P_k^s are defined as follows:

$$e_k^s \triangleq x_k - \hat{x}_k^s, \tag{3}$$
$$P_k^s \triangleq \mathbb{E}[(e_k^s)(e_k^s)' | y_0, \ldots, y_k]. \tag{4}$$

After $\hat{x}_k^s$ is obtained, the sensor decides whether to send it to the remote estimator. Let γ_k be the decision variable at time k , i.e., if $\gamma_k = 1$, $\hat{x}_k^s$ is sent, otherwise $\hat{x}_k^s$ is not sent.

Let $T \in \mathbb{N}$ be the time-horizon and define a schedule θ as

$$\theta = \{\gamma_1, \ldots, \gamma_T\} \in \{0, 1\}^T.$$

Under a given θ, the remote estimator calculates $\hat{x}_k$ and P_k, its own MMSE estimate of x_k and the associated error covariance. Define $J(\theta)$ as the trace of the average expected estimation error covariance, i.e.,

$$J(\theta) \triangleq \frac{1}{T} \sum_{k=1}^{T} \mathrm{Tr}\left(\mathbb{E}\left[P_k(\theta)\right]\right). \tag{5}$$

Consider the following problem from [11]:

Problem 2.1:

$$\min_{\theta} \quad J(\theta),$$
$$\text{s.t.} \quad \sum_{k=1}^{T} \gamma_k(\theta) = m.$$

III. Preliminaries

A. Kalman Filter Preliminaries

It is well known that $\hat{x}_k^s$ and P_k^s can be computed through a Kalman filter as follows:

$$\hat{x}_{k|k-1}^s = A\hat{x}_{k-1}^s, \tag{6}$$
$$P_{k|k-1}^s = AP_{k-1}^s A' + Q, \tag{7}$$
$$K_k = P_{k|k-1}^s C' [CP_{k|k-1}^s C' + R]^{-1}, \tag{8}$$
$$\hat{x}_k^s = A\hat{x}_{k-1}^s + K_k(y_k - CA\hat{x}_{k-1}^s), \tag{9}$$
$$P_k^s = (I - K_k C) P_{k|k-1}^s, \tag{10}$$

where the recursion starts from $\hat{x}_0^s = 0$ and $P_0^s = \Pi_0$. At the estimator side, it is straightforward to show that the optimal state estimate $\hat{x}_k$ is given by

$$\hat{x}_k = \begin{cases} A\hat{x}_{k-1}, & \text{if } \gamma_k = 0, \\ \hat{x}_k^s, & \text{if } \gamma_k = 1. \end{cases} \tag{11}$$

For simplicity, define functions h and $\tilde{g} : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$h(X) \triangleq AXA' + Q; \tag{12}$$
$$\tilde{g}(X) \triangleq X - XC'[CXC' + R]^{-1} CX. \tag{13}$$

Similar to [11], we assume the Kalman filter has entered steady state to simplify the discussion, i.e.,

$$P_k^s = \overline{P}, \ k \geq 1, \tag{14}$$

where $\overline{P}$ is the steady-state error covariance. As the unique positive semi-definite solution [12] of $\tilde{g} \circ h(X) = X$, $\overline{P}$ has the following property.

Lemma 3.1: For $0 \leq t_1 \leq t_2$, the following inequality holds:

$$h^{t_1}(\overline{P}) \leq h^{t_2}(\overline{P}). \tag{15}$$

In addition, if $t_1 < t_2$, then

$$\mathrm{Tr}\left(h^{t_1}(\overline{P})\right) < \mathrm{Tr}\left(h^{t_2}(\overline{P})\right). \tag{16}$$

Proof: See [11]. ■

B. Optimal Offline Sensor Schedule

Shi et al. [11] introduced the optimal offline sensor schedule to *Problem 2.1*. For simplicity, they considered $m = 2t - 1$ and $T = 4qt - 1$ for $t, q \in \mathbb{N}$. Other forms of T and m can be dealt similarly. As shown in the following proposition, the optimal offline schedule is a periodic schedule with period $2q$ and the m communication times between the sensor and the remote estimator are separated *as uniform as possible*.

