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Distributed event-triggered control for non-reliable networks ☆

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Abstract

This paper presents a distributed event-based control approach to cope with communication delays and packet losses affecting a networked dynamical system. Two network protocols are proposed to deal with these communication effects. The stability of the system is analyzed for constant and time-dependent trigger functions, showing that asymptotic stability can be achieved with the latter design, and this also guarantees a lower bound for the inter-event times. Analytical expressions for the delay bound and the maximum number of consecutive packet losses are derived for different scenarios. Finally, the results are illustrated through a simulation example.

1. Introduction

Event-triggered control has been developed to reduce the need for feedback while guaranteeing certain levels of performance and it has been proposed in Networked Control Systems (NCS) for allowing a more efficient usage of the limited network bandwidth [1–5].

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There is a natural interest in applying these techniques to decentralized NCS since the design of a centralized controller is inappropriate for a large number of subsystems due to the overload of the network by requesting and sending information from/to each node.

There are some recent contributions on distributed event-triggered control [6–12]. The basic idea is that each subsystem (also called agent or node) decides when to transmit the measurements based only on local information. Different approaches can be found in the literature such as deadband control [2], Lyapunov approaches to event-based control [3,13] or self-triggered control [14,15].

Even though event-based control has been shown to reduce the communication to face the problem of reduced bandwidth, network delays and packet losses cannot be avoided [16]. However, up to now, only a few papers have considered the effect of these issues on event-based control and even less have addressed a decentralized implementation.

Early papers [1,4] study simple stochastic systems and investigate the event-based control performance in dependence upon the medium access mechanism applied.

In [17,18], delays are compensated by model-based event-triggered approaches and the measurement of the delay. However, these schemes are difficult to implement in a distributed scenario since measuring transmission delays between two nodes requires clock synchronization in the entire network.

In distributed control, delays and packet losses are taken into account by [12]. As stated in this paper, one problem that might present trigger functions of the form $\|e(t)\| \leq \sigma\|x(t)\|$ is that for unreliable networks a lower bound for the broadcasting period cannot be guaranteed when the system approaches the origin, which is the main drawback of this approach. An extension of it has been presented in [19]. To avoid the Zeno behavior, the authors propose to define a minimal transmission period, so that if the time between two consecutive events is below it, the new event is ignored.

The proposed approach in this paper does not require this constraint to guarantee lower bounds for the minimal inter-event times nor clock synchronization between the nodes.

This paper extends our previous work [20], in which the problem of non-reliable networks is addressed for perfect decoupling, i.e., when the control law is able to perfectly compensate the effect of the coupling between neighboring subsystems, and for constant trigger functions. A network protocol is designed so that the synchronous update of all the nodes in a given neighborhood is not required, in contrast to [12]. Under certain requirements, upper bounds on the allowable delay and the maximum number of consecutive packet losses can be derived.

However, constant trigger functions do not allow asymptotic convergence to the equilibria. To address this, time-dependent trigger functions are proposed in this paper. We prove that the system is asymptotically stable and that the Zeno behavior is excluded with the proposed design. This is in contrast to trigger rules of the form $\|e(t)\| \leq \sigma\|x(t)\|$, which might not exclude the Zeno behavior [12]. Moreover, it is illustrated how time-dependent trigger functions can provide larger upper bounds on the delay than constant thresholds. Finally, the results are extended to the non-perfect decoupling.

The rest of the paper is organized as follows: Section 2 gives some insights of the problem of non-reliable networks in distributed event-triggered control. Section 3 presents two protocols to handle delays and packet losses. The performance of the system for perfect and non-perfect decoupling is analyzed in Sections 4 and 5, respectively. The results are illustrated through an example in Section 6. Finally, Section 7 presents the conclusions of the paper.

2. Preliminaries

2.1. System description

Consider the linear interconnected system

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \sum_{j \in N_i} H_{ij} x_j(t), \quad \forall i = 1, \dots, N, \tag{1}$$

and the control law

$$u_i(t) = K_i x_{b,i}(t) + \sum_{j \in N_i} L_{ij} x_{b,j}(t), \quad \forall i = 1, \dots, N. \tag{2}$$

where N_i is the set of "neighbors" of the subsystem i, i.e., the set of subsystems that directly drive agent i's dynamics, and H_{ij} is the interaction term between agent i and agent j, and $H_{ij} \neq H_{ji}$ might hold. The state x_i and the broadcasted state $x_{b,i}$ of the ith agent have dimension n_i, u_i is the m_i-dimensional local control signal of agent i, and A_i, B_i and H_{ij} are matrices of appropriate dimensions. K_i is the feedback gain for the nominal subsystem i. We assume that $A_i + B_i K_i$ is Hurwitz. L_{ij} is a set of decoupling gains.

Each agent i sends its state through the network at discrete time instances. Specifically, the agent i can only communicate with the set of agents on its neighborhood N_i. The transmission occurs when an event is triggered. We denote by $\{t_k^i\}_{k=0}^{\infty}$ the times at which an event is detected in the agent i, where $t_k^i < t_{k+1}^i$ for all k.

Remark 1. The control law (2) considers $x_{b,i}$ instead of the continuous state x_i so that the control signal is only updated at event times. Reducing actuation is important because some actuators are subject to wear. After some time in operation, this wear may result in phenomena that deteriorate the control performance, such as friction or hysteresis in mechanical actuators [21]. Continuous update of the control law is also less efficient in terms of energy waste.

2.2. Ideal vs. non-ideal network

In an ideal network scenario, the detection of an event, the broadcast of the corresponding state $x_{b,i}$, and its reception in all neighboring nodes are assumed to be simultaneous. If we define the error $e_i(t) = x_{b,i}(t) - x_i(t)$, Eq. (1) can be rewritten in terms of $e_i(t)$ and the control law (2), and we obtain

$$\dot{x}_i(t) = A_{K,i} x_i(t) + B_i K_i e_i(t) + \sum_{j \in N_i} \left(\Delta_{ij} x_j(t) + B_i L_{ij} e_j(t) \right), \tag{3}$$

where $A_{K,i} = A_i + B_i K_i$, and $\Delta_{ij} = B_i L_{ij} + H_{ij}$ are the coupling terms. In general, $\Delta_{ij} \neq 0$ since the interconnections between the subsystems may be not well known, there might be model uncertainties or the matrix B_i does not have full rank.

Definition 1. We say that the system is *perfectly decoupled* if $\Delta_{ij} = 0, \forall i, j = 1, \dots, N$, and non-perfectly decoupled otherwise.

For the ideal network, it also holds that $\dot{e}_i(t) = -\dot{x}_i(t), \ \forall t \in [t_k^i, t_{k+1}^i)$, since $x_{b,i}$ remains constant in the inter-event time.

However, in a non-reliable network, a broadcasted state may be received in the neighbors with delay, or even more, not be received at all. This may yield *state inconsistency*. In this context, this concept was introduced for the first time by [12].

Definition 2. A distributed event-based control design preserves *state consistency* if any broadcasted state is updated synchronously in each neighboring agent.

Example 1. In Fig. 1 an example of state inconsistency is presented. Assume that the piecewise constant signal $x_{b,1}$ is updated at event times denoted by t_k^1, $k \in \mathbb{N}$, and sent through the network to update the copy of the signal $x_{b,1\to 2}$ accordingly. We denote by $\tau_k^{1\to 2}$, $k \in \mathbb{N}$, the communication delay experienced in the broadcast. If the transmission was not subject to delay, both signals $x_{b,1}$ and $x_{b,1\to 2}$ would be identical. However, this is not the situation in the example of Fig. 1. In the time intervals $[t_1^1, t_1^1 + \tau_1^{1\to 2})$ and $[t_2^1, t_2^1 + \tau_2^{1\to 2})$ both signals are not equal. Hence, there is a state inconsistency since $x_{b,1}(t) \neq x_{b,1\to 2}(t) \quad \forall t \in [t_1^1, t_1^1 + \tau_1^{1\to 2}), \cup [t_2^1, t_2^1 + \tau_2^{1\to 2})$.

Therefore, a communication protocol should be defined to avoid state inconsistencies or to suitably deal with them. In this paper, two different protocols are proposed. The first one is designed to preserve state consistency by the transmission of additional signals to synchronize the nodes in the neighborhood. This constraint is relaxed by the second protocol which allows the neighboring agents to use different versions of the broadcasted states.

2.3. Trigger functions

The occurrence of an event, i.e., a broadcast over the network and a control law update, is defined by the trigger functions f_i which depend on local information of agent i only and take values in $\mathbb{R}$. The sequence of broadcasting times t_k^i is determined recursively by the event trigger function as $t_{k+1}^i = \inf\{t : t > t_k^i, f_i(t) > 0\}$. Particularly, we consider trigger functions of the form

$$f_i(t, e_i(t)) = \| e_i(t) \| - (c_0 + c_1 e^{-\alpha t}), \alpha > 0 \tag{4}$$

where $c_0 \geq 0, c_1 \geq 0$ but both parameters cannot be zero simultaneously.

