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Distributed Event-Based Control for Interconnected Linear Systems

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Abstract— **This paper presents a distributed event-based control strategy for a networked dynamical system consisting of N linear time-invariant interconnected subsystems. Each subsystem broadcasts its state over the network according to some triggering rules which depend on local information only. The system converges to an adjustable region around the equilibrium point under the proposed control design, and the existence of a lower bound for the broadcasting period is guaranteed. The effect of the coupling terms over the region of convergence and broadcasting period lower bound is analyzed, and a novel model-based approach is derived to reduce the communications. Simulation results show the effectiveness of the proposed approaches and illustrate the theoretical results.**

I. INTRODUCTION

Event-based methods invoke a communication between the components of a control loop only when something significant occurred in the system [1], [2]. In recent work, event- and self-triggered policies have been proposed to use more efficiently the limited bandwith available in networked control [3]-[8]. When the number of systems increases, the bandwidth saving becomes more important.

The centralized control of large-scale systems would require an accurate knowledge of the interaction between the subsystems, also called "agents", and the consumption of a lot of network resources. On the other hand, in a distributed control approach, each agent could collect information from its neighboring nodes and trigger controller updates according to some rules. The limitations imposed by the network renders the frequency at which the system communicates an important issue. A reduction in the transmission frequency implies bandwidth saving but a certain level of performance must be guaranteed. A more natural choice is to transmit only when a certain condition depending on the state is satisfied, that is, when an event occurs.

Distributed event-triggered control for multi-agent system has already been examined in [9]. In [10], distributed event-based control for average consensus problems for both single- and double-integrator multi-agent systems is proposed. We extend here this event-based control strategy to interconnected linear systems, with symetric interconnections. The agents communicate through a network and the overall system converges to non-cooperative equilibrium points, whereas in [10] the equilibria are cooperative.

Thus our contribution sums up to a novel distributed event-based control design for interconnected linear systems. A neighboring relationship is defined in the sense of dynamical interaction between the nodes. The distributed control of interconnected dynamical systems has been discussed in the past [11]. However, the timing issue imposed by the network makes necessary a redesign of the control structure. A similar idea was presented in [7]. Each subsystem broadcasts its state to its neighbors only at event times. With regards to [7], the main contribution of this work is that the triggering mechanism does not continuously depend on the state of the system but on the error between the current and the latest broadcasted state. In a first approach, the control law is updated whenever the agent sends or receives a new measurement value, and so both the control law and the broadcasted states are piecewise constant functions. In a second approach, the broadcasted states are used by each agent to generate the control signal according to a local model of its neighborhood. This leads to substantial reduction in the number of events if the model is accurate enough. Different triggering conditions are proposed to guarantee the convergence of the system to an arbitrary small region around the equilibrium and the existence of a state independent strictly positive lower bound for the broadcast period.

The rest of the paper is organized as follows: Section II contains the problem statement for this work. The event-based control strategy is presented in Section III. The effect of the coupling terms over the system stability is analyzed in Section IV. Section V presents the model-based extension of the work. Numerical simulations in Section VI show the efficiency of the proposed strategy with respect to previous results. The conclusions in Section VII end the paper.

II. PROBLEM STATEMENT

Consider a system of N linear time-invariant subsystems. The dynamics of each subsystem are given by:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \sum_{j \in N_i} H_{ij} x_j(t) \qquad (1)$$

$\forall i = 1, \dots, N$, where N_i is the set of "neighbors" of subsystem i and H_{ij} is the interaction term between agent i and agent j. We assume that neighboring is a symmetric relation, and so $H_{ij} = H_{ji}$. The state x_i has dimension n_i, u_i is the m_i-dimensional local control signal of agent i, and A_i, B_i and H_{ij} are matrices of appropriate dimensions.

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A digital communication network is used, and so every subsystem can only broadcast its state to its neighbors in a discrete-time manner. The control law is given by:

$$u_i(t) = K_i\tilde{x}_i(t) + \sum_{j\in N_i} L_{ij}\tilde{x}_j(t), \quad \forall i = 1, ..., N \tag{2}$$

where K_i is the feedback gain for the nominal subsystem i, i.e., we assume that $A_i + B_iK_i$ is Hurwitz, L_{ij} is a set of decoupling gains, and $\tilde{x}_j(t)$ is the latest state broadcasted by agent j at time t. The times at which the agents broadcast their state generates a sequence of broadcasting times $\{t_k^i\}_{k=0}^{\infty}$, where $t_k^i < t_{k+1}^i$ for all k. Let also $n = \sum\limits_{i=0}^{N} n_i$.

Let us define the error e_i as $e_i = \tilde{x}_i - x_i$, and rewrite (1) in terms of e_i and the control law (2):

$$\dot{x}_i(t) = A_{K,i}x_i(t) + B_iK_ie_i(t) + \sum_{j\in N_i}\left(\Delta_{ij}x_j(t) + B_iL_{ij}e_j\right) \tag{3}$$

$\forall i = 1, ..., N$, where $A_{K,i} = A_i + B_iK_i$, and $\Delta_{ij} = B_iL_{ij} + H_{ij}$. We first assume that the perfect decoupling condition $\Delta_{ij} = 0\ \forall i, j$ holds. In this case, (3) becomes $\dot{x}_i(t) = A_{K,i}x_i(t) + B_iK_ie_i(t) + \sum_{j\in N_i} B_iL_{ij}e_j$.