Proposition 3.2: The optimal offline schedule $\theta^\star_{\text{off}} \in \{0,1\}^T$ that minimizes $J(\theta)$ in (5) is given by:

$$\gamma_{2lq} = 1 \ \forall\ l = 1, \ldots, 2t-1, \text{ and } \gamma_k = 0 \text{ otherwise.}$$

Under $\theta^\star_{\text{off}}$, P_k evolves as

$$P_k = \begin{cases} h(P_{k-1}), & \text{if } \gamma_k = 0, \\ \overline{P}, & \text{if } \gamma_k = 1. \end{cases}$$

The corresponding minimum $J(\theta)$ is given by

$$J(\theta^\star_{\text{off}}) = \frac{2t}{T}\sum_{i=0}^{2q-1} \text{Tr}\left(h^i(\overline{P})\right) - \frac{1}{T}\text{Tr}\left(\overline{P}\right). \tag{17}$$

Proof: See [11]. ■

IV. A Hybrid Sensor Schedule

In this section, we consider the hybrid schedule proposed in [11]. We assume $m = 2t-1$ and $T = 4qt-1$ for $t, q \in \mathbb{N}$. The lemmas in this section, if without proof, follow directly from [11].

A. A Hybrid Sensor Schedule

First note that the sensor is able to calculate $\hat{x}_k$ as it has access to all γ_k's. Define ε_k as

$$\varepsilon_k \triangleq \hat{x}^s_k - A\hat{x}_{k-1}, \tag{18}$$

where $A\hat{x}_{k-1}$ is the predicted state estimate at the estimator based on the previous optimal state estimate $\hat{x}_{k-1}$. Thus if $\hat{x}^s_k$ is not sent at time k, ε_k will indicate how close is the state estimate at the estimator from the optimal state estimate at the sensor.

Lemma 4.1: The following statements on e^s_k hold:

1) e^s_k is independent of $\hat{x}^s_k$, hence $\mathbb{E}[(e^s_k)(\hat{x}^s_k)'] = 0$.
2) e^s_k is zero-mean Gaussian.
3) e^s_k is independent of $A^d\hat{x}^s_{k-d}$ and $\hat{x}^s_k - A^d\hat{x}^s_{k-d}$ for any $d \in \mathbb{N}$.

Lemma 4.2: e^s_k and ε_k are independent and $\mathbb{E}[e^s_k\varepsilon'_k] = 0$.

Lemma 4.3: ε_{2lq} is zero-mean Gaussian and its covariance is independent of l.

Lemma 4.4: $\mathbb{E}[\varepsilon_{2lq}\varepsilon'_{2lq}] = h^{2q}(\overline{P}) - \overline{P}$.

Let the rank of $h^{2q}(\overline{P}) - \overline{P}$ be r. From Lemma 3.1, $h^{2q}(\overline{P}) - \overline{P} \geq 0$, hence there exists an orthonormal matrix $U \in \mathbb{R}^{n\times n}$ such that

$$U'(h^{2q}(\overline{P}) - \overline{P})U = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{\text{r}})$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ are the r nonzero eigenvalues of $h^{2q}(\overline{P}) - \overline{P}$. Define $F \in \mathbb{R}^{n\times n}$ as:

$$F \triangleq U\begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & I_{n-r} \end{bmatrix}. \tag{19}$$

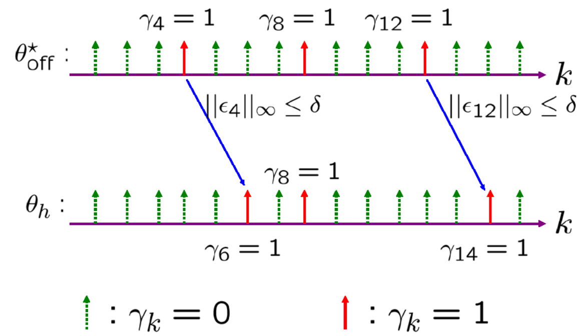


Fig. 2. Realization of $\theta^\star_{\text{off}}$ and θ_h

Then

$$F'(h^{2q}(\overline{P}) - \overline{P})F = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Define ϵ_{2lq} as $\epsilon_{2lq} \triangleq F'\varepsilon_{2lq}$. We propose the hybrid schedule θ_{h} as follows: *for a given threshold $\delta > 0$, θ_{h} is identical to $\theta^\star_{\text{off}}$ except at $k = 2lq$ when l is odd, in which instances, if $\|\epsilon_{2lq}\|_\infty \leq \delta$, set $\gamma_{2lq} = 0$ and $\gamma_{(2l+1)q} = 1$.*