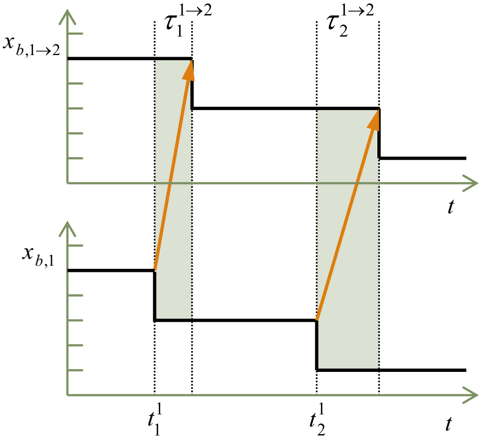

Fig. 1. Example of state inconsistency of the signal $x_{b,1}$ and its copy $x_{b,1\to 2}$ in other node of the network.

This type of trigger functions has been motivated in [11] and [9] for multi-agent systems and interconnected systems, respectively. Whereas constant trigger functions, that is, $c_1 = 0$, have been vastly studied in the literature, see e.g. [2,5], showing a trade-off between performance and number of generated events, time-dependent trigger functions (4) can give good performance while decreasing the number of events and guaranteeing a minimum inter-event time even if $c_0 = 0$, if the parameters are adequately selected. This last property, i.e., the exclusion of the Zeno behavior is not guaranteed, for instance, by event-triggering rules derived using Lyapunov analysis of the form $\|e_i(t)\| \le \sigma_i \|x_i(t)\|$ [12,22].

3. Transmission protocols

Before describing the proposed protocols, let us first introduce some notation.

Definition 3. Let us denote by $\tau_k^{i\to j}$ the delay in the transmission of the state $x_i(t_k^i)$ of agent i to its neighbor j, $j \in N_i$, at time t_k^i, and by $\overline{\tau}_k^i$

$$\overline{\tau}_k^i = \max\left\{\tau_k^{i\to j}, j \in N_i\right\}.$$

Definition 4. Let us denote by $P_{i\to j}^k$ the number of successive packet losses in the transmission of the state $x_i(t_k^i)$ of agent i to its neighbors j, $j \in N_i$, in the interval of time $[t_k^i, t_{k+1}^i)$, and by P_i^k the maximum of $P_{i\to j}^k$ for all $j \in N_i$.

We now introduce the basic assumption that imposes constraints on delays and the number of consecutive packet dropouts.

Assumption 1. We assume that the maximum delay and the number of successive packet dropouts which occur in the transmission of information from the subsystem i to its neighbors $j \in N_i$, denoted by $(\tau^\star)^i$ and $P_i^\star$, respectively, are such that no event is generated before all the neighbors have successfully received the broadcasted state $x_{b,i}$.

Later in the paper we provide bounds on the delay and consecutive packet losses which guarantee that this assumption holds.

The second important consideration is that the sender i knows that the data has been successfully received by j by getting an acknowledgment signal (ACK). If an ACK is not received before a *waiting time* denoted by T_W^i, the packet is treated as lost. How to determine T_W^i is analyzed later on, but it seems logical to set this value larger than the maximum delay.

If agent i has not received an acknowledgment of the reception of all the neighbors after the waiting time T_W^i, we propose two alternatives denoted by *Wait for All* (WfA) and *Update when Receive* (UwR).

3.1. WfA protocol

The state at $t_k^i + T_W^i$ is broadcasted again to all the neighbors. If after waiting T_W^i an ACK is not received from all $j \in N_i$, the retransmission takes place again, and so on. This process can occur at most $P_i^\star + 1$ times. Once all the neighbors have successfully received the data, agent i sends a *permission signal* (PERM) so that the previously transmitted data can be used to update the control law (2). Both signals ACK and PERM are assumed to be delivered with a delay approximated by zero over a reliable channel. This assumption is similar to the "pool", and "request" and "warning" signals used in [23] and [24], respectively.

A very similar protocol is presented in [12]. As stated there, the reason to use a PERM signal and to retransmit the state to all the neighbors instead of only retransmitting to those from which an ACK signal has not been received, is to preserve the *state consistency* (see Definition 2). Since the broadcasted data is not valid until a PERM signal is received from agent i, all the neighboring agents update the value at the same time and therefore, the value of the error e_i is the same in all nodes. This allows us to define stack vectors for the state $x(t)$ and the error signal $e(t)$ so that the stability of the overall system can be studied as in the ideal network case.

3.2. UwR protocol

The previous protocol simplifies the analysis but it has some drawbacks. First, all nodes in the neighborhood have to wait for the slower connection (longer delay) to process the received data. Secondly, the WfA protocol may involve unnecessary transmissions, since if an agent did not receive the measurement, the broadcast would take place again with an updated measurement for all the neighbors. Finally, the ACK signal is vastly used in network protocols to guarantee reliability of packet transfers, but the PERM demands a more involved communication protocol. In order to overcome these drawbacks, in the new protocol:

- Agent i waits T_W^i to get acknowledgments from the neighbors. To those agents $j \in N_i$ from which it did not receive the ACK signal, it sends the state $x_i(t_k^i + T_W^i)$ at time $t_k^i + T_W^i$. Agent i may transmit before the next event at most $P_i^\star + 1$ times.
- Let us denote by $\mathcal{N}_i(t) \subseteq N_i$ the agents to which the subsystem i transmits information at time t. In contrast to WfA, agent i only transmits a new measurement to those agents from which it has not received the ACK signal. If the last event occurred at time t_k^i and $t \in [t_k^i, t_{k+1}^i)$, thus

  $$\forall j \in N_i, \notin \mathcal{N}_i(t) \quad \exists t_k^{i \to j} : t_k^i \le t_k^{i \to j} < t,$$

  where $t_k^{i \to j}$ is the time of the successful broadcast to agent j. Hence if at time t the node j is not in $\mathcal{N}_i(t)$ it means that it has correctly received a broadcasted state after the occurrence of the last event and it has confirmed this reception with an ACK signal.
- The number of consecutive packet losses and the network delay are upper-bounded for each agent i, according to Assumption 1. Thus, it must hold

  $$\mathcal{N}_i((t_{k+1}^i)^-) = \varnothing,$$

  where $(t_{k+1}^i)^-$ refers to the instant time before t_{k+1}^i. I.e., all neighbors have successfully received the state of agent i before the next event occurrence.

Example 2. In order to clarify the difference between both protocols, a simple example is given in Fig. 2.

A system with two agents is depicted. Assume that Agent 1 detects an event at time t_k^1 and broadcasts its state $x_1(t_k^1)$ to its neighbor Agent 2. The transmission is delayed by τ_k^1 and Agent 2 sends then the ACK signal. In the scenario of WfA protocol, once the ACK signal is received by Agent 1 (see Fig. 2a), the PERM signal is sent (both signals are assumed to be sent and received instantaneously), and both agents update the broadcasted state in the control law at the same time

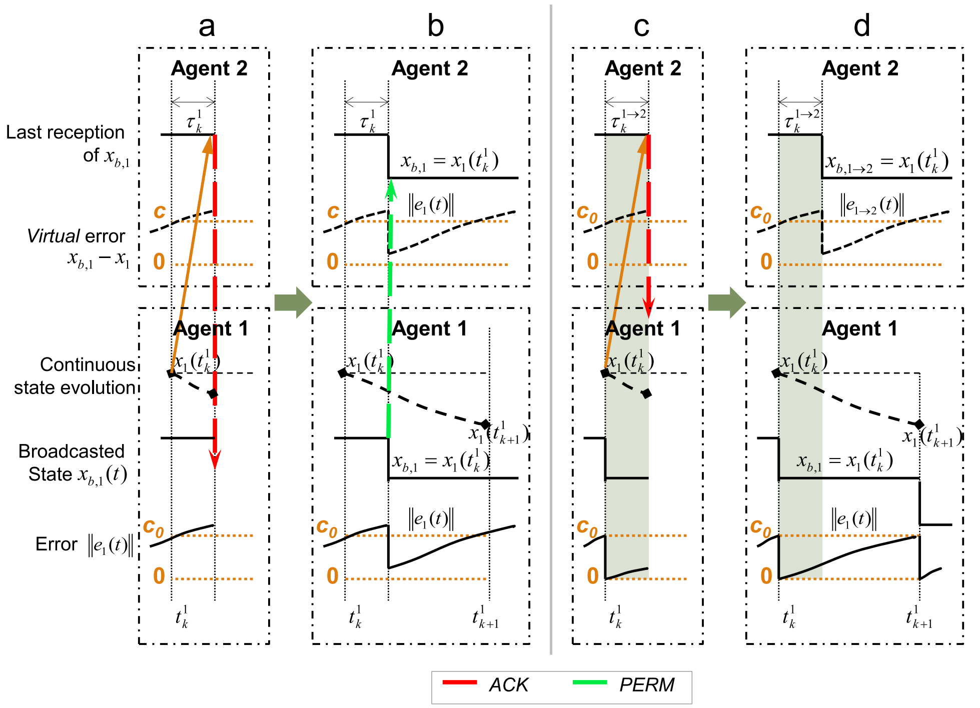

Fig. 2. Update mechanism of WfA (a) and b)) and UwR (c) and d)) protocols.

$t_k^1 + \tau_k^1$ (see Fig. 2b). Thus, $x_{b,1}$ takes the same value at any time in both agents and, hence, $e_1(t)$ is the same in the dynamics of Agent 1 and 2.