We also define $A_K = \text{diag}(A_{K,1}, A_{K,2}, ..., A_{K,N})$, $B = \text{diag}(B_1, B_2, ..., B_N)$, $M = \begin{pmatrix} B_1K_1 & -H_{12} & \cdots & -H_{1N} \\ -H_{21} & B_2K_2 & \cdots & -H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -H_{N1} & -H_{N2} & \cdots & B_nK_N \end{pmatrix}$ and $\hat{K} = \begin{pmatrix} K_1 & L_{12} & \cdots & L_{1N} \\ L_{21} & K_2 & \cdots & L_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \cdots & K_N \end{pmatrix}$.

Note that $M = B\hat{K}$. Define the stack vectors $x = (x_1^T, x_2^T, ..., x_N^T)^T$ and $e = (e_1^T, e_2^T, ..., e_N^T)^T$ as the state and error vectors of the overall system, respectively. Thus

$$\dot{x}(t) = A_Kx(t) + Me(t) = A_Kx(t) + B\hat{K}e(t). \tag{4}$$

As the broadcasted states $\tilde{x}_i$ remain constant between consecutive events, $\dot{e}(t)$ in each interval is given by:

$$\dot{e}(t) = -A_Kx(t) - Me(t) = -A_Kx(t) - B\hat{K}e(t) \tag{5}$$

III. Event-based control strategy

The occurrence of an event is defined by trigger functions f_i which depend on local information of agent i only and take values in $\mathbb{R}$. The sequence of broadcasting times t_k^i are determined recursively by as $t_{k+1}^i = \inf\{t : t > t_k^i, f_i(t) > 0\}$. The next subsections present some results for the problem stated above under different trigger functions.

A. Static trigger function

In this subsection, we first propose a set of static trigger functions as follows

$$f_i(e_i(t)) = \|e_i(t)\| - c \tag{6}$$

where c is a positive constant. Note that the trigger function for each subsystem only depends on its state.

The following theorem proves that system (4) with trigger functions (6) asymptotically converges to a region around the equilibrium point which, without loss of generality, is assumed to be $(0, \ldots, 0)^T$. The functions (6) bound the errors by c, since an event is triggered as soon as the norm of e_i becomes larger than c.

Assumption 1: We assume that $A_{K,i}$, $i = 1, \ldots, N$ is diagonalizable so that the Jordan form of $A_{K,i}$ is diagonal and its elements are the eigenvalues of $A_{K,i}$, $\lambda_k(A_{K,i})$ $k = 1, \ldots, n$. This assumption facilitates the calculations, but the extension to general Jordan blocks is straightforward.

Theorem 2: Consider the closed-loop system (4) and trigger functions of the form (6). Then, for all initial conditions $x(0) \in \mathbb{R}^n$, and $t > 0$, it holds

$$\|x(t)\| \le k_0\Big(\frac{\|M\|\sqrt{N}c}{|\lambda_{max}(A_K)|} + e^{-|\lambda_{max}(A_K)|t}\big(\|x(0)\| - \frac{\|M\|\sqrt{N}c}{|\lambda_{max}(A_K)|}\big)\Big) \tag{7}$$

where $\lambda_{max}(A_K)$ denotes the maximum real part of the eigenvalues of A_K, and k_0 is a positive constant $k_0 = \|V\|\|V^{-1}\|$, where V is the matrix of the eigenvectors of A_K. Furthermore, the closed-loop system does not exhibit Zeno behavior.

Proof: The analytical solution of (4) is given by $x(t) = e^{A_Kt}x(0) + \int\limits_0^t e^{A_K(t-\tau)}Me(\tau)d\tau$.

Remark 3: The integrability of $e(t)$ is justified by the definition of $f_i(e_i(t))$, which guarantees that $e_i(t)$ cannot be updated to zero immediately after it had done so. Thus there is an arbitrarily small, yet positive lower bound on the interexecution times. Thus the right hand side of (4) is piecewise continuous.

Then, the state is bounded by $\|x(t)\| \le \|e^{A_Kt}x(0)\| + \|\int\limits_0^t e^{A_K(t-\tau)}Me(\tau)d\tau\| \le \|e^{A_Kt}x(0)\| + \int\limits_0^t \|e^{A_K(t-\tau)}Me(\tau)d\tau\| \le \|e^{A_Kt}\|\|x(0)\| + \int\limits_0^t \|e^{A_K(t-\tau)}\|\|M\|\|e(\tau)\|d\tau$.