Since w_k's and v_k's are random, the instances for the sensor to send $\hat{x}^s_k$ to the remote estimator under θ_h also become random. On the contrary, under $\theta^\star_{\text{off}}$, the communication instances are fixed a priori. Fig. 2 shows a sample realization of the sensor communication instances under both $\theta^\star_{\text{off}}$ and θ_h for $t = 2$, $q = 2$, $m = 3$ and $T = 15$.

Remark 4.5: In [11], $|\varepsilon_{2lq}| \leq \delta$ was used to define the event, i.e., when l is odd, if $|\varepsilon_{2lq}| \leq \delta$, then $\gamma_{2lq} = 0$. Since the entries of ε_{2lq} might be mutually correlated, it is difficult to analyze the estimation quality under the hybrid sensor schedule. For example, when an event is triggered, no closed-form but only an upper bound of P_{2lq} could be obtained, and consequently, selecting the optimal threshold δ has to rely on Monte Carlo simulations. In this paper, through a linear transformation F', ε_{2lq} is changed to ϵ_{2lq}, a random vector with mutually independent zero-mean Gaussian entries, which makes the analysis considerably simpler.

Lemma 4.6:

$$p_\delta \triangleq \Pr(\|\epsilon_{2lq}\|_\infty > \delta) = 1 - [1 - 2Q(\delta)]^r, \tag{20}$$
$$\mathbb{E}[\varepsilon_{2lq}\varepsilon'_{2lq} | \|\epsilon_{2lq}\|_\infty \leq \delta] = [1 - \beta(\delta)][h^{2q}(\overline{P}) - \overline{P}],$$

where $Q(\delta)$ is the Q-function defined by

$$Q(\delta) \triangleq \int_\delta^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x$$

and

$$\beta(\delta) \triangleq \frac{2}{\sqrt{2\pi}}\delta e^{-\frac{\delta^2}{2}}[1 - 2Q(\delta)]^{-1}. \tag{21}$$

Proof: Let $\epsilon_{2lq} = \begin{bmatrix} \xi^1_{2lq} \\ \xi^2_{2lq} \end{bmatrix}$ where $\xi^1_{2lq} \in \mathbb{R}^r$ and $\xi^2_{2lq} \in \mathbb{R}^{n-r}$. Since $\mathbb{E}[\varepsilon_{2lq}\varepsilon'_{2lq}] = h^{2q}(\overline{P}) - \overline{P}$, we have

$$\mathbb{E}[\epsilon_{2lq}\epsilon'_{2lq}] = F'\mathbb{E}[\varepsilon_{2lq}\varepsilon'_{2lq}]F = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence ξ^1_{2lq} is a zero mean Gaussian multivariate random variable with unit variance, and $\xi^2_{2lq} = 0$ almost surely. Lemma 1.2 leads to the following:

$$1-p_\delta = \Pr(\|\epsilon_{2lq}\|_\infty \le \delta) = \Pr(\|\xi^1_k\|_\infty \le \delta)) = [1-2Q(\delta)]^r,$$

which shows (20), and

$$\mathbb{E}[\epsilon_{2lq}\epsilon_{2lq}^{'}|\|\epsilon_{2lq}\|_\infty \le \delta] = [1-\beta(\delta)]\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $U^{'} = U^{-1}$, one has

$$\begin{aligned} & \mathbb{E}[\varepsilon_{2lq}\varepsilon_{2lq}'|\|\epsilon_{2lq}\|_\infty \le \delta] \\ = \; & (F^{'})^{-1}\mathbb{E}[\epsilon_{2lq}\epsilon_{2lq}^{'}|\|\epsilon_{2lq}\|_\infty \le \delta]F^{-1} \\ = \; & [1-\beta(\delta)]U\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}U^{'} \\ = \; & [1-\beta(\delta)][h^{2q}(\overline{P})-\overline{P}], \end{aligned}$$

which completes the proof. ■

Corollary 4.7: γ_{2lq}'s are i.i.d Bernoulli random variables with $\Pr(\gamma_{2lq}=0) = 1-p_\delta$, and $\Pr(\gamma_{2lq}=1) = p_\delta$.

