Stochastic self-triggered model predictive control for linear systems with probabilistic constraints

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ABSTRACT

A stochastic self-triggered model predictive control (SSMPC) algorithm is proposed for linear systems subject to exogenous disturbances and probabilistic constraints. The main idea behind the self-triggered framework is that at each sampling instant, an optimization problem is solved to determine both the next sampling instant and the control inputs to be applied between the two sampling instants. Although the self-triggered implementation achieves communication reduction, the control commands are necessarily applied in open-loop between sampling instants. To guarantee probabilistic constraint satisfaction, necessary and sufficient conditions are derived on the nominal systems by using the information on the distribution of the disturbances explicitly. Moreover, based on a tailored terminal set, a multi-step open-loop MPC optimization problem with infinite prediction horizon is transformed into a tractable quadratic programming problem with guaranteed recursive feasibility. The closed-loop system is shown to be stable. Numerical examples illustrate the efficacy of the proposed scheme in terms of performance, constraint satisfaction, and reduction of both control updates and communications with a conventional time-triggered scheme.

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1. Introduction

Networked control systems are usually subject to constraints and uncertainties. The constraints include not only the traditional system constraints, such as state constraints, but also communication constraints, such as a limited bandwidth in wireless communication networks. For such systems, an integrative model predictive control (MPC) and event-based control approach is a natural idea which could ensure the system constraint satisfaction and trade off the performance of control systems and the usage of communication resources. Thus, the research of event-based MPC is of great interest.

Two specific types of event-based control are event-triggered and self-triggered control. Different from event-triggered control which requires the continuous monitoring of system states, self-triggered control determines the next update time in advance based on the information at the current sampling instant. Also, self-triggered control allows the shut-down of the sensors between two updates, resulting in a lower sampling frequency to prolong the lifespan of sensors powered by batteries. Please refer to Heemels, Johansson, and Tabuada (2012) and Hetel et al. (2017) for an overview of event-based control.

This paper considers a self-triggered implementation of stochastic MPC (SMPC) for linear systems with stochastic disturbances. One main feature of SMPC is the presence of probabilistic constraints, which require the constraints to be satisfied with given probability thresholds. Such constraints can mitigate the conservativeness introduced by hard constraints of robust MPC (RMP). SMPC has found applications in diverse fields, e.g., building climate control (Long, Liu, Xie, & Johansson, 2014) or chemical processes (Qin & Badgwell, 2003). To the best of our knowledge, stochastic self-triggered MPC (SSMPC) has not been explored up to now. One remarkable challenge is how to characterize the ‘propagation’ of uncertainties during two sampling instants and formulate a computationally tractable optimization problem for determining sampling instants and control design.

Some developments of self-triggered MPC are available. Many of these results are proposed for systems without uncertainties (Barradas Berglind, Gommans, & Heemels, 2012; Hashimoto, Adachi, & Dimarogonas, 2017; Henriksson, Quevedo, Sandberg, ...
Section 5 presents numerical simulations and Section 6 concludes. For systems with uncertainties, most results account for the synthesis of self-triggered control and RMPC which aims to guarantee robust constraint satisfaction. The interested reader can refer to Aydiner, Brunner, and Heemels (2015) and Brunner, Heemels, and Allgöwer (2014, 2016). By maximizing the inter-sampling time subject to constraints on the cost function, a robust self-triggered MPC (RSMPMC) algorithm is presented for constrained linear systems with bounded additive disturbances in Brunner et al. (2014), which employs the robust Tube MPC method in Mayne, Seron, and Raković (2005) to guarantee constraint satisfaction. In Brunner et al. (2014), all constraint parameters are determined by fixing the maximal inter-sampling time, which has the drawback of leading to a conservative region of attraction. To alleviate the conservatism, a RSMPMC algorithm is proposed in A. R. D. M. Valério, 2014, which extends the linear Tube method (Raković, Kouvaritakis, Findeisen, & Cannon, 2012) to the presence of bounded disturbances. By combining with the self-triggering mechanism in Aydiner et al. (2015), a recent RSMPMC method is presented in Brunner et al. (2016) with the focus of extending the Tube method in Chisci, Rossiter, and Zappa (2001) to evaluate the effect of the uncertainty on the prediction of the self-triggered setup.