For the UwR protocol, the update in Agent 1 is applied immediately at time t_k^1 (see Fig. 2c), whereas the receiver updates the state information at time $t_k^1 + \tau_k^{1\to 2}$ (τ_k^1 and $\tau_k^{1\to 2}$ are the same), as illustrated in Fig. 2d. Thus, in the interval $[t_k^1, t_k^1 + \tau_k^{1\to 2})$ the broadcasted state $x_{b,1}$ has different values in the two nodes and consequently the error e_1 considered in Agent 1 differs from the error affecting the dynamics of Agent 2.

Note that Agent 2 does not monitor e_1 since it only knows the state of Agent 1 at event times. It is drawn in the figure to clarify the difference between the two protocols.

The performance of both protocols is analyzed next. We first assume that perfect decoupling ($\Delta_{ij} = 0$) can be achieved, since the analysis is simplified and moreover, upper bounds on the delay and packet losses can be derived for each agent, giving less conservative results. The results for the general case when the matching condition does not hold are derived afterwards.

4. Performance analysis for perfect decoupling

We firstly investigate the performance of the event-based control with constant threshold obtained by using the WfA protocol. After that, we extend these results to the situation which uses UwR. Finally, we discuss an extension to time-dependent trigger functions.

In the sequel, the following assumption holds.

Assumption 2. We assume that $A_{K,i}$, $i=1,\dots,N$ is diagonalizable so that there exists a matrix $D_i=\text{diag}(\lambda_k(A_{K,i}))$ and an invertible matrix of eigenvectors V_i such that $A_{K,i}=V_iD_iV_i^{-1}$. Moreover, we denote $\lambda_{max}(A_{K,i})=\max\{\Re e(\lambda):\lambda\in\lambda(A_{K,i})\}$ and $\kappa(V_i)=\|V_i\|\,\|V_i^{-1}\|$.

This assumption facilitates the calculations, but the extension to general Jordan blocks is achievable as discussed in [9].

4.1. Properties of event-triggered control using WfA protocol

Let us consider trigger functions (4) with $c_1=0$ and $c_0>0$, that is

$$f_i(e_i(t))=\|e_i(t)\|-c_0, \quad c_0>0. \tag{5}$$

Let us first assume that the communication can only experience delays but no packet dropouts.

4.1.1. Communication with delays

Proposition 1. *Let us consider trigger functions of the form (5) and the WfA protocol. If Assumption 1 holds, the error of any subsystem i is upper bounded by* $\|e_i(t)\|<2c_0$.

Proof. Assume that the last event occurred at time t_k^i and that the maximum transmission delay to its neighbors is $\overline{\tau}_k^i$. It follows that

$$\|e_i(t_k^i+\overline{\tau}_k^i)-e_i(t_k^i)\|<c_0, \tag{6}$$

has to be satisfied (see Eq. (5)) because no event is generated in the time interval $[t_k^i,t_{k+1}^i)$ from Assumption 1. Since an event has occurred at time t_k^i, $\|e_i(t_k^i)\|=c_0$ holds and, from Eq. (6) it holds $\|e_i(t_k^i+\overline{\tau}_k^i)\|<2c_0$, which is independent of the broadcasting time t_k^i and, hence, valid for all time. □

The previous result allows stating the next theorem. An analytical upper bound on the delay is derived, which is also the lower bound on the inter-event time, while the convergence of $x_i(t)$ to a region around the equilibrium is guaranteed.

Remark 2. Assumption 1 imposes that $\overline{\tau}_k^i<t_{k+1}^i-t_k^i$, $\forall k$, and this has allowed to establish a bound on the error $e_i(t)$, for all t. Alternatively, the upper bound on the delay could be set to an arbitrary integer number ν of minimum inter-event times:

$$\overline{\tau}_k^i<\nu(t_{k+1}^i-t_k^i), \quad \nu\in\mathbb{N}.$$

Thus, an equivalent result to Proposition 1 would be derived:

$$\|e_i(t)\|\le(\nu+1)c_0.$$

Note, however, that if the error was increased, the performance would degrade. The analytical results will illustrate this trade-off between maximum delay and performance.

Theorem 2. *If the network delay is upper bounded by*

$$(\tau^\star)^i=\frac{c_0}{\|A_{K,i}\|\kappa(V_i)\|x_i(0)\|+\mu_i\left(1+\dfrac{\|A_{K,i}\|\kappa(V_i)}{|\lambda_{max}(A_{K,i})|}\right)2c_0}, \tag{7}$$

where $\mu_i = \|B_iK_i\| + \sum_{j\in N_i}\|B_iL_{ij}\|$, *then any broadcasted state* $x_{b,i}$ *of any subsystem* $i \in 1,\dots,N$ *is successfully received by the neighbors* $j\in N_i$ *before a new event occurs, and the inter-event time is lower bounded* $t^i_{k+1} - t^i_k \geq (\tau^\star)^i$.

Moreover, for all initial conditions $x_i(0)$ *and* $t>0$ *it holds*

$$\|x_i(t)\| \leq \kappa(V_i)\left(\frac{\mu_i 2c_0}{|\lambda_{max}(A_{K,i})|} + e^{-|\lambda_{max}(A_{K,i})|t}\left(\|x_i(0)\| - \frac{\mu_i 2c_0}{|\lambda_{max}(A_{K,i})|}\right)\right). \tag{8}$$

Proof. In order to prove the theorem, let us assume that Assumption 1 holds.

The analysis will derive an upper bound for the delay which preserves this assumption. The error in the time interval $[t^i_k, t^i_k + \overline{\tau}^i_k)$ is given by

$$e_i(t^i_k + \overline{\tau}^i_k) - e_i(t^i_k) = x_i(t^i_k) - x_i(t^i_k + \overline{\tau}^i_k),$$

since the broadcasted state $x_{b,i}$ is not updated in any agent before the time instance $t^i_k + \overline{\tau}^i_k$ according to the WfA protocol, so that $x_{b,i}(t^i_k + \overline{\tau}^i_k) = x_{b,i}(t^i_k) = x_i(t^i_{k-1})$ holds. This yields

$$e_i(t^i_k + \overline{\tau}^i_k) - e_i(t^i_k) = \left(I - e^{A_{K,i}\overline{\tau}^i_k}\right)x_i(t^i_k) + \int_0^{\overline{\tau}^i_k} e^{A_{K,i}s}\left(B_iK_ie_i(s) + B_i\sum_{j\in N_i} L_{ij}e_j(s)\right)ds,$$

based on which the upper bound for the delay $\overline{\tau}^i_k$ can be derived as

$$(\tau^\star)^i_k = \arg\min_{\overline{\tau}^i_k \geq 0}\left\{\left\|\left(I - e^{A_{K,i}\overline{\tau}^i_k}\right)x_i(t^i_k) + \int_0^{\overline{\tau}^i_k} e^{A_{K,i}s}\left(B_iK_ie_i(s) + B_i\sum_{j\in N_i} L_{ij}e_j(s)\right)ds\right\| = c_0\right\}.$$

Note that this bound depends on $x_i(t^i_k)$. In order to guarantee the existence of the bound for the delay, we need to find an upper bound of the state for any t^i_k. The state at any time is given by $x_i(t) = e^{A_{K,i}t}x_i(0) + \int_0^t e^{A_{K,i}(t-s)}\left(B_iK_ie_i(s) + B_i\sum_{j\in N_i}L_{ij}e_j(s)\right)ds$. The error is bounded by $\|e_i(t)\| < 2c_0, \forall i$ by Proposition 1. Thus, a bound on $x_i(t)$ can be calculated following the methodology of [9] as Eq. (8).

Note that Eq. (8) is upper bounded by

$$\|x_i(t)\| \leq \kappa(V_i)\left(\frac{\|B_iK_i\|2c_0 + (\sum_{j\in N_i}\|B_iL_{ij}\|)2c_0}{|\lambda_{max}(A_{K,i})|} + \|x_i(0)\|\right), \quad \forall t, \tag{9}$$

if the negative terms are omitted, and using that $e^{-|\lambda_{max}(A_{K,i})|t} \leq 1,\ \forall t \geq 0$.