Proof: Note that $\Pr(\gamma_{2lq}=0) = \Pr(\|\epsilon_{2lq}\|_\infty \le \delta)$. The assertion is a direct result from (20). ■

We now introduce one of the main results of this paper, in which a closed-form expression on the error covariance matrix P_k is obtained.

Theorem 4.8: The error covariance P_k under $\theta_{\rm h}$ is given by

$$P_k = \begin{cases} h(P_{k-1}), & \text{if } \gamma_k = 0, \\ \overline{P}, & \text{if } \gamma_k = 1, \end{cases}$$

except at those time instances $k = 2lq$ when l is odd and $\|\epsilon_{2lq}\|_\infty \le \delta$, in which cases, $\gamma_{2lq} = 0$ and P_{2lq} is given by

$$P_{2lq} = \overline{P} + [1-\beta(\delta)][h^{2q}(\overline{P}) - \overline{P}]. \tag{22}$$

Proof: We only prove the exceptional case as the other cases are straightforward. At the estimator, if no packet is received at time $k = 2lq$, then $\gamma_{2lq} = 0$, which also implies $\|\epsilon_{2lq}\|_\infty \le \delta$. Therefore

$$\begin{aligned} P_{2lq} &= \mathbb{E}[(x_{2lq} - \hat{x}_{2lq})(\cdot)'|\|\epsilon_{2lq}\|_\infty \le \delta] \\ &= \mathbb{E}[(x_{2lq} - \hat{x}^s_{2lq} + \hat{x}^s_{2lq} - \hat{x}_{2lq})(\cdot)'|\|\epsilon_{2lq}\|_\infty \le \delta] \\ &= \mathbb{E}[(e^s_{2lq} + \varepsilon_{2lq})(e^s_{2lq} + \varepsilon_{2lq})'|\|\epsilon_{2lq}\|_\infty \le \delta] \\ &= \mathbb{E}[(e^s_{2lq})(e^s_{2lq})'|\|\epsilon_{2lq}\|_\infty \le \delta] + \\ &\quad\; \mathbb{E}[\varepsilon_{2lq}\varepsilon_{2lq}'|\|\epsilon_{2lq}\|_\infty \le \delta] \\ &= \overline{P} + [1-\beta(\delta)][h^{2q}(\overline{P}) - \overline{P}], \end{aligned}$$

where the second last equality is from Lemma 4.2 and the last equality is from Lemma 4.6. ■

V. Performance Analysis

In this section we compare the performance of $\theta_{\rm h}$ with that of $\theta^\star_{\rm off}$. The following lemma on properties of $\beta(\delta)$ in (21) is useful to derive the main result. The proof is straightforward and is omitted.

Lemma 5.1:

1) $\beta(\delta) > 1 - \delta^2$.
2) $\beta(\delta)$ is strictly decreasing in δ.
3) $\lim_{\delta\to 0}\beta(\delta) = 1$.
4) $\lim_{\delta\to +\infty}\beta(\delta) = 0$.

The following result provides a sufficient and necessary condition on δ such that $\theta_{\rm h}$ outperforms $\theta^\star_{\rm off}$.

Theorem 5.2: The following statements hold.

1) $J(\theta_{\rm h}) \le J(\theta^\star_{\rm off})$ iff $\delta \in [0, \delta_{\max}]$, where $\delta_{\max}$ is the unique solution to
$$\beta(\delta)\left(\Gamma_2 - \Gamma_0\right) = \Gamma_2 - \Gamma_1 \tag{23}$$
and
$$\Gamma_j = \mathrm{Tr}\left[\sum_{i=jq}^{(j+1)q-1} h^i(\overline{P})\right],\ j = 0,1,2.$$
2) For any $\delta \in (0, \delta_{\max})$ and for any realization ϕ of $\theta_{\rm h}$,
$$J(\phi) \le J(\theta^\star_{\rm off}). \tag{24}$$
Furthermore there exists a positive probability of ϕ such that (24) becomes strict, hence $J(\theta_{\rm h}) < J(\theta^\star_{\rm off})$.