Inspired by Aydiner et al. (2015) and Brunner et al. (2016), we design a self-triggered strategy for SMPC. Notice that inherent differences between SMPC and RMPC make our SMPC algorithm largely different from the ones presented in Aydiner et al. (2015) and Brunner et al. (2016). Following the ideas of Tube MPC (Kouvaritakis, Cannon, Raković, & Cheng, 2010), we construct stochastic tubes as tight as possible by explicitly using the distributions of the disturbances. Since a crucial assumption of feedback at every time step in Kouvaritakis et al. (2010) is not satisfied in the self-triggered setting (which allows open-loop operations between sampling instants), some appropriate and non-trivial modifications are needed: (i) by considering the multi-step open-loop operation between control updates, three predicted controllers are defined for different phases of the prediction horizon, making it more complex than (Kouvaritakis et al., 2010) to evaluate the effect of the uncertainty on predictions and construct equivalent deterministic constraints; (ii) the inter-sampling time as an optimizing variable is included in the cost function and a tuning parameter is introduced to provide a trade-off between performance and communication; (iii) an improved terminal set, which is adapted to different inter-sampling times, is designed to make the constraints recursively feasible.

The present paper is the first work on SSMPC, which extends the existing literatures on MPC considerably. The main contributions are summarized in the following. (i) Our joint design of the self-triggering mechanism and the SMPC controller effectively reduces the amount of communication, while guaranteeing control performance with specific level of trade-off; (ii) The MPC optimization problem is transformed into a tractable quadratic programming problem by using information on the disturbance distribution. (iii) For the self-triggering mechanism, the probability of constraint violation can be tight to the specified limit. (iv) Both recursive feasibility and closed-loop stability are guaranteed. To illustrate the effectiveness of the algorithm, numerical experiments are carried out to compare the proposed SSMPC with a periodically-triggered SMPC (PSMPC), RSMPC, and unconstrained MPC (LQO).

The remainder of this paper is structured as follows. Problem formulation is set up in Section 2. In Section 3, a multi-step open-loop MPC optimization problem is formulated incorporating probabilistic constraints and specific terminal sets. In Section 4, a SSMPMC algorithm is developed and main results are established. Section 5 presents numerical simulations and Section 6 concludes.

2. Problem formulation

The self-triggered MPC framework of this paper is shown in Fig. 1, in which the notations are introduced below. Consider a linear time-invariant system

\[ x(k+1) = Ax(k) + Bu(k) + w(k), \quad k \in \mathbb{N}, \]

where \( x(k) \in \mathbb{R}^n \) is the state, \( u(k) \in \mathbb{R}^m \) the control input, \( w(k) \in \mathbb{R}^n \) the stochastic disturbance, and \( (A, B) \) a stabilizable pair. Notice that \( N_T = N_w \). Assume that \( u(k), k \in \mathbb{N} \) are independent and identically distributed (i.i.d.) and the elements of \( u(k) \) have zero mean. The distribution \( F \) of the ith element \( u_i(k) \) is assumed to be known and continuous with a bounded support \([-\sigma_i, \sigma_i] \), \( \sigma_i > 0 \), and correspondingly we have \( u(k) \in \mathbb{W} \). Moreover, system (1) is subject to \( n_t \) probabilistic constraints \( Pr[g_i(x(k) \leq h_i)] \geq \rho_i, \quad \ell \in \{1, \ldots, n_t\}, \quad k \in \mathbb{N} \), where \( g_i \in \mathbb{R}^{n_t}, h_i k \in \mathbb{R} \), and \( \rho_i \in [0, 1] \). In the sequel, we will focus on one probabilistic constraint

\[ Pr[g_i^T(x(k) \leq h_i)] \geq p_i, \quad k \in \mathbb{N}, \]

as the other constraints can be treated in a similar way.