The matrix A_K is diagonalizable by construction. Thus

$$A_K = VDV^{-1} \implies e^{A_K} = Ve^DV^{-1} \tag{8}$$

where D is the diagonal matrix of the eigenvalues of A_K and V is the matrix with the corresponding eigenvectors as its columns. Denote as $\lambda_{max}(A_K)$ the maximum real part of the eigenvalues of A_K. Thus we can bound $\|e^{A_Kt}\| \le \|V\|\|V^{-1}\|e^{\lambda_{max}(A_K)t}$and the state as $\|x(t)\| \le \|V\|\|V^{-1}\|(e^{\lambda_{max}(A_K)t}\|x(0)\| + \int\limits_0^t e^{\lambda_{max}(A_K)(t-\tau)}\|M\|\|e(\tau)\|d\tau)$.

The trigger condition $f_i(e_i(t)) > 0$ enforces $\|e_i(t)\| \le c$ so that $\|e(t)\| \le \sqrt{N}c$. Defining $k_0 = \|V\|\|V^{-1}\|$, we have $\|x(t)\| \le k_0\Big(e^{\lambda_{max}(A_K)t}\|x(0)\| + \frac{\|M\|\sqrt{N}c}{\lambda_{max}(A_K)}(e^{\lambda_{max}(A_K)t} - 1)\Big)$, which can be rewritten as (7) since the eigenvalues of A_K all have negative real part ($\lambda_{max}(A_K) < 0$).

In order to prove that Zeno behavior is excluded, let's assume that agent i's latest event trigger occurs at $t = t^* > 0$. Then $\|e_i(t^*)\| = 0$, and so $f_i(0) = -c < 0$. Therefore agent i cannot trigger at the same time instant. Between two consecutive events it holds that $\dot{e}_i(t) = -\dot{x}_i(t)$. From definition of the error $\|e_i(t)\| \leq \|e(t)\|$, and from (5) we derive $\|\dot{e}(t)\| \leq \|A_K\|\|x(t)\| + \|M\|\|e(t)\| \leq \|A_K\|\|x(t)\| + \|M\|\sqrt{N}c$.

From (7) it follows that the state is bounded by $\|x(t)\| \leq k_0\Big(\frac{\|M\|\sqrt{N}c}{|\lambda_{max}(A_K)|} + \|x(0)\|\Big), \forall t \geq t^*$. Denote this bound as $x_{max,1}$. Then for any t between t^* and the next event time for agent i, $\|e_i(t)\| \leq \|e(t)\| \leq \int\limits_{t^*}^{t} \|\dot{e}(\tau)\|d\tau \leq \int\limits_{t^*}^{t} (\|A_K\|\|x(\tau)\| + \|M\|\sqrt{N}c)d\tau \leq (\|A_K\|x_{max,1} + \|M\|\sqrt{N}c)(t-t^*)$. From (8), it is also derived that $\|A_K\| \leq \|V\|\|D\|\|V^{-1}\| \leq k_0|\lambda_{min}(A_K)|$, where $\lambda_{min}(A_K)$ is the eigenvalue of A_K with largest modulus. If all eigenvalues are real, $\lambda_{min}(A_K)$ is the minimum eigenvalue of A_K.

The next event is not triggered before $e_i(t)$ reaches the value of c. Thus a lower bound on the inter-event times is

$$\tau = \frac{c}{k_0^2|\lambda_{min}(A_K)|\|x(0)\| + \|M\|\sqrt{N}c\Big(k_0^2\frac{|\lambda_{min}(A_K)|}{|\lambda_{max}(A_K)|} + 1\Big)} \quad (9)$$

which is strictly positive. Thus Zeno behavior is excluded. ■

Remark 4: The norm of A_K is computed as the spectral norm. Since A_K is diagonalizable, the norm of D can be bounded by the largest modulus of the eigenvalues ($\|D\| \leq |\lambda_{min}(A_K)|$). If they are all real, the largest modulus of the eigenvalues is the minimum eigenvalue since they all have negative real part.

B. Time-dependent trigger condition

This subsection presents a set of time-dependent trigger functions defined as

$$f_i(e_i(t)) = \|e_i(t)\| - (c_0 + c_1 e^{-\alpha t}) > 0, \alpha > 0 \quad (10)$$

where $c_0 \geq 0, c_1 > 0$. When the parameters α and c_1 are adequately selected, the time-dependency can decrease the number of events without degrading the performance, meanwhile c_0 guarantees the convergence to an arbitrary small region. The following theorem holds:

Theorem 5: Consider the closed-loop system (4) and the trigger functions of the form (10), with $0 < \alpha < |\lambda_{max}(A_K)|$. Then, for all initial conditions $x(0) \in \mathbb{R}^n$ and $t > 0$, it holds

$$\|x(t)\| \leq k_0\Big(\frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + e^{-|\lambda_{max}(A_K)|t}\big(\|x(0)\| - \|M\|\sqrt{N}\big(\frac{c_0}{|\lambda_{max}(A_K)|} + \frac{c_1}{|\lambda_{max}(A_K)| - \alpha}\big)\big) + \frac{e^{-\alpha t}\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \alpha}\Big) \quad (11)$$

and the closed-loop system does not exhibit Zeno behavior.