In order to derive an upper bound for the delay for any t, we recall that

$$\dot{e}_i(t) = -A_{K,i}x_i(t) - B_iK_ie_i(t) - \sum_{j\in N_i} B_iL_{ij}e_j(t)$$

holds in the interval $t \in [t^i_{k-1} + \overline{\tau}^i_{k-1}, t^i_k + \overline{\tau}^i_k)$ for any two consecutive events t^i_{k-1}, t^i_k, and, in particular, it holds in the subinterval $[t^i_k, t^i_k + \overline{\tau}^i_k) \subset [t^i_{k-1} + \overline{\tau}^i_{k-1}, t^i_k + \overline{\tau}^i_k)$. Hence, $\dot{e}_i(t)$ can be bounded as

$$\begin{aligned}\|\dot{e}_i(t)\| &= \|A_{K,i}x_i(t) + B_iK_ie_i(t) + \sum_{j\in N_i} B_iL_{ij}e_j(t)\| \\ &\leq \|A_{K,i}\|\,\|x_i(t)\| + \|B_iK_i\|\,\|e_i(t)\| + \sum_{j\in N_i}\|B_iL_{ij}\|\,\|e_j(t)\|.\end{aligned} \tag{10}$$

The state $x_i(t)$ can be bounded according to Eq. (9), and for the error it holds that $\| e_i(t) \| < 2c_0$ (see Proposition 1). Thus, Eq. (10) can be easily integrated in the interval $[t_k^i, t_k^i + \overline{\tau}_k^i)$, and it yields

$$\| e_i\left(t_k^i + \overline{\tau}_k^i\right) - e_i\left(t_k^i\right) \| \leq \left(\| A_{K,i} \| \kappa(V_i) \left(\| x_i(0) \| + \frac{(\| B_i K_i \| + \sum_{j \in N_i} \| B_i L_{ij} \|) 2c_0}{|\lambda_{max}(A_{K,i})|} \right) \right.$$
$$\left. + (\| B_i K_i \| + \sum_{j \in N_i} \| B_i L_{ij} \|) 2c_0 \right) \overline{\tau}_k^i.$$

Thus, the delay bound (7) for agent i ensures that Assumption 1 is not violated, and this concludes the proof. □

Remark 3. Note that Eq. (7) sets the maximum allowable delay for a given c_0, i.e., a certain level of performance. An alternative way to proceed would be to obtain the minimum value of c_0 so that the system could tolerate a given delay bound $(\tau^\star)^i$ (which would be, of course, constrained somehow by the dynamics of the system).

4.1.2. Communication with delays and packet losses

The previous analysis was focused on the effect of delays exclusively. However, in practice, delays and packet losses may occur simultaneously.

Corollary 3. *Assume that the maximum number of consecutive packet losses is upper bounded by $P_i^\star$, and the transmission delay τ_k^i is upper bounded by a constant $\overline{\tau}^i$ given by*

$$\overline{\tau}^i = \frac{(\tau^\star)^i}{P_i^\star + 1}, \tag{11}$$

where $(\tau^\star)^i$ is given by Eq. (7). Assume also that the waiting time T_W^i of the WfA protocol is set to $\overline{\tau}^i$. Then, there is a successful broadcast before the occurrence of a new event and the state of each agent i is bounded by Eq. (8).

Proof. Assuming that an event was triggered at time t_k^i, the accumulated error after $P_i^\star$ consecutive packet losses and a transmission delay $\overline{\tau}_k^i \leq \overline{\tau}^i$ is

$$\underbrace{(e_i(t_k^i + T_W^i) - e_i(t_k^i)) + (e_i(t_k^i + 2T_W^i) - e_i(t_k^i + T_W^i)) + \ldots}_{P_i^\star \, times}$$
$$+ (e_i(t_k^i + P_i^\star T_W^i + \overline{\tau}_k^i) - e_i(t_k^i + P_i^\star T_W^i))$$
$$= e_i(t_k^i + P_i^\star T_W^i + \overline{\tau}_k^i) - e_i(t_k^i). \tag{12}$$

Since $P_i^\star T_W^i + \overline{\tau}_k^i \leq P_i^\star T_W^i + \overline{\tau}^i = (P_i^\star + 1)\overline{\tau}^i = (\tau^\star)^i$, and $(\tau^\star)^i$ is also the minimum inter-event time for the system, this implies that $\| e_i(t_k^i + P_i^\star T_W^i + \overline{\tau}_k^i) - e_i(t_k^i) \| < c_0$. Hence, $\| e_i(t) \| < 2c_0$ holds and so does the bound (8). □

Remark 4. Note that the maximum number of consecutive packet dropouts $P_i^\star$ and the maximum tolerable delay $\overline{\tau}^i$ are correlated. A large value of $P_i^\star$ implies small values of $\overline{\tau}^i$ and vice versa. This way, there is a trade-off between both parameters.

4.2. Properties of event-triggered control using UwR protocol

In this section we study the UwR protocol, where the main issue is to keep track of the different versions of the broadcasted states. First, some definitions are introduced to adapt the previous analysis to this new scenario.

Definition 5. We denote by $\{t_k^{i\to j}\}$ the set of successful broadcasting times from agent i to agents $j\in N_i$, and the error

$$e_{i\to j}(t)=x_{b,i\to j}(t_k^{i\to j})-x_i(t),\quad t\in[t_k^{i\to j},t_{k+1}^{i\to j}),\tag{13}$$

where $x_{b,i\to j}(t_k^{i\to j})$ is the last successful broadcasted state from agent i to agent j, $j\in N_i$.

With this definition, the dynamics of agent i is given by

$$\dot{x}_i(t)=A_{K,i}x_i(t)+B_iK_ie_i(t)+\sum_{j\in N_i}B_iL_{ij}e_{j\to i}(t)\tag{14}$$

with $e_i(t)$ the agent i's version of the error. We assume that agent i automatically updates its broadcasted state in its control law and does not need to wait to receive an acknowledgment of successful receptions from its neighbors. With these prerequisites the following theorem is obtained.

Theorem 4. *If the network delay is upper bounded by*

$$\overline{\tau}^i=\frac{(\tau^\star)^i}{P_i^\star+1},\tag{15}$$

where $P_i^\star$ is the maximum number of consecutive packet losses and

$$(\tau^\star)^i=\frac{c_0}{\|A_{K,i}\|\kappa(V_i)\|x_i(0)\|+\overline{\mu}_i\left(1+\dfrac{\|A_{K,i}\|\kappa(V_i)}{|\lambda_{max}(A_{K,i})|}\right)2c_0},\tag{16}$$

with $\overline{\mu}_i=\frac{1}{2}\|B_iK_i\|+\sum_{j\in N_i}\|B_iL_{ij}\|$, and the waiting time T_W^i of the UwR protocol is set to $\overline{\tau}^i$, then any broadcasted state $x_{b,i}$ is successfully received by all the neighbors of the subsystem i before a new event occurs. Moreover, the local inter-event times $t_{k+1}^i-t_k^i$ are lower bounded by Eq. (16), and for any initial condition $x_i(0)$ and for any $t>0$, it holds

$$\|x_i(t)\|\le\kappa(V_i)\left(\frac{\overline{\mu}_i2c_0}{|\lambda_{max}(A_{K,i})|}+e^{-|\lambda_{max}(A_{K,i})|t}\left(\|x_i(0)\|-\frac{\overline{\mu}_i2c_0}{|\lambda_{max}(A_{K,i})|}\right)\right).\tag{17}$$

Proof. According to the UwR protocol, $\|e_i(t)\|\le c_0$ holds and $e_i(t)\ne e_{i\to j}(t)$, in general. However, as Assumption 1 applies, $\|e_{i\to j}(t_k^{i\to j})-e_i(t_k^i)\|<c_0$ yields $\|e_{i\to j}(t)\|<2c_0$.

Thus, a bound on the state can be derived from Eq. (14) in a similar way as in Theorem 2 and Eq. (17) holds. The proof of the first part of the theorem can be obtained by following the proof of Theorem 2, since in the interval $[t_k^i,t_k^{i\to j})$ the state information $x_{b,i\to j}$ remains constant in the agent j, so that $\dot{e}_{i\to j}(t)=-\dot{x}_i(t)$ holds. If the error $e_{i\to j}(t)$ is integrated in the interval $[t_k^i,t_k^{i\to j})$ considering that the state is bounded by Eq. (17), and that the error is bounded as discussed above, then Eq. (16) is derived. Finally, Eq. (15) can be derived as in Corollary 3. □

Remark 5. Note that the delay bound for WfA and UwR protocols are different (see Eqs. (7), (16)). Since $\overline{\mu}_i<\mu_i$, under the same initial conditions UwR allows larger delays.

4.3. Time-dependent trigger functions

Let us consider trigger functions (4) with $c_0 = 0$ and $c_1 > 0$

$$f_i(t, e_i(t)) = \|e_i(t)\| - c_1 e^{-\alpha t}, \quad \alpha > 0. \tag{18}$$

The case $c_0, c_1 > 0$ is equivalent to having a constant threshold from the analytical point of view.

4.3.1. Performance of WfA protocol

Proposition 5. *Let us consider trigger functions of the form (18) and WfA protocol. If Assumption 1 holds, the error of any subsystem i is upper bounded by $\|e_i(t)\| < c_1(1 + e^{\alpha\tau^\star})e^{-\alpha t}$, where $\tau^\star > 0$ is the maximum transmission delay in the system.*

Proof. The proof can be found in Appendix A.

Note that the value of $\tau^\star$ is unknown. Its existence is assumed, and the following theorem will prove it, giving the expression to compute it numerically.

If the results of Propositions 1 and 5 are compared, it can be noticed that the error bound not only depends on time but on the maximum delay in the latter case. Moreover, for constant trigger functions the bound on the error is the double than for ideal networks, whereas for time-dependent trigger functions the ratio of these two bounds (reliable and non-reliable network) is always greater than 2 for $\tau^\star > 0$.