In a periodically-triggered MPC scheme, the predictive control input at time \( k \) can be designed as

\[ u(k+i|k) = Kx(k+i|k) + c(k+i|k), \quad i \in \mathbb{N}, \]

where \( K \in \mathbb{R}^{n \times n} \) is chosen offline such that the matrix \( \Phi = A + BK \) is Schur stable and for a prediction horizon \( N \in \mathbb{N} \geq 1 \), \( c(k+i|k) \) are optimization variables and \( c(k+i|k) = 0 \) for \( i \in \mathbb{N} \). At each time instant \( k \), \( u(k) = Kx(k) + c(k|k) \) is applied to the system.

To reduce the amount of communication, in the self-triggered scheme, the states \( x(k) \) are only measured and transmitted to the controller at sampling instants \( k_j \in \mathbb{N}, j \in \mathbb{N} \), which evolve as \( k_{j+1} = k_j + M_j \) with \( k_0 = 0 \). The inter-sampling time \( M_j \in \mathbb{N} \) is determined by a self-triggering mechanism based on the state at sampling instant \( k_j \). Since the values of \( x(k_j + i|k_j), \quad i \in \mathbb{N} \), cannot be determined at time \( k_j \), the presence of

![Fig. 1. The self-triggered MPC framework.](image-url)
stochastic disturbances, the predictive control sequence in (3) is not applicable to control the system at time instants between \(k_i\) and \(k_{i+1}\) in an open-loop fashion and we redefine it in the self-triggered setup as

\[
\begin{align*}
  u(k_i + l | k_i) &= K z(k_i + l | k_i) + c(k_i + l | k_i), \\
  i &\in \mathbb{N}_{> M-1},
\end{align*}
\]

(4)

\[
\begin{align*}
  u(k_i + l | k_i) &= K x(k_i + l | k_i) + c(k_i + l | k_i), \\
  i &\in \mathbb{N}_{> M,N-1},
\end{align*}
\]

(5)

\[
\begin{align*}
  u(k_i + l | k_i) &= K x(k_i + l | k_i), \\
  i &\in \mathbb{N}_{> N},
\end{align*}
\]

(6)

where the nominal state \(z(k_i + l | k_i) \eqdef \mathbb{E}[x(k_i + l | k_i)]\) evolves as

\[
\begin{align*}
z(k_i + l + 1 | k_i) &= \mathcal{P} z(k_i + l | k_i) + B c(k_i + l | k_i), \\
i &\in \mathbb{N},
\end{align*}
\]

(7)

with \(z(k_i | k_i) = x(k_i)\). The predictive controller (4) is designed with respect to nominal state predictions, which are deterministic and only depend on \(x(k_i)\). And (5) is designed with respect to disturbed state predictions, which are introduced to affect the design of the uncertainty in the prediction. After the Nth prediction time, the predictive controller (6) is given by the state feedback law. Notice that the number of decision variables is a finite \(N\). As shown in Fig. 1, after solving an MPC optimization problem at sampling instant \(k_i\), the first \(M_i\) predictive control inputs, i.e., \(u(k_j | k_i)\), \(j \in \{0, 1, 2, \ldots, M_i - 1\}\), are transmitted to the actuator and are applied to the system until the next sampling instant \(k_{i+1}\).

The goal is to design \(\mathbf{c}(k_i) = [c^T(k_i | k_i) c^T(k_i + 1 | k_i) \ldots c^T(k_i + N - 1 | k_i)]^T \in \mathbb{R}^{N \times N}\) and to maximize \(M_i\) at each sampling instant \(k_i\), such that a low frequency of control updates and communication is achieved, while stabilizing a neighborhood of the origin and guaranteeing probabilistic constraint satisfaction.