Proof: (1) Recall that $p_\delta = \Pr\left(\|\epsilon_{2lq}\|_\infty > \delta\right)$. For any odd number l,

$$\begin{aligned} & \sum_{k=2(l-1)q}^{2(l+1)q-1} \mathbb{E}[P_k(\theta_{\rm h})] \\ = \; & \sum_{i=0}^{2q-1} h^i(\overline{P}) + p_\delta \sum_{i=0}^{2q-1} h^i(\overline{P}) + (1-p_\delta)\Bigg[\beta(\delta)\sum_{i=0}^{q-1} h^i(\overline{P}) \\ & + [1-\beta(\delta)]\sum_{i=2q}^{3q-1} h^i(\overline{P}) + \sum_{i=0}^{q-1} h^i(\overline{P})\Bigg], \end{aligned}$$

which is independent of l. Define D_δ as

$$\begin{aligned} & D_\delta \\ & \triangleq \mathrm{Tr}\left[\sum_{i=q}^{2q-1} h^i(\overline{P}) - \beta(\delta)\sum_{i=0}^{q-1} h^i(\overline{P}) - [1-\beta(\delta)]\sum_{i=2q}^{3q-1} h^i(\overline{P})\right] \\ & = \Gamma_1 - \beta(\delta)\Gamma_0 - [1-\beta(\delta)]\,\Gamma_2. \end{aligned}$$

With some manipulation, we obtain

$$J(\theta^\star_{\rm off}) - J(\theta_{\rm h}) = \frac{1}{4q}(1-p_\delta)D_\delta.$$

Since $1 - p_\delta \ge 0$, $J(\theta^\star_{\rm off}) \ge J(\theta_{\rm h})$ iff $D_\delta \ge 0$. One can easily verify that $\Gamma_2 > \Gamma_1 > \Gamma_0$. From Lemmas 3.1 and 5.1, D_δ is strictly decreasing in δ and

$$D_0 = \Gamma_1 - \Gamma_0 > 0, \quad D_\infty = \Gamma_1 - \Gamma_2 < 0.$$

Hence there is a unique $\delta_{\max}$ such that $D_{\delta_{\max}} = 0$, which corresponds to the $\delta_{\max}$ in (23). One notes that $\left(\Gamma_2 - \Gamma_1\right)/\left(\Gamma_2 - \Gamma_0\right) \in (0,1)$, thus (23) must have a solution. Furthermore, for all $\delta < \delta_{\max}$, $D_\delta > 0$ and for all $\delta > \delta_{\max}$, $D_\delta < 0$.

(2) Let $\delta_{\max} > 0$. For any $\delta \in (0, \delta_{\max})$ and for any realization ϕ of $\theta_{\rm h}$, if $\|\epsilon_{2lq}\|_\infty > \delta$ for all odd number l, then ϕ is the same as $\theta^\star_{\rm off}$. Hence $J(\phi) = J(\theta^\star_{\rm off})$. Otherwise if there exists an odd number l such that $\|\epsilon_{2lq}\|_\infty \le \delta$, then similar to the proof of the first statement, one easily verifies

that $J(\phi) < J(\theta^\star_{\text{off}})$. Notice that the probability of those ϕ's with at least one l such that $\|\epsilon_{2lq}\|_\infty \le \delta$ is positive, hence

$$J(\theta_{\text{h}}) = \sum_\phi \Pr(\phi)J(\phi) < \sum_\phi \Pr(\phi)J(\theta^\star_{\text{off}}) = J(\theta^\star_{\text{off}}). \qquad \blacksquare$$

Remark 5.3: Since $\beta(\delta)$ is monotonically decreasing in δ from Lemma 5.1 and Γ_0, Γ_1, Γ_2 can be easily computed, (23) can be solved efficiently via Newton's method.

To minimize $J(\theta_{\text{h}})$, we simply compute the δ that maximizes $(1-p_\delta)D_\delta$ so that the difference between $J(\theta_{\text{h}})$ and $J(\theta^\star_{\text{off}})$ is maximum. Let $\delta^\star$ be the optimal δ, i.e.,

$$\delta^\star \triangleq \arg\min_{\delta>0} J(\theta_{\text{h}}) = \arg\max_{\delta>0} [1-2Q(\delta)]^r D_\delta. \qquad (25)$$

The optimal $\delta^\star$ for the schedule θ_{h} is presented in the following theorem.