Proof: Proceeding as before we get $\|x(t)\| \leq k_0\Big(\|x(0)\|e^{\lambda_{max}(A_K)t}\int\limits_0^t e^{\lambda_{max}(A_K)(t-\tau)}\|M\|\|e(\tau)\|d\tau\Big)$. Note that now we can compute $\|e(\tau)\| \leq \sqrt{N}(c_0 + c_1 e^{-\alpha\tau})$, by the definition of the trigger function. Thus $\|x(t)\| \leq k_0\Big(\int\limits_0^t e^{\lambda_{max}(A_K)(t-\tau)}\sqrt{N}(c_0 + c_1 e^{-\alpha\tau})\|M\|d\tau + e^{\lambda_{max}(A_K)t}\|x(0)\|\Big) = k_0\Big(\frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|}(1 - e^{-|\lambda_{max}(A_K)|t}) + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \alpha}(e^{-\alpha t} - e^{-|\lambda_{max}(A_K)|t}) + \|x(0)\|e^{-|\lambda_{max}(A_K)|t}\Big)$, which by reordering terms proves the first part. We next show that broadcasting period is lower bounded. Specifically, (11) can be bounded by

$$\|x(t)\| \leq k_0\Big(\|x(0)\|e^{-|\lambda_{max}(A_K)|t} + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \alpha}e^{-\alpha t}\Big). \quad (12)$$

Denote this bound as $x_{max,2}(t)$.

Let's prove now that the Zeno behavior is excluded, that is, there exists a positive lower bound for the inter-event times. Proceeding as before, it holds that $\|\dot{e}(t)\| \leq \|A_K\|\|x(t)\| + \|M\|\|e(t)\| \leq \|A_K\|\|x(t)\| + \|M\|\sqrt{N}(c_0 + c_1 e^{-\alpha t}) \leq k_0^2|\lambda_{min}(A_K)|\Big(\frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \|x(0)\|e^{-|\lambda_{max}(A_K)|t} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \alpha}e^{-\alpha t}\Big) + \|M\|\sqrt{N}(c_0 + c_1 e^{-\alpha t})$. Denoting $k_1 = k_0^2|\lambda_{min}(A_K)|\|x(0)\|$, $k_2 = \|M\|\sqrt{N}c_1\big(\frac{k_0^2|\lambda_{min}(A_K)|}{|\lambda_{max}(A_K)| - \alpha} + 1\big)$ and $k_3 = \|M\|\sqrt{N}c_0\big(\frac{k_0^2|\lambda_{min}(A_K)|}{|\lambda_{max}(A_K)|} + 1\big)$, it follows that

$$\|\dot{e}(t)\| \leq k_1 e^{-|\lambda_{max}(A_K)|t} + k_2 e^{-\alpha t} + k_3. \quad (13)$$

Assume that $c_0 \neq 0$. Then, $k_3 \neq 0$ and $\|\dot{e}(t)\| \leq k_1 + k_2 + k_3$. The fact that $\|e_i(t)\| \leq \|e(t)\|$ allows to bound the error of agent i as $\|e_i(t)\| \leq \|e(t)\| \leq \int\limits_{t^*}^{t} \|\dot{e}(\tau)\|d\tau \leq \int\limits_{t^*}^{t} (k_1 + k_2 + k_3)d\tau = (k_1 + k_2 + k_3)(t - t^*)$. The trigger condition is not fulfilled before $\|e_i(t)\| = c_0 \leq c_0 + c_1 e^{-\alpha t}$. Thus a lower bound τ on the inter-execution time is given by

$$\tau = c_0/(k_1 + k_2 + k_3). \quad (14)$$

This is a positive quantity, and so, the inter-event times are lower bounded.

Let's assume now the case $c_0 = 0$. In that case $k_3 = 0$. Then, the state is bounded by $\|x(t)\| \leq k_0\Big(\|x(0)\|e^{-|\lambda_{max}(A_K)|t} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \alpha}(e^{-\alpha t} - e^{-|\lambda_{max}(A_K)|t})\Big)$, and so the overall system converges asymptotically to the equilibrium point. In order to determine the lower bound of the inter-event times for this case, let us bound (13) with $k_3 = 0$ as $\|\dot{e}(t)\| \leq k_1 e^{-|\lambda_{max}(A_K)|t} + k_2 e^{-\alpha t} \leq k_1 e^{-|\lambda_{max}(A_K)|t^*} + k_2 e^{-\alpha t^*}$, since $t^* < t$. With that bound and considering that $\|e_i(t)\| \leq \|e(t)\|$, it follows that $\|e_i(t)\| \leq \int\limits_{t^*}^{t} k_1 e^{-|\lambda_{max}(A_K)|t^*} + k_2 e^{-\alpha t^*} d\tau = (k_1 e^{-|\lambda_{max}(A_K)|t^*} + k_2 e^{-\alpha t^*})(t - t^*)$. According to (10) with $c_0 = 0$, the next event will not be triggered before $\|e_i(t)\| = c_1 e^{-\alpha t}$. Thus, a lower bound on the inter-event intervals is given by

$$\Big(\frac{k_1}{c_1}e^{(\alpha - |\lambda_{max}(A_K)|)t^*} + \frac{k_2}{c_1}\Big)\tau = e^{-\alpha\tau}. \quad (15)$$