3. Optimization problem formulation

In this section, the problem described in Section 2 is formulated to a computationally tractable MPC optimization problem with a fixed inter-sampling time \(M \in \{1, N-1\}\). Define the prototype optimization problem \(\Pi_0^M(\mathbf{c}(k_i))\) as follows.

\[
\begin{align*}
\min_{{\mathbf{c}(k_i)}} J^M(\mathbf{c}(k_i)) &\eqdef \frac{1}{\alpha} \sum_{i=0}^{M-1} \mathbb{E}_{k_i}[\|x(k_i + l | k_i)\|^2_Q] \\
+ \|u(k_i + l | k_i)\|^2_L - \ell_i A_i + \sum_{i=M}^{\infty} \mathbb{E}_{k_i}[\|x(k_i + l | k_i)\|^2_Q] \\
+ \|u(k_i + l | k_i)\|^2_L - \ell_i A_i \\
\text{subject to (4)-(6) with } M_i = M, z(k_i | k_i) = x(k_i), \text{and} \quad (7)
\end{align*}
\]

\[
\begin{align*}
\forall i \in \mathbb{N}_{> M-2} : &\quad z(k_i + 1 | k_i) = Az(k_i + l | k_i) + Bu(k_i + l | k_i) \\
\forall i \in \mathbb{N} : &\quad x(k_i + l | k_i) = Ax(k_i + l | k_i) + Bu(k_i + l | k_i) + u(k_i + l | i) \\
\forall i \in \mathbb{N}_{> 1} : &\quad \Pr[g^T x(k_i + l | i) \leq h_i] \geq p.
\end{align*}
\]

(8a)

(8b)

(8c)

Therein, \(Q > 0\) and \(R > 0\) are weighting matrices and scalar \(\alpha \geq 1\) is a tuning parameter. The constant \(\ell_i \eqdef \lim_{n \to -\infty} \mathbb{E}_{k_i}[\|x(k_i + l | k_i)\|^2_Q + \|u(k_i + l | k_i)\|^2_L] \) can be computed offline by Theorem 2 of Cannon, Kouvaritakis, and Wu (2009). Although the infinite-horizon cost in (7) can be expressed as a quadratic function (see Remark 3.2), solving \(\Pi_0^M(\mathbf{c}(k_i))\) online is still unrealistic due to the presence of an infinite number of probabilistic constraints.

Remark 3.1. Based on the sampling interval, cost function (7) is divided into two parts similar to Aydiner et al. (2015) and Brunner et al. (2016). In Aydiner et al. (2015) and Brunner et al. (2016), the cost function consists of finite-horizon costs and a terminal cost which are defined by nominal systems. Considering the presence of stochastic uncertainties, we define (7) in expectation.

Remark 3.2. Using the probabilistic distribution of \(w\) and extending the proof of Theorem 2 in Cannon et al. (2009), cost function (7) can be written as a quadratic form of \(\mathbf{c}(k_i)\) given by \(J^M(\mathbf{c}(k_i)) = \mathbf{c}^T(k_i) P_c \mathbf{c}(k_i) + \mathbf{c}^T(k_i) P_{a} x(k_i) + f_{a}(x(k_i)) + f_{w}\), where \(P_{c} > 0\), \(P_{a}\), and \(P_{a}\) are constant matrices, and function \(f_{a}(x(k_i))\) and constant \(f_{w}\) are determined, respectively, by the state \(x(k_i)\) and the distribution of \(w\) regardless of the choice of \(\mathbf{c}(k_i)\).

3.1. Probabilistic constraint handling strategy

To render \(\Pi_0^M(\mathbf{c}(k_i))\) computationally tractable, we will convert probabilistic constraints (8c) to deterministic ones, such that the observed probability of constraint satisfaction is tight to the specified value and the derived constraints are recursively feasible for the closed-loop system. Under the assumption that the first \(M\) inputs in the sequence are applied in an open-loop fashion, Lemma 3.1 gives an equivalent form of (8c).