Theorem 5.4: The optimal $\delta^\star$ is the unique solution to

$$\delta^2 + (r-1)\left[1-\beta(\delta)\right] - r\frac{\Gamma_1-\Gamma_0}{\Gamma_2-\Gamma_0} = 0. \qquad (26)$$

In particular, if $r=1$,

$$\delta^\star = \sqrt{\frac{\Gamma_1-\Gamma_0}{\Gamma_2-\Gamma_0}}. \qquad (27)$$

Proof: (1) When $r=1$,

$$\frac{\mathrm{d}\{[1-2Q(\delta)]D_\delta\}}{\mathrm{d}\delta} = \frac{2e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}\left[\Gamma_1-(1-\delta^2)\Gamma_0-\delta^2\Gamma_2\right], \qquad (28)$$

which is an increasing function. Setting (28) to zero, one obtains (27).

(2) When $r \ge 2$,

$$\begin{aligned}
&\frac{\mathrm{d}\{[1-2Q(\delta)]^r D_\delta\}}{\mathrm{d}\delta}\\
=\;& \frac{2e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}[1-2Q(\delta)]^{r-2}\Big\{[1-2Q(\delta)](\Gamma_2-\Gamma_0)(1-\delta^2)\\
&+(\Gamma_2-\Gamma_0)(r-1)\frac{2\delta e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}} - [1-2Q(\delta)](\Gamma_2-\Gamma_1)r\Big\}\\
=\;& \frac{2e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}[1-2Q(\delta)]^{r-1}\Big\{(\Gamma_2-\Gamma_0)\left[1-\delta^2+(r-1)\beta(\delta)\right]\\
&-(\Gamma_2-\Gamma_1)r\Big\}.
\end{aligned}$$

Define

$$\varphi(\delta) \triangleq (\Gamma_2-\Gamma_0)\left[1-\delta^2+(r-1)\beta(\delta)\right] - (\Gamma_2-\Gamma_1)r.$$

From Lemma 5.1, $\varphi(\delta)$ is strictly decreasing in δ with

$$\lim_{\delta\to 0}\varphi(\delta) = r$$

and

$$\begin{aligned}
\varphi(\delta_{\max}) &< (\Gamma_2-\Gamma_0)\left[\beta(\delta_{\max})+(r-1)\beta(\delta_{\max})\right] - (\Gamma_2-\Gamma_1)r\\
&= 0.
\end{aligned}$$

Note that $\frac{2e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}[1-2Q(\delta)]^{r-1} > 0$, therefore, the optimal $\delta^\star$ exists and is the unique solution to (26). $\blacksquare$

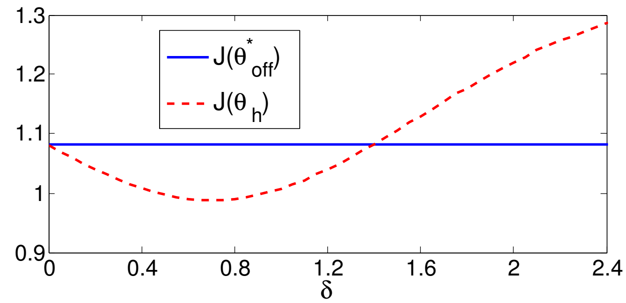


Fig. 3. Comparison of θ_{h} and $\theta^\star_{\text{off}}$ when $r=1$

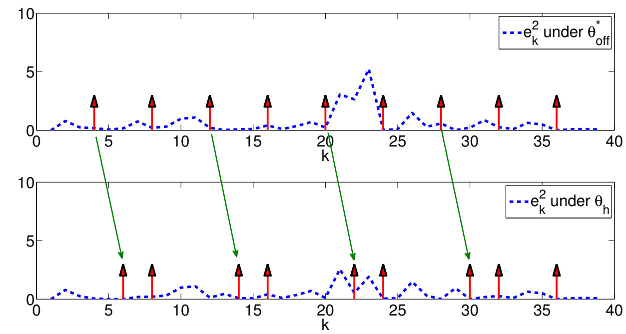


Fig. 4. A sample path of the estimation errors under θ_{h} and $\theta^\star_{\text{off}}$