Theorem 6. *Let $\alpha < |\lambda_{max}(A_{K,i})|$, $\forall i = 1, \ldots, N$. If the network delay for any broadcast in the system (1) is upper bounded by*

$$\tau^\star = \min\{(\tau^\star)^i, \quad i = 1, \ldots, N\} \tag{19}$$

being $(\tau^\star)^i$ the solution of

$$\left(\frac{k_{1,i}}{c_1} + \frac{k_{2,i}}{c_1}\left(1 + e^{\alpha(\tau^\star)^i}\right)\right)(\tau^\star)^i = e^{-\alpha(\tau^\star)^i}, \tag{20}$$

and

$$k_{1,i} = \|A_{K,i}\|\kappa(V_i)\|x_i(0)\| \tag{21}$$

$$k_{2,i} = \left(\|A_{K,i}\|\kappa(V_i)\frac{1}{|\lambda_{max}(A_{K,i})| - \alpha} + 1\right)\mu_i c_1, \tag{22}$$

then any broadcasted state $x_{b,i}$ is successfully received by the neighbors $j \in N_i$ before a new event occurs. Hence, the inter-event times are lower bounded $t^i_{k+1} - t^i_k \geq \tau^\star$. Moreover, for all initial conditions $x_i(0)$ and $t > 0$ it holds.

$$\|x_i(t)\| \leq \kappa(V_i)\left(\frac{\mu_i c_1(1 + e^{\alpha\tau^\star})e^{-\alpha t}}{|\lambda_{max}(A_{K,i})| - \alpha} + e^{-|\lambda_{max}(A_{K,i})|t}\left(\|x_i(0)\| - \frac{\mu_i c_1(1 + e^{\alpha\tau^\star})e^{-\alpha t}}{|\lambda_{max}(A_{K,i})| - \alpha}\right)\right). \tag{23}$$

Proof. The proof can be found in Appendix A.

4.3.2. Performance of UwR protocol

Under this protocol, the system dynamics is given by Eq. (14). Note that equivalently to the results for constant threshold, it holds that $\|e_i(t)\| \le c_1 e^{-\alpha t}$ and $\|e_{i\to j}(t)\| < c_1(1+e^{\alpha\tau^\star})e^{-\alpha t}$, where $\tau^\star > 0$ is the upper bound on the delay derived in the next theorem.

Theorem 7. *Let $\alpha < |\lambda_{max}(A_{K,i})|$, $\forall i = 1,\dots,N$. If the network delay for any broadcast in the system (1) is upper bounded by*

$$\tau^\star = \min\{(\tau^\star)^i, \quad i = 1,\dots,N\} \tag{24}$$

being $(\tau^\star)^i$ the solution of

$$\left(\frac{k_{1,i}}{c_1} + \frac{k_{2,i}}{c_1} + \frac{k_{3,i}}{c_1}\left(1 + e^{\alpha(\tau^\star)^i}\right)\right)(\tau^\star)^i = e^{-\alpha(\tau^\star)^i}, \tag{25}$$

and

$$k_{1,i} = \|A_{K,i}\|\kappa(V_i)\|x_i(0)\| \tag{26}$$

$$k_{2,i} = \|B_iK_i\|\left(1 + \frac{\kappa(V_i)\|A_{K,i}\|}{|\lambda_{max}(A_{K,i})| - \alpha}\right)c_1 \tag{27}$$

$$k_{3,i} = \sum_{j\in N_i}\|B_iL_{ij}\|\left(1 + \frac{\kappa(V_i)\|A_{K,i}\|}{|\lambda_{max}(A_{K,i})| - \alpha}\right)c_1, \tag{28}$$

then any broadcasted state $x_{b,i}$ is successfully received by the neighbors $j \in N_i$ before a new event occurs. Hence, the inter-event times are lower bounded $t^i_{k+1} - t^i_k \ge \tau^\star$. Moreover, for all initial conditions $x_i(0)$ and $t > 0$ it holds

$$\|x_i(t)\| \le \kappa(V_i)\left(\frac{\overline{\mu}_i(\tau^\star)c_1 e^{-\alpha t}}{|\lambda_{max}(A_{K,i})| - \alpha} + e^{-|\lambda_{max}(A_{K,i})|t}\left(\|x_i(0)\| - \frac{\overline{\mu}_i(\tau^\star)c_1 e^{-\alpha t}}{|\lambda_{max}(A_{K,i})| - \alpha}\right)\right), \tag{29}$$

where $\overline{\mu}_i(\tau^\star) = \|B_iK_i\| + \sum_{j\in N_i}\|B_iL_{ij}\|(1+e^{\alpha\tau^\star})$.

Proof. The proof can be found in Appendix A.

Note that trigger functions (18) ensures the asymptotic convergence to the origin while guaranteeing a lower bound for the minimum inter-event time if the delay is below $\tau^\star$. This cannot be achieved if the triggering conditions are of the form $\|e_i(t)\| \le \sigma_i\|x_i(t)\|$, as pointed out in [12].

Remark 6. The solutions of Eqs. (20) and (25) has to be computed numerically. However, approximated solutions can be derived so that the value of $(\tau^\star)^i$ is given explicitly. For instance, for small values, if the following approximations are taken $e^{\alpha(\tau^\star)^i} \approx 1 + \alpha(\tau^\star)^i$ and $e^{-\alpha(\tau^\star)^i} \approx 1 - \alpha(\tau^\star)^i$, it yields

$$(\tau^\star)^i \approx \frac{k_{1,i} + 2k_{2,i} + \alpha c_1}{2k_{2,i}\alpha}\left(-1 + \sqrt{1 + \frac{4k_{2,i}\alpha c_1}{(k_{1,i} + 2k_{2,i} + \alpha c_1)^2}}\right).$$

Moreover, if c_1 is small enough compared to $k_{1,i}, k_{2,i}$, the square root can also be approximated as $\sqrt{1+x} \approx 1 + x/2$, and it results in

$$(\tau^\star)^i \approx \frac{c_1}{k_{1,i} + 2k_{2,i} + \alpha c_1}. \tag{30}$$

Note that from the expression above, the influence of α on $(\tau^\star)^i$ is little compared to the effect of c_1, since the value of c_1 is small compared to $k_{1,i}, k_{2,i}$.

5. Performance analysis for non-perfect decoupling

If perfect decoupling cannot be assumed, the formulation changes. In order to illustrate it, let us consider an ideal network first. As it has been shown in Section 2.2, the dynamics of each agent can be rewritten in terms of the error as

$$\dot{x}_i(t) = A_{K,i}x_i(t) + B_iK_ie_i(t) + \sum_{j \in N_i} \left(\Delta_{ij}x_j(t) + B_iL_{ij}e_j(t)\right).$$

Note that if $\Delta_{ij} \neq \mathbf{0}$, the dynamics of $\dot{x}_i(t)$ explicitly depends on $x_j(t)$, $\forall j \in N_i$. Thus, $\|x_i(t)\|$ cannot be upper bounded if $\|x_j(t)\|$ is not. But at the same time, the dynamics of $x_j(t)$ depends on the neighborhood, and then there is a vicious circle.

Hence, one possible solution to this problem is to rewrite the equations in terms of the overall system state and error as

$$\dot{x}(t) = (A_K + \Delta)x(t) + BKe(t), \tag{31}$$

where

$$A_K = \text{diag}(A_{K,1}, A_{K,2}, \ldots, A_{K,N}) \tag{32}$$

$$B = \text{diag}(B_1, B_2, \ldots, B_N) \tag{33}$$

$$K = \{K_{ij}\}, \quad K_{ij} = \begin{cases} K_i & \textit{if } i = j \\ L_{ij} & \textit{for } i \neq j \end{cases} \tag{34}$$

$$\Delta = \{\Delta_{ij}\}, \quad \Delta_{ij} = \mathbf{0} \quad \text{if } i = j \text{ or } j \notin N_i, \tag{35}$$

and $x = (x_1^T, x_2^T, \ldots, x_N^T)^T$, $e = (e_1^T, e_2^T, \ldots, e_N^T)^T$.

Let us assume that the communication is subject to delays and packet losses. If the state consistency is preserved, for instance if WfA protocol is considered, Eq. (31) holds because the update of broadcasted states is synchronized. Under certain assumptions on the error bound (e.g., Proposition 1), an equivalent analysis to the perfect decoupling case can be inferred for Eq. (31). However, if the state consistency cannot be guaranteed (UwR protocol), a different approach is required to handle the problem.

For the sake of simplicity, we next show the formulation which solves this situation for constant trigger functions (5).

5.1. Dealing with the state inconsistency

Let us recall the definition of the error (13). If perfect decoupling does not hold and the transmissions over the network are governed by the UwR protocol, the dynamics of each subsystem is given by

$$\dot{x}_i(t) = A_{K,i}x_i(t) + \sum_{j \in N_i} \Delta_{ij}x_j(t) + B_iK_ie_i(t) + \sum_{j \in N_i} B_iL_{ij}e_{j\to i}(t). \tag{36}$$

Let us define the following set of matrices

$$\mathrm{M}_i = B_i(L_{i1}\ L_{i2}\ \dots\ L_{ii-1}\ K_i\ L_{ii+1}\ \dots\ L_{iN}), \quad \forall i = 1, \dots, N, \tag{37}$$

with $L_{ij} = 0$ if $j \notin N_i$, and the matrix

$$\mathrm{M} = \begin{pmatrix} \mathrm{M}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathrm{M}_2 & \dots & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathrm{M}_N \end{pmatrix}, \tag{38}$$

where $\mathbf{0}$ is a $n \times nN$ matrix whose elements are all zero.