Lemma 3.1. For any \(M \in \{1, N-1\}\) and any sampling instant \(k_i \in \mathbb{N}\), probabilistic constraints \(\Pr[g^T x(k_i + l | k_i) \leq h_i] \geq p_i \) in \(\Pi_0^M(\mathbf{c}(k_i))\) are satisfied if and only if \(\mathbf{c}(k_i)\) satisfies

\[
\begin{align*}
g^T \mathbf{P} x(k_i) + g^T H_i \mathbf{c}(k_i) &\leq h_i - \gamma_i^M, \quad i \in \mathbb{N}_2, \quad (9)
\end{align*}
\]

where \(H_i \eqdef [\mathbf{P} \mathbf{I} - 2 \mathbf{P} \mathbf{B} \mathbf{P} \mathbf{I} - 2 \mathbf{B} 0 \ldots 0] \) and \(\gamma_i^M\) is defined as the minimum value such that

\[
\begin{align*}
\Pr\left[\begin{align*}
g^T A^{-1} w(k_i) + \cdots + g^T w(k_i + i - 1 | k_i) \\
\leq \gamma_i^M \end{align*}\right] &= p_i, \quad i \in \mathbb{N}_{1,M}. \quad (10)
\end{align*}
\]

Proof. For \(i \in \mathbb{N}_{1,M}\), it holds by (4) that \(x(k_i + l | k_i) = \mathbf{P} x(k_i) + H_i \mathbf{c}(k_i) + \mathbf{P} A^{-1} w(k_i) + \cdots + w(k_i + i - 1 | k_i)\). Further, for \(i \in \mathbb{N}_{M+1}\), it follows from (5) that \(x(k_i + l | k_i) = \mathbf{P} x(k_i) + H_i \mathbf{c}(k_i) + \mathbf{P} A^{-1} w(k_i + i | k_i) + \sum_{i=0}^{m-1} \mathbf{P}^i w(k_i + i - 1 - i | k_i)\). Hence, it follows directly by (10) that (8c) are equivalent to deterministic constraints (9).

In practice, the values of \(\gamma_i^M\) can be approximated using univariate convolutions with arbitrarily small approximation error (see Dai, Xia, Gao, & Cannon, 2017, Remark 3.2). The approximation can be performed offline in polynomial time. By modifying Lemma 3.1, recursively feasible constraints are derived in Theorem 3.1 ensuring the satisfaction of (2).

Theorem 3.1. Given any \(M \in \{1, N-1\}\) and any sampling instant \(k_i\), \(i \in \mathbb{N}\), let for all \(k_i \in \mathbb{N}_{1,k_i+1}\)

\[
\begin{align*}
x(k_i + 1) &= A x(k_i) + Bu(k_i) + u(k_i) \\
u(k_i) &= K z(k_i | k_i) + c(k_i | k_i).
\end{align*}
\]

(11)

Suppose \(\mathbf{c}(k_i)\) satisfies

\[
\begin{align*}
g^T \mathbf{P} x(k_i) + g^T H_i \mathbf{c}(k_i) &\leq h_i - \gamma_i^M, \quad i \in \mathbb{N}_2, \quad (12)
\end{align*}
\]

where \(\gamma_i^M\) is defined as the maximum element of the ith column of \(\Gamma\) in (13) with \(d_i^M \eqdef \max_{w \in \mathbb{V}^N} g^T \mathbf{P}^{i-1} w, b_i^M \eqdef \max_{w \in \mathbb{V}^N} g^T \mathbf{P}^{i-M} w\).
Box I.

\[
\sum_{\ell=0}^{M-1} A^\ell w, \text{ and } \xi_i \text{ the minimum value such that } Pr\{\sum_{\ell=0}^{M-1} A^\ell \Phi_i^T w \\leq \xi_i\} = p. \text{ Then, for the closed-loop system (11), (i) there exists at least one solution } c(k_i+1) \text{ satisfying (12) for } M = 1 \text{ at all future sampling instants } k_i+1, i \in \mathbb{N}_{\geq 1}; \text{ (ii) probabilistic constraints (2) are satisfied for all } k \in \mathbb{N}.\]