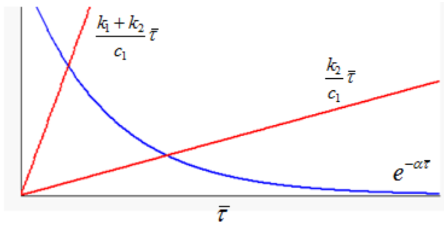

Fig. 1. Graphical solution of (15)

The right hand side of (15) is always positive. Moreover, for $\alpha < |\lambda_{max}(A_K)|$ the left hand side is strictly positive as well, and the term in brackets is upper bounded by $\frac{k_2+k_1}{c_1}$ and lower bounded by k_2/c_1, and this yields to a positive value of τ for all $t^* \geq 0$. The existence of the solution τ can also be depicted graphically (see Figure 1). The solution is given by the intersection of the exponential curve and the straight line between the two bounds whose slope depends on t^*. Thus, there is no Zeno behavior. ■

IV. Coupling Stabilization

At the beginning of Section II, it has been assumed that for each i, $A_{K,i}$ is Hurwitz. The motivation of this section is to relax the previous condition. Specifically, the case when the design of the nominal control law makes the system closed-loop stable due to the interconnections but each isolated subsystem can still be unstable, is presented.

Consider (1). This initial set of equations can be rewritten in terms of the state of the overall system as $\dot{x}(t) = \hat{A}x(t) + Bu(t)$, where $u = (u_1^T, u_2^T, \ldots, u_N^T)$ and $\hat{A} = \begin{pmatrix} A_1 & H_{12} & \cdots & H_{1N} \\ H_{21} & A_2 & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & A_N \end{pmatrix}$.

Assume that we can design a state-feedback controller based on the latest broadcasted state $\tilde{x}$, that is $u(t) = \hat{K}\tilde{x}(t)$. Now $\hat{K}$ is defined as $\hat{K} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{pmatrix}$ that renders the nominal system given by $\dot{x}(t) = (\hat{A} + B\hat{K})x(t) = \hat{A}_K x(t)$ stable and $\hat{A}_K$ diagonalizable. However, $A_i + B_iK_{ii}$ is not necessarily Hurwitz in this case. Then we have

$$\dot{x}(t) = \hat{A}_K x(t) + B\hat{K}e(t). \tag{16}$$

The following can then be stated:

Theorem 6: The state norm of (16) with trigger functions (10), and for all initial conditions $x(0) \in \mathbb{R}^n$ and $t > 0$, fulfills the following condition

$$\|x(t)\| \leq \hat{k}_0\Big(\frac{\|B\hat{K}\|\sqrt{N}c_0}{|\lambda_{max}(\hat{A}_K)|} + e^{-|\lambda_{max}(\hat{A}_K)|t}\big(\|x(0)\| - \|B\hat{K}\|\sqrt{N}\big(\frac{c_0}{|\hat{\lambda}_{max}(\hat{A}_K)|} + \frac{c_1}{|\hat{\lambda}_{max}(\hat{A}_K)|-\alpha}\big)\big) + \frac{e^{-\alpha t}\|B\hat{K}\|\sqrt{N}c_1}{|\hat{\lambda}_{max}(\hat{A}_K)|-\alpha}\Big)$$

where $\hat{\lambda}_{max}(\hat{A}_K)$ represents the maximum real part of the eigenvalues of $\hat{A}_K$. Furthermore, the closed-loop system does not exhibit Zeno-behavior.

Proof: The solution of (16) is $x(t) = e^{\hat{A}_K t}x(0) + \int_0^t e^{\hat{A}_K(t-\tau)}B\hat{K}e(\tau)d\tau$. Using similar calculations as in the previous sections, the state norm can be bounded as $\|x(t)\| \leq \|e^{\hat{A}_K t}x(0)\| + \|\int_0^t e^{\hat{A}_K(t-\tau)}B\hat{K}e(\tau)d\tau\|$. The matrix $\hat{A}_K$ is stable and thus all its eigenvalues have negative real part. Moreover, $\hat{A}_K$ is diagonalizable and can be decomposed as $\hat{A}_K = \hat{V}\hat{D}\hat{V}^{-1}$, where $\hat{D}$ is the diagonal matrix and $\hat{V}$ contains the eigenvectors. Then, by defining $\hat{k}_0 = \|\hat{V}^{-1}\|\|\hat{V}\|$ we get that $\|e^{\hat{A}_K t}\| \leq \hat{k}_0 e^{-|\hat{\lambda}_{max}(\hat{A}_K)|t}$, where $\hat{\lambda}_{max}(\hat{A}_K)$ is maximum real part of the eigenvalues of $\hat{A}_K$.