Denote by

$$\overrightarrow{e}_i^T = \left(e_{1\to i}^T\ e_{2\to i}^T\ \dots\ e_{i-1\to i}^T\ e_i^T\ e_{i+1\to i}^T\ \dots\ e_{N\to i}^T\right), \quad \forall i = 1, \dots, N, \tag{39}$$

with $e_{j\to i} = \mathbf{0}$ if $j \notin N_i$, and

$$\overrightarrow{e}^T = \left(\overrightarrow{e}_1^T\ \dots\ \overrightarrow{e}_N^T\right). \tag{40}$$

With these definitions, the dynamics of the overall system is

$$\dot{x}(t) = (A_K + \Delta)x(t) + \mathrm{M}\overrightarrow{e}(t). \tag{41}$$

Proposition 8. *If Assumption 1 holds and trigger functions (5) and the UwR protocol are considered, the error (40) is bounded by*

$$\|\overrightarrow{e}(t)\| \le c_0\sqrt{N + 4\sum_{i=1^N|N_i|}} = \overline{c}_0, \tag{42}$$

where $|N_i|$ denotes the number of elements of the set N_i.

Proof. From Eq. (40) it follows that

$$\|\overrightarrow{e}(t)\| \le \sqrt{\sum_{i=1}^{N} \|e_i(t)\|^2 + \sum_{i=1}^{N}\sum_{j \in N_i} \|e_{i\to j}(t)\|^2}.$$

Under the UwR protocol, $\|e_i(t)\| \le c_0$ and $\|e_{i\to j}(t)\| < 2c_0$ hold. It yields

$$\|\overrightarrow{e}(t)\| < \sqrt{\sum_{i=1}^{N} c_0^2 + \sum_{i=1}^{N}\sum_{j \in N_i} (2c_0)^2} = \sqrt{c_0^2(N + 4\sum_{i=1}^{N} |N_i|)},$$

which is equivalent to Eq. (42). □

The previous result shows that due to the state inconsistency, the bound on the error increases. For instance, if WfA protocol is used, the error is bounded by $\|e(t)\| < 2\sqrt{N}c_0$, which is a lower upper bound than Eq. (42). Otherwise, if the opposite is assumed, it follows that $\frac{3}{4}N > \sum_{i=1}^{N}|N_i|$

must hold by enforcing $\overline{c}_0 = c_0\sqrt{N+4\sum_{i=1}^{N}|N_i|} < 2\sqrt{N}c_0$. However, this cannot be satisfied for a connected topology.

Larger upper bounds on the error involve more conservative upper bounds on the maximum delay. Hence, it can be expected that the analytic results for the state inconsistency and non-perfect decoupling are more tight. The outcome is enounced in the next theorem.

Theorem 9. If the network delay is upper bounded by

$$\tau^{\star} = \frac{c_0}{\|A_K+\Delta\|\kappa(V)\|x(0)\| + \mu_{max}\left(1+\frac{\|A_K+\Delta\|\kappa(V)}{|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|}\right)\overline{c}_0}, \tag{43}$$

where $\mu_{max} = \max\{\|M_i\|, i=1,\dots,N\}$, then any broadcasted state $x_{b,i}$ is successfully received by the neighbors $j \in N_i$ before a new event occurs. Hence, the inter-event times are lower bounded $t_{k+1}^i - t_k^i \geq \tau^{\star}$. Moreover, for all initial conditions $x(0)$ and $t>0$ it holds

$$\|x(t)\| \leq \kappa(V)\left(\frac{\mu_{max}\overline{c}_0}{|\lambda_{max}(A_{K,i})|-\kappa(V)\|\Delta\|} + e^{-(|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|)t}\left(\|x(0)\| - \frac{\mu_{max}\overline{c}_0}{|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|}\right)\right). \tag{44}$$

Proof. The proof can be found in Appendix A.

Remark 7. The conservativism of Eq. (43) comes from the fact that the individual dynamics of the subsystems cannot be decoupled and the system has to be treated as a whole. However, this does not mean that the system, in practice, cannot tolerate longer delays, simply the analytical approach taken only guarantees stability for $\tau \leq \tau^{\star}$.

6. Simulation results

6.1. System description

The system considered is a collection of N inverted pendulums of mass m and length l coupled by springs with rate k. Each subsystem can be described by

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} - \frac{a_i k}{ml^2} & 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u_i + \sum_{j \in N_i} \begin{pmatrix} 0 & 0 \\ \frac{h_{ij}k}{ml^2} & 0 \end{pmatrix} x_j$$

where $x_i = (x_{i_1}\ x_{i_2})^T$, a_i is the number of springs connected to the ith pendulum and $h_{ij} = 1, \forall j \in N_i$ and 0 otherwise. K_i and L_{ij} gains are designed to decouple the system and place the poles at -1, -2. Therefore, $K_i = (-3\,ml^2\ a_i k - ml^2/4(8+4\,g/l))$ and $L_{ij} = (c-k\ 0)$.

The same system has been used in [9] to demonstrate the event-based control strategy assuming an ideal network.

6.2. Performance

To illustrate the theoretical results, the system behavior is investigated in three situations:

1. Ideal communication channel.
2. Non-ideal network using WfA protocol.
3. Non-ideal network using UwR protocol.

Consider that the number of subsystems is $N=4$ and the initial conditions are $x(0)=(-0.9425\ 0\ 1.0472\ 0\ 0.6283\ 0\ -1.4137\ 0)^T$. The upper bounds on the delay are computed for WfA and UwR protocols and for different values of the parameter c_0 of the trigger function (5). The results are illustrated in Table 1. Note that the difference between the value of $\tau^\star$ given by the two protocols increases with c_0 and that the UwR protocol always allows larger (less conservative) values on the delay.

Let $c_0=0.05$ and a delay generated randomly between zero and the corresponding upper bound specified in Table 1 (1.140 ms for WfA and 1.329 for UwR). The state of subsystem 2, the events time and the control input $u(t)$ are depicted in Fig. 3 for the three situations stated above. The behavior of the subsystem is similar in the three cases as the effect of delays in the performance is mitigated by means of the two proposed protocols.

Note that even though the delay does not significantly affect the performance, it has an impact on the sequence of events. This is an interesting property of event-based control, because the delay in one transmission affects the occurrence of future events.

Table 1
Delays bounds (7) and (16) for different values of c_0 and for WfA and UwR protocols.

c_0	0.01	0.02	0.05	0.1
$(\tau^\star)^i_{\text{WfA}}$ (ms)	0.347	0.613	1.140	1.624
$(\tau^\star)^i_{\text{UwR}}$ (ms)	0.363	0.666	1.329	2.054

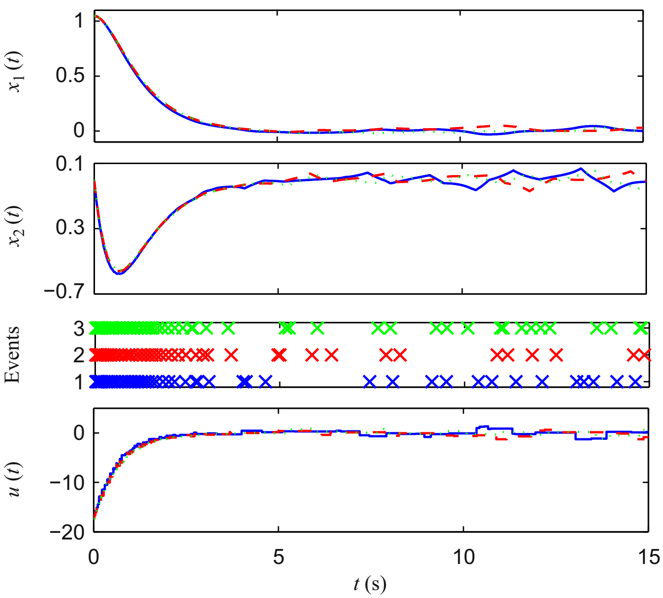

Fig. 3. Behavior of the subsystem 2 with the WfA (dashed red line) and the UwR (dotted green line) protocols, and a ideal network (solid blue line). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

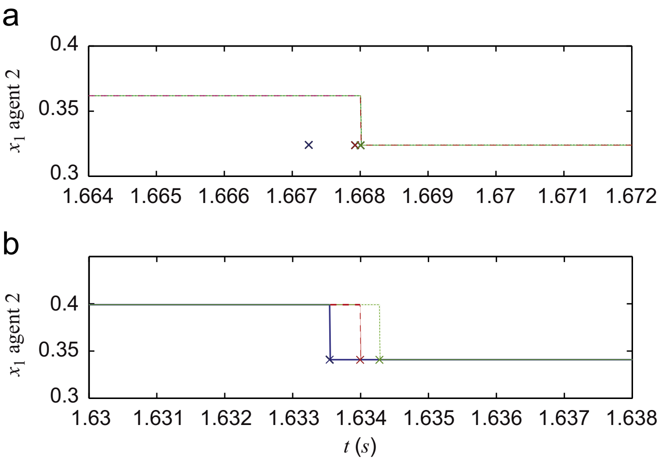

Fig. 4. Difference between a) WfA and b) UwR protocols in updating the state information. Only the first component of the state is depicted: $x_{b,2}$ (solid blue line), $x_{b,2\rightarrow 1}$ (dashed red line), and $x_{b,2\rightarrow 3}$ (dotted green line). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

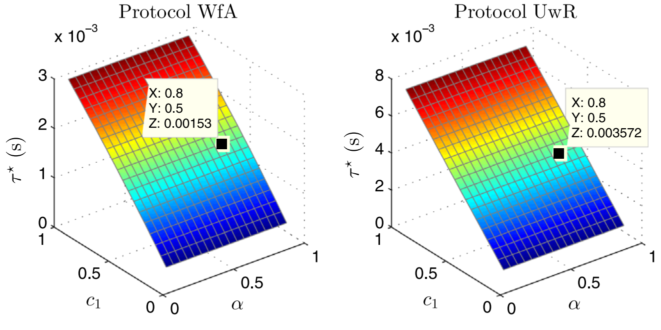

Fig. 5. Influence of c_1 and α on the delay bound (19) (left) and (24) (right). The case $c_1 = 0.5$, $\alpha = 0.8$ are 1.53 ms and 3.57 ms, respectively.