VI. Examples

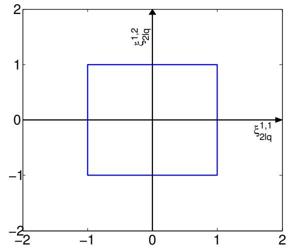
A. First-order Systems

Consider the following parameters for system (1)-(2): $A = 1.01$, $C=1$, $Q=R=0.5$, $m=99$, $T=399$, $q=2$. The optimal offline schedule $\theta^\star_{\text{off}}$ is periodic with period 4. The local Kalman filter converges to its steady-sate value, $\overline{P} = 0.3101$, from which one obtains $h^{2q}(\overline{P}) = 2.3969$, $\Gamma_0 = 1.1264$, $\Gamma_1 = 3.1922$ and $\Gamma_2 = 5.3419$. Then we solve for $\delta_{\max} = 1.3809$ and $\delta^\star = 0.7$. Fig. 3 plots $J(\theta^\star_{\text{off}})$ and $J(\theta_{\text{h}})$ for various values of δ. From the figure, $\delta_{\max}$ is 1.38, and the optimal δ is 0.7 where the difference between $J(\theta_{\text{h}})$ and $J(\theta^\star_{\text{off}})$ achieves its maximum. The empirical results from Fig. 3 therefore agrees well with Theorem 5.2, and with Theorem 5.4 for $r=1$.

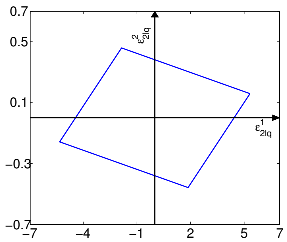
We further plot e_k^2 in a sample path of $\theta^\star_{\text{off}}$ and θ_h (taking $\delta = 0.7$) from $k=0$ to $k=39$ in Fig. 4, where a red arrow indicates a particular time k when $\hat{x}_k^s$ is sent. Clearly, by rescheduling the transmission at appropriate times (e.g., even transmission instances under $\theta^\star_{\text{off}}$), the estimation error is reduced. These four instances indicated in the plot demonstrate intuitively why and how the estimation error can be reduced by the proposed hybrid schedule, as one can note that the hybrid schedule allocates these four samples at more suitable times than the offline schedule.

B. Second-order System

Consider the following system (1)-(2) with parameters: $A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1.05 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, $R=0.5$, $m=49$, $T=199$, $q=2$. The optimal offline schedule $\theta^\star_{\text{off}}$ is again periodic with period 4.



(a)



(b)

Fig. 5. The triggering set of ε_{2lq} when $r = 2$

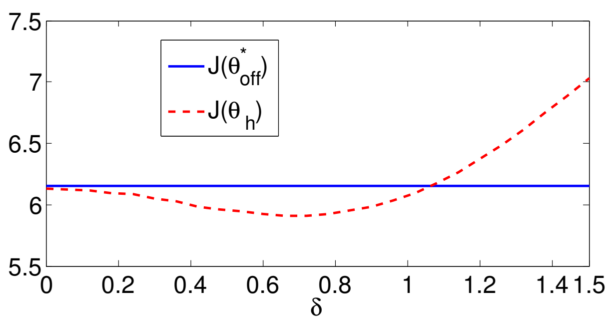


Fig. 6. Comparison of θ_{h} and $\theta^{\star}_{\mathrm{off}}$ when $r = 2$

The local Kalman filter converges to its steady-sate value

$$\overline{P} = \begin{bmatrix} 0.3802 & 0.2840 \\ 0.2840 & 1.6894 \end{bmatrix},$$

from which one obtains

$$h^{2q}(\overline{P}) = \begin{bmatrix} 13.3431 & 6.5106 \\ 6.5106 & 4.8251 \end{bmatrix},$$