Thus, assuming trigger functions as in (10) with $c_0, c_1 \neq 0$, we have $\|x(t)\| \leq \hat{k}_0\Big(e^{-|\hat{\lambda}_{max}(\hat{A}_K)|t}\|x(0)\| + \int_0^t e^{-|\hat{\lambda}_{max}(\hat{A}_K)|(t-\tau)}\|B\hat{K}\|\sqrt{N}(c_0 + c_1e^{-\alpha t})d\tau\Big)$. Solving the integral and grouping terms, it follows that $\|x(t)\| \leq \hat{k}_0\Big(\frac{\|B\hat{K}\|\sqrt{N}c_0}{|\hat{\lambda}_{max}(\hat{A}_K)|} + e^{-|\hat{\lambda}_{max}(\hat{A}_K)|t}\Big(\|x(0)\| - \|B\hat{K}\|\sqrt{N}\big(\frac{c_0}{|\hat{\lambda}_{max}(\hat{A}_K)|} + \frac{c_1}{|\hat{\lambda}_{max}(\hat{A}_K)|-\alpha}\big)\Big) + \frac{e^{-\alpha t}\|B\hat{K}\|\sqrt{N}c_1}{|\hat{\lambda}_{max}(\hat{A}_K)|-\alpha}\Big)$, which proves the first part of the theorem.

Similar calculations to the previous sections yield a lower bound for the broadcasting period of $\tau = c_0/(\hat{k}_1 + \hat{k}_2 + \hat{k}_3)$, where $\hat{k}_1 = \hat{k}_0^2|\hat{\lambda}_{min}(\hat{A}_K)|\|x(0)\|$, $\hat{k}_2 = \|B\hat{K}\|\sqrt{N}c_1\big(\frac{\hat{k}_0^2|\hat{\lambda}_{min}(\hat{A}_K)|}{|\hat{\lambda}_{max}(\hat{A}_K)|-\alpha} + 1\big)$ and $\hat{k}_3 = \|B\hat{K}\|\sqrt{N}c_0\big(\frac{\hat{k}_0^2|\hat{\lambda}_{min}(\hat{A}_K)|}{|\hat{\lambda}_{max}(\hat{A}_K)|} + 1\big)$. Hence, the system does not exhibit Zeno-behavior. ■

Remark 7: The case $c_0 = 0$ can be studied as in Section III-B. In that case, $\hat{k}_3 = 0$ and the lower bound on the broadcasting period is the solution of

$$\Big(\frac{\hat{k}_1}{c_1}e^{(\alpha-|\hat{\lambda}_{max}(A_K)|)t^*} + \frac{\hat{k}_2}{c_1}\Big)\tau = e^{-\alpha\tau}. \tag{17}$$

The existence of a positive solution in (17) can be proved graphically as in Figure 1.

V. Model-Based Control

The event-based strategy analyzed previously is based on a control law which maintains its value between two consecutive events and is based on the latest broadcasted state. One alternative to this control law can be achieved in the event that each agent has knowledge of the dynamics of its neighborhood.

In particular, let us define the control law for each agent based on a model as $u_i(t) = K_i\tilde{x}_i(t) + \sum_{j\in N_i} L_{ij}\tilde{x}_j(t)$, where $\tilde{x}_i$ now represents the state estimation of x_i given by the model $(\tilde{A}_i, \tilde{B}_i)$ of each isolated agent $\dot{\tilde{x}}_i(t) = \tilde{A}_{K,i}\tilde{x}_i(t)$, where $\tilde{A}_{K,i} = \tilde{A}_i + \tilde{B}_iK_i$. Let us also define $\tilde{A}_K = \text{diag}(\tilde{A}_{K,1}, \ldots, \tilde{A}_{K,n})$. The error e_i is defined as previously and is also reset at events' occurrence. Let's introduce a sequence of functions $\hat{x}_{k,i} : \mathbb{R} \to \mathbb{R}^{n_i}$ supported over the time interval between two consecutive events for agent i. Let's denote this interval as $[t_k^i, t_k^{i+1})$. The value of $\hat{x}_{k,i}$ at time $t \in \mathbb{R}$ is $x_i(t_k^i)e^{\tilde{A}_{K,i}(t-t_k^i)}$ if $t \in [t_k^i, t_k^{i+1})$ and zero otherwise. With this notation, the state estimation $\forall t$ is

$$\tilde{x}_i(t) = \sum_{k=0}^{\infty} \hat{x}_{k,i}(t). \tag{18}$$

In this approach, if the difference between the system state and the model differs from a given quantity, which depends on the trigger function, an event is triggered and the model is updated to the state at the event time

If we consider the trigger function defined in (10), the state will be also bounded by (11). However, the lower bound for the inter-event time will have a different expression.

Assumption 8: Let be $\bar{A}_K = \tilde{A}_K - A_K$ be the difference between the model and the real plant closed loop dynamics. We assume that the following is satisfed:

$$\frac{\sqrt{N}(c_0 + c_1)/k_0}{\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}} < \frac{\|A_K\| - \|\bar{A}_K\|}{\|\tilde{A}_K\|}. \tag{19}$$

Theorem 9: If assumptions 8 holds, the lower bound of the broadcasting period for the system (4) when the control law for each agent is based on state estimations of the form (18), with triggering functions (10), $0 < \alpha < |\lambda_{max}(A_K)|$, is greater than (14).