6.3. WfA vs. UwR

In order to illustrate the difference between WfA and UwR in more detail, Fig. 4 extracts a short time interval showing how the broadcasted state $x_{b,2}$ of Agent 2 is used in the system. Since Agent 2 is an inner pendulum, it has two neighbors. For WfA protocol the three copies of $x_{b,2}$ (one in Agent 2, one in Agent 1, and the third in Agent 3) are identical. All the neighbors wait for the last reception ($x_{b,2\rightarrow 3}$ in the depicted case) at time $t = 1.668$ s to update the value of x_b (Fig. 4a), which is depicted by the dashed green line. In contrast, using the UwR protocol (Fig. 4b), whenever an event is triggered in Agent 2, its state is broadcasted and immediately updated in Agent 2. The neighbors also update as soon as they receive the broadcasted state. Note that the events times are not the same in the two protocols because the time of one update affects the generation of future events, as mentioned before.

6.4. Time-dependent trigger function

Trigger functions (18) depend on two parameters c_1 and α. Fig. 5 depicts the bounds on the delay for a set of values of $c_1 \in [0.1, 1]$ and $\alpha \in [0.1, 0.95]$ so that $\alpha < |\lambda_{max}(A_K)| = 1$ is satisfied. The figure on the left shows the results for the WfA protocol (solution of Eq. (20)), and the one on the right for UwR (solution of Eq. (25)). Observe that $\tau^\star$ is always greater when the transmissions are ruled by the UwR protocol. In both cases, as commented on Remark 6, the influence of c_1 over the delay bound is much more appreciable than the variations on α.

If the solutions given for constant trigger functions and time-dependent trigger functions are compared, it can be noticed that the results are better in the second case. Furthermore, if we take the values of the parameters used in the previous section, constant thresholds ($c_0 = 0.02$) gives values of $\tau^\star$ around 0.6 ms, whereas for the exponential threshold ($c_1 = 0.5$ and $\alpha = 0.8$), $\tau^\star$ is three (WfA) and five times (UwR) greater.

It can be concluded that time-dependent trigger functions are a better choice because they provide asymptotic convergence and they also allow longer delays in the network.

Let us consider the case $c_1 = 0.5$ and $\alpha = 0.8$. The upper bound on the delay is 1.43 ms (WfA) and 3.57 ms (UwR), according to Fig. 5. The performance of the system under the time-dependent trigger functions is compared with the behavior using the static-trigger functions for $(\tau^\star)^i = 3.57$ ms. The results are shown in Fig. 6. The state of agent 2 (x_{2_1}, x_{2_2}) is depicted in Fig. 6a, and b shows the broadcasted states $(x_{b,2_1}, x_{b,2_2})$. The broadcasted state for the constant

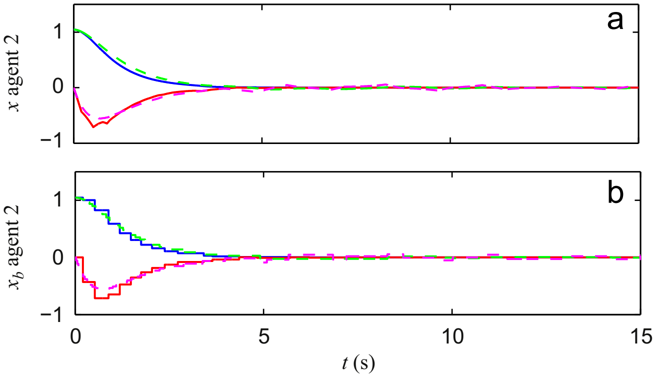

Fig. 6. Behavior of the agent 2 with trigger functions (5) ($c_0 = 0.05$) (dashed line) and (18) ($c_1 = 0.5$, $\alpha = 0.8$) (solid line), with 3.57 ms as upper bound on the delay. a) (x_{2_1}, x_{2_2}), b) $(x_{b,2_1}, x_{b,2_2})$. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

Table 2
Delays for different values of c_0 and N.

N	c_0			
	0.01	0.02	0.05	0.1
10	0.089	0.110	0.191	0.284
20	0.063	0.077	0.129	0.196
50	0.040	0.048	0.080	0.122
100	0.028	0.034	0.057	0.086
200	0.020	0.024	0.040	0.061

threshold looks like a continuous function due to the high frequency of events detection, whereas piecewise constant functions are clearly appreciated in the time-dependent trigger function case.

Note that the number of updates in the broadcasted state (number of events) decreases with trigger functions (18) and the performance around the equilibria is better with respect to Eq. (5). Moreover, the minimum and mean inter-event times have been computed according for these simulation results, resulting in 3.9 ms and 353 ms, respectively, for Eq. (5), and 1.2 ms, which agrees with the results of Table 1, and 215 ms for Eq. (18). Hence, the time-dependent trigger functions are an interesting alternative in non-ideal networks.

6.5. *Non-perfect decoupling*

The upper bound on the delay $\tau^\star$ computed according to Eq. (43) for different values of c_0 and N is given in Table 2. The values are expressed in milliseconds. Note that the resulting values are more conservative than in the perfect decoupling case. Moreover, given that $\overline{c}_0$ depends on N, the tolerable delay is reduced when the number of agents increases. This fact did not have influence in the case of perfect decoupling. The increase of the dimension of the matrices with N also influences the bound negatively.

7. Conclusions

This paper has presented an extension of the distributed control design of [9] to non-reliable networks. Two transmission protocols have been proposed as means of dealing with the effects of non reliable networks. Upper bounds on the delay and maximum number of consecutive packet dropouts have been derived for different situations. The existence of a lower bound on the inter-event times and the asymptotic convergence to the origin, if time-dependent trigger functions are used, have been proved. A simulation example has illustrated the theoretical results.

Appendix A. Proofs

A.1. *Proof of Proposition 5*

Assume that the last event occurred at time t_k^i and that the maximum transmission delay to its neighbors is $\overline{\tau}_k^i$. From Assumption 1, it follows that

$$\left\| \int_{t_k^i}^{t_k^i+\overline{\tau}_k^i} \dot{e}_i(s)\, ds \right\| = \| e_i(t_k^i+\overline{\tau}_k^i) - e_i(t_k^i) \| < c_1 e^{-\alpha(t_k^i+\overline{\tau}_k^i)}, \tag{A.1}$$

has to be satisfied (see Eq. (18)) because no event is generated in the time interval $[t_k^i, t_{k+1}^i)$. Since an event has occurred at time t_k^i, $\| e_i(t_k^i) \| = c_1 e^{-\alpha t_k^i}$ holds and, thus

$$\| e_i(t_k^i+\overline{\tau}_k^i) \| < c_1 e^{-\alpha t_k^i} + c_1 e^{-\alpha(t_k^i+\overline{\tau}_k^i)} = c_1 (1+e^{\alpha \overline{\tau}_k^i}) e^{-\alpha(t_k^i+\overline{\tau}_k^i)},$$

must hold. Because this result is valid for any time t and $e^{\alpha\overline{\tau}_k^i} < e^{\alpha\tau^\star}$, $\forall \overline{\tau}_k^i < \tau^\star$, it follows

$$\| e_i(t) \| < c_1 (1+e^{\alpha\tau^*}) e^{-\alpha t}.$$

A.2. Proof of Theorem 6

The state at any time is given by

$$x_i(t) = e^{A_{K,i}t}x_i(0) + \int_0^t e^{A_{K,i}(t-s)}\left(B_iK_ie_i(s) + B_i\sum_{j\in N_i}L_{ij}e_j(s)\right)ds.$$

According to Eq. (5), the error is bounded by $\|e_i(t)\| < c_1(1+e^{\alpha\tau^*})e^{-\alpha t}$. Thus, a bound on $x_i(t)$ can be calculated following the methodology of [9] as

$$\|x_i(t)\| \le \kappa(V_i)\left(\frac{\mu_i c_1(1+e^{\alpha\tau^\star})e^{-\alpha t}}{|\lambda_{max}(A_{K,i})|-\alpha} + e^{-|\lambda_{max}(A_{K,i})|t}\left(\|x_i(0)\| - \frac{\mu_i c_1(1+e^{\alpha\tau^\star})e^{-\alpha t}}{|\lambda_{max}(A_{K,i})|-\alpha}\right)\right),$$

which proves the second part of the theorem.