$\Gamma_0 = 6.0188$, $\Gamma_1 = 18.5135$, $\Gamma_2 = 45.5479$ and

$$F = \begin{bmatrix} -0.2251 & 1.2726 \\ -0.1091 & -2.6253 \end{bmatrix}.$$

Given a δ (e.g., $\delta = 1$), the event is triggered when ξ_k^1 drop inside the rectangle that is shown in Fig. 5(a), or equivalently ε_{2lq} is inside the parallelogram drawn in Fig. 5(b) From part 1) of Theorem 5.2 (with $r = 2$), we obtain $\delta_{\max} = 1.0493$ and from Theorem 5.4, $\delta^{\star} = 0.6940$. Fig. 6 plots $J(\theta^{\star}_{\mathrm{off}})$ and $J(\theta_{\mathrm{h}})$ for different values of δ. This figure shows that $\delta_{\max}$ is about 1.05 and the optimal δ is approximately 0.7. Again the empirical results match well with the theoretical ones.

VII. Conclusion

In this paper, we present an improved hybrid sensor schedule to tackle the problem of remote state estimation with limited sensor communications. This schedule leads to better performance when compared with the optimal offline schedule and has a small implementation cost. Future work include extensions to closed-loop control data scheduling and multiple sensor scheduling.

Appendix

The following two lemmas are straightforward to verify and the proofs are omitted.

Lemma 1.1: Let $\mathbf{x} \in \mathbb{R}$ be a Gaussian random variable with zero mean and variance $\mathbb{E}[\mathbf{x}^2] = 1$. For $\delta > 0$, we have

$$\mathbb{E}[\mathbf{x^2} | |\mathbf{x}| \leq \delta] = 1 - \beta(\delta).$$

Lemma 1.2: Let $\xi \in \mathbb{R}^r$ be a Gaussian random variable with zero mean and $\mathbb{E}[\xi\xi'] = I_r$. Let $\delta \geq 0$. Then

$$\Pr(\|\xi\|_\infty \leq \delta) = [1 - 2Q(\delta)]^r,$$

$$\mathbb{E}[\xi\xi' \mid \|\xi\|_\infty \leq \delta] = [1 - \beta(\delta)]I_r.$$

References

[1] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[2] Y. Mo, L. Shi, R. Ambrosino, and B. Sinopoli, "Network lifetime maximization via sensor selection," in *Proceedings of the Asian Control Conference*, 2009, pp. 441–446.

[3] R. Ambrosino, B. Sinopoli, and K. Poolla, "Optimal sensor scheduling for remote estimation over wireless sensor networks," *Modelling, Estimation and Control of Networked Complex Systems*, vol. 1, pp. 127–142, 2009.

[4] V. Gupta, T. Chung, B. Hassibi, and R. M. Murray, "On a stochastic sensor selection algorithm with applications in sensor scheduling and dynamic sensor coverage," *Automatica*, vol. 42, no. 2, pp. 251–260, 2006.

[5] Y. Chen, Q. Zhao, V. Krishnamurthy, and D. Djonin, "Transmission scheduling for optimizing sensor network lifetime: A stochastic shortest path approach," *IEEE Transactions on Signal Processing*, vol. 55, no. 5, pp. 2294 –2309, May 2007.

[6] L. Shi, P. Cheng, and J. Chen, "Sensor data scheduling over a packet-dropping network," in *the 8th World Congress on Intelligent Control and Automation*, July 2010, pp. 933–937.

[7] K. J. Åström and B. M. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2002, pp. 2011–2016.

[8] O. Imer and T. Başar, "Optimal estimation with limited measurements," in *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference*, December 2005, pp. 1029 – 1034.

[9] L. Li, M. Lemmon, and X. Wang, "Event-triggered state estimation in vector linear processes," in *Proceedings of the American Control Conference*, July 2010, pp. 2138 –2143.

[10] R. Cogill, "Event-based control using quadratic approximate value functions," in *Proceedings of the 48th IEEE Conference on Decision and Control held jointly with the 28th Chinese Control Conference*, December 2009, pp. 5883 –5888.

[11] L. Shi, K. H. Johansson, and L. Qiu, "Time and event-based sensor scheduling for networks with limited communication resources," in *World Congress of the International Federation of Automatic Control (IFAC)*, vol. 18, no. 1, 2011, pp. 13 263–13 268.

[12] B. Anderson and J. Moore, *Optimal Filtering*. Prentice Hall, 1979.