Proof: Define the overall system state estimation as $\tilde{x}(t) = (\tilde{x}_1^T, \ldots, \tilde{x}_N^T)$. Let's prove that the bound for the inter-events time is larger in the model-based approach. We have $\dot{e}(t) = \dot{\tilde{x}}(t) - \dot{x}(t) = \tilde{A}_K\tilde{x}(t) - (A_K x(t) + Me(t)) = (\tilde{A}_K - A_K)x(t) + (\tilde{A}_K - M)e(t) = \bar{A}_K x(t) + (\tilde{A}_K - M)e(t)$. Then, $\|\dot{e}(t)\| \leq \|\bar{A}_K\|\|x(t)\| + \|\tilde{A}_K - M\|\|e(t)\| \leq \|\bar{A}_K\|\|x(t)\| + \|\tilde{A}_K - M\|\sqrt{N}(c_0 + c_1 e^{-\alpha t})$.

Assume that the last event occurred at a time $t^* \leq t$ and consider the case when $c_0, c_1 \neq 0$. It follows that $c_0 + c_1 e^{-\alpha t} \leq c_0 + c_1 e^{-\alpha t^*} \leq c_0 + c_1$. Moreover, the state norm can be bounded as in (12), and so:

$$\|x(t)\| \leq k_0\Big(\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \alpha}\Big).$$

Then $\|e_i(t)\| \leq \|e(t)\| \leq \int_{t^*}^{t} \|\dot{e}(\tau)\|d\tau \leq \Big(\|\bar{A}_K\|k_0\big(\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}\big) + \|\tilde{A}_K - M\|\sqrt{N}(c_0 + c_1)\Big)(t - t^*)$.

The next event will not occur before $\|e_i(t)\| \leq c_0 \leq c_0 + c_1 e^{-\alpha t}$. This condition gives a lower bound for the broadcasting period τ' that will be larger than (14) if $\|\bar{A}_K\|k_0\Big(\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}\Big) + \|\tilde{A}_K - M\|\sqrt{N}(c_0 + c_1) < \|A_K\|k_0\Big(\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}\Big) + \|M\|\sqrt{N}(c_0 + c_1)$, or equivalently $(\|\tilde{A}_K - M\| - \|M\|)\sqrt{N}(c_0 + c_1) < (\|A_K\| - \|\bar{A}_K\|)\Big(\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}\Big)$. After some manipulations

$$\frac{\sqrt{N}(c_0 + c_1)}{\|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}} < k_0\frac{\|A_K\| - \|\bar{A}_K\|}{\|\tilde{A}_K - M\| - \|M\|}. \tag{20}$$

The denominator on the right hand side is bounded as $\|\tilde{A}_K - M\| - \|M\| \leq \|\tilde{A}_K\| + \|M\| - \|M\| = \|\tilde{A}_K\|$. If Assumption 8 holds, (20) is fulfilled, and the broadcasting period lower bound is larger for the model-based approach. ■

Remark 10: Assumption 8 is not a strong assumption since when $\|\bar{A}_K\|$ goes to zero, $\|\tilde{A}_K\| \simeq \|A_K\|$, and the right hand side of (19) can be approximated to one. For initial conditions satisfying $k_0\|x_0\| > \sqrt{N}(c_0 + c_1)$, it holds $\frac{\sqrt{N}(c_0+c_1)}{k_0} < \|x(0)\| + \frac{\|M\|\sqrt{N}c_0}{|\lambda_{max}(A_K)|} + \frac{\|M\|\sqrt{N}c_1}{|\lambda_{max}(A_K)|-\alpha}$.

VI. Simulation Results

This section presents some simulation results in order to demonstrate the event-based control strategy. The system considered is a collection of N inverted pendulums of mass m and length l coupled by springs with rate k. Each subsystem can be described as follows:

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} - \frac{a_i k}{ml^2} & 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u_i + \sum_{j \in N_i} \begin{pmatrix} 0 & 0 \\ \frac{h_{ij}k}{ml^2} & 0 \end{pmatrix} x_j$$

where $x_i = \begin{pmatrix} x_{i_1} & x_{i_2} \end{pmatrix}^T$, a_i is the number of springs connected to the ith pendulum and $h_{ij} = 1, \forall j \in N_i$ and 0 otherwise. K_i and L_{ij} gains are designed to decouple the system and place the poles at -1, -2. This yields the control law $u_i = \Big(-3ml^2 \quad a_i k - \frac{ml^2}{4}(8 + \frac{4g}{l})\Big)\tilde{x}_i + \sum_{j \in N_i} \begin{pmatrix} -k & 0 \end{pmatrix} \tilde{x}_j$, where $\tilde{x}_i = \begin{pmatrix} \tilde{x}_{i_1} & \tilde{x}_{i_2} \end{pmatrix}^T$. In the following, the system parameters are set to $g = 10, m = 1, l = 2$ and $k = 5$, as in [7].

A. Static trigger function

The output of the system and the sequence of events for $N = 4$ with trigger functions (6) with $c = 0.05$ is shown in Figure 2 for initial conditions $x(0) = \begin{pmatrix} \pi/2 & 0 & \pi/3 & 0 & -\pi/2 & 0 & -\pi/4 & 0 \end{pmatrix}$. The lower graph shows the generation of events for each agent, whereas the upper graph depicts their output.