Note that Eq. (23) can be upper bounded as

$$\|x_i(t)\| \le \kappa(V_i)\left(\frac{\mu_i c_1(1+e^{\alpha\tau^\star})e^{-\alpha t}}{|\lambda_{max}(A_{K,i})|-\alpha} + e^{-|\lambda_{max}(A_{K,i})|t}\|x_i(0)\|\right). \tag{A.2}$$

Moreover, in the interval $t\in[t^i_{k-1}+\overline{\tau}^i_{k-1}, t^i_k+\overline{\tau}^i_k)$ it holds that

$$\dot{e}_i(t) = -A_{K,i}x_i(t) - B_iK_ie_i(t) - \sum_{j\in N_i}B_iL_{ij}e_j(t),$$

and this is particularly true in the subinterval $[t^i_k, t^i_k+\overline{\tau}^i_k)$. Thus

$$\begin{aligned}\|\dot{e}_i(t)\| &= \left\|A_{K,i}x_i(t) + B_iK_ie_i(t) + \sum_{j\in N_i}B_iL_{ij}e_j(t)\right\| \\ &\le \|A_{K,i}\|\,\|x_i(t)\| + \|B_iK_i\|\,\|e_i(t)\| + \sum_{j\in N_i}\|B_iL_{ij}\|\,\|e_j(t)\|.\end{aligned}$$

Therefore, integrating the error in the interval $[t^i_k, t^i_k+\overline{\tau}^i_k)$ and noting that $\|x_i(t)\| \le \|x_i(t^i_k)\|$ in Eq. (A.2) in this interval

$$\begin{aligned}\left\|e_i\left(t^i_k+\overline{\tau}^i_k\right) - e_i\left(t^i_k\right)\right\| \le &\left(\|A_{K,i}\|\kappa(V_i)\left(\frac{\mu_i c_1(1+e^{\alpha\tau^\star})e^{-\alpha t^i_k}}{|\lambda_{max}(A_{K,i})|-\alpha} + e^{-|\lambda_{max}(A_{K,i})|t^i_k}\|x_i(0)\|\right)\right. \\ &\left. + \mu_i c_1(1+e^{\alpha\tau^\star})e^{-\alpha t^i_k}\right)\overline{\tau}^i_k.\end{aligned}$$

Denote $k_{1,i} = \|A_{K,i}\|\kappa(V_i)\|x_i(0)\|$ and $k_{2,i} = (\|A_{K,i}\|\kappa(V_i)(1/(|\lambda_{max}(A_{K,i})|-\alpha))+1)\mu_i c_1$. From Eq. (A.1) in Preposition 5, it follows that the upper bound on the delay satisfies

$$\left(k_{1,i}e^{-|\lambda_{max}(A_{K,i})|t^i_k} + k_{2,i}(1+e^{\alpha\tau^\star})e^{-\alpha t^i_k}\right)\overline{\tau}^i_k = c_1 e^{-\alpha(t^i_k+\overline{\tau}^i_k)}.$$

It yields

$$\left(\frac{k_{1,i}}{c_1}e^{-(|\lambda_{max}(A_{K,i})|-\alpha)t^i_k} + \frac{k_{2,i}}{c_1}\left(1+e^{\alpha\tau^\star}\right)\right)\overline{\tau}^i_k = e^{-\alpha\overline{\tau}^i_k}.$$

The right hand side is always positive and takes values in the interval $[0,1)$. The left hand side is also positive and its image is $[0,+\infty)$. Hence, there is a positive solution for the upper bound on the delay. Moreover, the left hand side is upper bounded by $(k_{2,i}/c_1 + k_{2,i}/c_1(1+e^{\alpha\tau^\star}))\overline{\tau}^i_k$ for $\alpha < |\lambda_{max}(A_{K,i})|$. Hence, the most conservative bound on the delay $\tau^\star$ is given by

$$\tau^\star = \min\{(\tau^\star)^i, \quad i=1,\ldots,N\},$$

where $(\tau^\star)^i$ are the solutions of

$$\left(\frac{k_{1,i}}{c_1}+\frac{k_{2,i}}{c_1}\left(1+e^{\alpha(\tau^\star)^i}\right)\right)(\tau^\star)^i=e^{-\alpha(\tau^\star)^i}.$$

A.3. Proof of Theorem 7

The state at any time is given by

$$x_i(t)=e^{A_{K,i}t}x_i(0)+\int_0^t e^{A_{K,i}(t-s)}\left(B_iK_ie_i(s)+B_i\sum_{j\in N_i}L_{ij}e_{j\to i}(s)\right)ds.$$

Under the UwR protocol, it holds that $\|e_i(t)\|\le c_1e^{-\alpha t}$, and $\|e_{j\to i}(t)\|<c_1(1+e^{\alpha\tau^\star})e^{-\alpha t}$. hence, following the same steps than in the proof of Theorem 6, it yields

$$\|x_i(t)\|\le\kappa(V_i)\left(\frac{\overline{\mu}_i(\tau^\star)c_1e^{-\alpha t}}{|\lambda_{max}(A_{K,i})|-\alpha}+e^{-|\lambda_{max}(A_{K,i})|t}\left(\|x_i(0)\|-\frac{\overline{\mu}_i(\tau^\star)c_1e^{-\alpha t}}{|\lambda_{max}(A_{K,i})|-\alpha}\right)\right),$$

where $\overline{\mu}_i(\tau^\star)=\|B_iK_i\|+\sum_{j\in N_i}\|B_iL_{ij}\|(1+e^{\alpha\tau^\star})$.

In the interval $[t_k^i,t_k^{i\to j})$, $\dot{e}_{i\to j}(t)=-\dot{x}_i(t)$ holds. Thus, it can be derived easily that

$$\|e_{i\to j}(t_k^{i\to j})-e_{i\to j}(t_k^i)\|\le\left(k_{1,i}e^{-|\lambda_{max}(A_{K,i})|t_k^i}+\left(k_{2,i}+k_{3,i}(1+e^{\alpha\tau^\star})\right)e^{-\alpha t_k^i}\right)\tau_k^{i\to j},$$

$k_{1,i},k_{2,i}$ and $k_{3,i}$ defined in Eqs. (26)–(28).

According to Proposition 5, $\|e_{i\to j}(t_k^{i\to j})-e_{i\to j}(t_k^i)\|<c_1e^{-\alpha t_k^{i\to j}}$. And the upper bound on the delay is the minimum value of $(\tau^\star)^i$ which solves

$$\left(\frac{k_{1,i}}{c_1}+\frac{k_{2,i}}{c_1}+\frac{k_{3,i}}{c_1}\left(1+e^{\alpha(\tau^\star)^i}\right)\right)(\tau^\star)^i=e^{-\alpha(\tau^\star)^i}.$$

A.4. Proof of Theorem 9

From Eq. (41), the state at any time is given by

$$x(t)=e^{(A_K+\Delta)t}x(0)+\int_0^t e^{(A_K+\Delta)(t-s)}\mathrm{M}\overrightarrow{e}(s)\,ds.$$

According to Lemma 8, the error $\overrightarrow{e}(s)$ is bounded by $\overline{c}_0$. Moreover, since A_K is diagonalizable, $e^{(A_K+\Delta)t}$ can be bounded as $\|e^{(A_K+\Delta)t}\|\le\kappa(V)e^{-(|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|)t}$, from a result of semigroup theory [25]. Thus, it follows

$$\|x(t)\|\le\kappa(V)\Big(\|x(0)\|e^{-(|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|)t}+\frac{\|\mathrm{M}\|\overline{c}_0}{|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|}\left(1-e^{-(|\lambda_{max}(A_K)|-\kappa(V)\|\Delta\|)t}\right)\Big).$$

Reordering terms and noting that $\|\mathrm{M}\|$ is bounded by $\mu_{\max}$ because is a block diagonal matrix, it falls out (44).

The upper bound on the delay can be derived easily noting that if the last event occurred at $t=t_k^i$, it holds that

$$\| e_{i\to j}(t_k^{i\to j}) - e_{i\to j}(t_k^i) \| \le \int_{t_k^i}^{t_k^{i\to j}} \| \dot{e}_{i\to j}(s) \| ds \le \int_{t_k^i}^{t_k^{i\to j}} \| \dot{x}_i(s) \| ds \le \int_{t_k^i}^{t_k^{i\to j}} \| \dot{x}(s) \| ds,$$

since $x_{b,i\to j}$ remain constant in the interval and $\| \dot{x}_i(s) \| \le \| \dot{x}(s) \|$.

Because $\| \dot{x}(s) \| \le \| A_K + \Delta \| \| x(s) \| + \| \mathrm{M} \| \| \overrightarrow{e}(s) \|$, following equivalent steps as in Theorem 2 in [26], it yields

$$\| e_{i\to j}(t_k^{i\to j}) - e_{i\to j}(t_k^i) \| \le \left(\| A_K + \Delta \| \kappa(V) \left(\| x(0) \| + \frac{\| \mathrm{M} \| \overline{c}_0}{|\lambda_{max}(A_K)| - \kappa(V) \| \Delta \|} \right) + \| \mathrm{M} \| \overline{c}_0 \right) (t_k^{i\to j} - t_k^i).$$

According to Assumption 1, no event occurs before the broadcasted state is successfully received and, therefore the increase of the error in the interval $[t_k^i, t_k^{i\to j})$ is bounded by c_0, giving the upper bound on the delay (43).

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