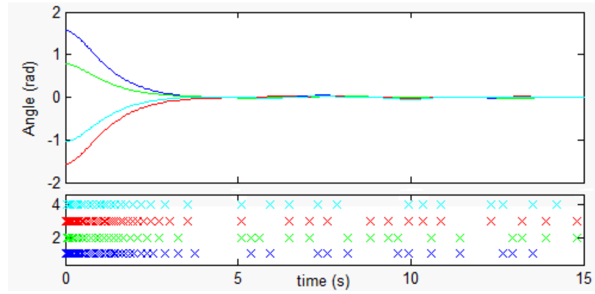

Fig. 2. Simulation result with trigger functions (6), $c = 0.05$

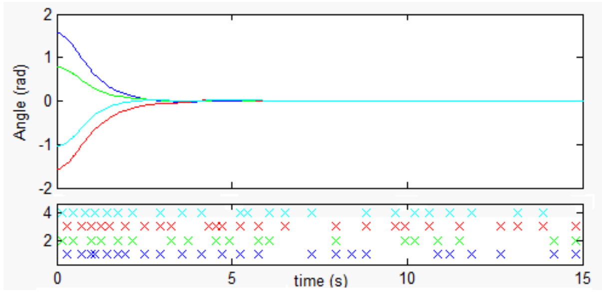

Fig. 3. Simulation result with trigger functions (10), $c_0 = 0.01, c_1 = 0.5$ and $\alpha = 0.7$

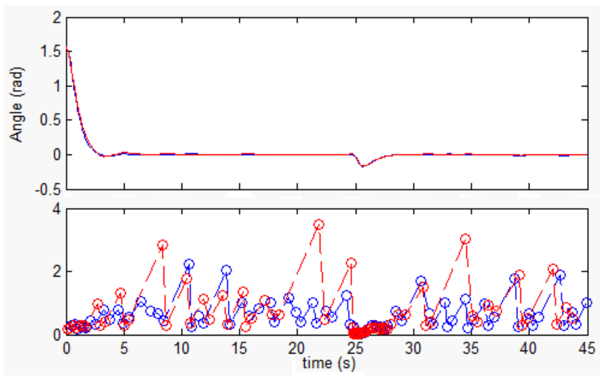

Fig. 4. Simulation result with trigger functions (10) for the approaches of the sections III-B and V

B. Time-dependent trigger function

Figure 3 shows the system response under the previously described conditions but with trigger functions (10) with parameters $c_0 = 0.01, c_1 = 0.5$ and $\alpha = 0.7$. The convergence of the system to the equilibrium point is guaranteed whereas the number of events generated (see the lower graph) decreases significantly, especially for small values of time.

C. Model based control

In Section V a model-based approach was presented as a means of improving the event-based control of Section III. Figure 4 compares the output of agent 1 when a simulation under the conditions of the previous section is performed for both approches. Additionally, a disturbance is induced to the agent at $t = 25s$. The lower graph shows the evolution of the broadcasting period. We observe how the model based approach gives larger values, especially around the equilibrium point.

Table I extends this study for a larger N. Several simulations were performed for different initial conditions for each value of N. Mininum and mean values of the inter-event times (τ_k^i) were calculated for the set of the simulations with the same number of agents. We see that the model-based approach gives around 50% larger broadcasting periods, remaining almost constant when N increases. If we compare these results to [7], we see that the proposed scheme can provide around six times larger τ_k^i. For example, for $N = 100$, trigger functions of the form (10) give mean values

TABLE I

INTER-EVENT TIMES FOR DIFFERENT N. VALUES GIVEN IN SECONDS

	Trig. cond. (10), **approach III-B**		**Trig. cond.** (10), **approach V**		**Trig. cond. of [7]**
N	$\{\tau_k^i\}_{\text{min}}$	$\{\tau_k^i\}_{\text{mean}}$	$\{\tau_k^i\}_{\text{min}}$	$\{\tau_k^i\}_{\text{mean}}$	$\{\tau_k^i\}_{\text{mean}}$
10	0.263	0.688	0.185	0.900	0.1149
50	0.205	0.620	0.184	0.903	0.1175
100	0.185	0.627	0.211	0.978	0.1152
150	0.184	0.664	0.210	0.975	0.1180
200	0.183	0.650	0.181	0.967	0.1177

of τ_k^i of 0.627 and 0.978 for the approaches of sections III-A and V, respectively, while the trigger functions in [7] gives 0.1152. Though the approach in [7] ensures asymptotic stability, we guarantee the convergence to an arbitrary small region around the origin with $c_0 \neq 0$. Alternatively, one can choose $c_0 = 0$ to get rid of this drawback.

VII. CONCLUSIONS

We presented a novel distributed event-based control strategy for linear interconnected subsystems. The events are generated by the agents based on local information only, broadcasting their state over the network. The proposed trigger functions preserve the desired convergence properties and guarantee the existence of a strictly positive lower bound for the broadcast period, excluding the Zeno behavior. A model-based approach was also presented as a means of reducing the number of events. The contribution of the current paper with respect to previous work are verified through computer simulations.

The inclusion in the formalism of network problems as time-delays and dropouts, the consideration of the used protocol, and the application to discrete-time systems are part of the future work.

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