**Proof.** For (i), if \( \beta_i^M \) is defined as the first row of (13) given in Box I, then (12) is equivalent to (9). Next, consider the feasibility of (12) for \( M = 1 \) at time \( k_i+1 \). Define \( T \) as the shift matrix with ones on the superdiagonal and zeros elsewhere. Based on the information available at current sampling instant \( k_i \) and constructing a candidate solution \( \hat{c}(k_i+1) \) as \( T^T \hat{c}(k_i) = [c^T(k_i + M k_i) \ldots c^T(k_i + N - 1 | k_i) \ldots 0] \), it yields that \( x(k_i+1) + i(k_i+1) = \Phi^i \hat{c}(k_i) + H_i \hat{c}(k_i) + \Phi^i e(k_i+1) + \Phi^i w(k_i+1) + \ldots + w(k_i+1) + i \). Since \( e(k_i+1) \) depends on \( \hat{c}(k_i) \), \ldots , \( w(k_i + M - 1 | k_i) \) which have already been realized at time \( k_i+1 \), the worst-case bound on \( E(k_i+1|k_i) \) needs to be considered explicitly at time \( k_i \). Hence, to ensure \( \hat{c}(k_i+1) \) is a feasible solution at time \( k_i+1 \), we must require at time \( k_i \)

\[
g^T \Phi^i x(k_i) + g^T H_i \hat{c}(k_i) \leq -b^T + \xi_i^M, \quad i \in \mathbb{N}_{\geq M+1},
\]

which corresponds to defining \( \beta^M_i \) in (12) as the second row of (13). By the same arguments, the feasibility of (12) for \( M = 1 \) at time \( k_i+1 \), \( i \in \mathbb{N}_{\geq 2} \), can be ensured if \( \beta_i^M \) is defined as the \((\ell+1)\)th row of (13) and it holds at time \( k_i \) that

\[
g^T \Phi^i x(k_i) + g^T H_i \hat{c}(k_i) \leq -b^T + \xi_i^M + \xi_i^{M+1} + \ldots + \xi_i^{M+\ell-2} + \xi_i^{M+\ell-1}, \quad i \in \mathbb{N}_{\geq M+\ell}.\]

Taking the intersection of (9), (14), and (15) for \( i \in \mathbb{N}_{\geq 2} \), the feasibility at all sampling instants \( k_i, k_i+1, k_i+2, \ldots \) can be ensured if \( \beta_i^M \) in (12) is chosen as the maximum element of the \( i \)th column of (13), thereby proving (i).

For (ii), from Lemma 3.1, constraints (12) ensure that \( Pr\{g^T x(k_i + i | k_i) \leq h_i \} \geq p \) is satisfied for all \( i \in \mathbb{N}_{[1, M]} \), which implies the satisfaction of (2) for all \( k \in \mathbb{N}_{[k_i+1, k_i+1]} \). By recursive feasibility of (12) in (i), it holds that (2) is satisfied for all \( k \in \mathbb{N}_{[k_i+1, k_i+1]} \) and all \( i \in \mathbb{N}. \)

Parameters \( \gamma_i^M \) in (9) and \( \beta_i^M \) in (12) are determined by not only the length \( i \) of the predicted time steps but also the length \( M \) of the open-loop steps. By setting \( M = 1 \), the results in Lemma 3.1 and Theorem 3.1 are reduced to Theorems 1 and 3 in Kouvaritakis et al. (2010). In the following, two properties of the sequence \( \beta_i^M \), \( i \in \mathbb{N}_{\geq 1} \), are established.

**Lemma 3.2.** For all \( M \in \mathbb{N}_{[1, N-1]} \), it holds that

\[
\beta_i^M = \begin{cases} 
\gamma_i^M, & i \in \mathbb{N}_{[1, M]}, \\
b_i^M + \sum_{\ell=M+2}^{i} d_i^M, & i \in \mathbb{N}_{\geq M+1}.
\end{cases}
\]

**Proof.** For \( i \in \mathbb{N}_{[1, M]} \), (16) holds directly. For \( i \in \mathbb{N}_{\geq M+1} \), we have \( \gamma_i^M \leq b^M + \xi_i^M. \) Further, it holds for \( i \in \mathbb{N}_{\geq M+2} \) that \( \xi_i^M \leq d_i^M + \xi_i^{M+1}. \) Then, it can be concluded that \( \beta_i^M \) is equal to the last non-zero element in the \( i \)th column of (13), which gives (16).

**Lemma 3.3.** For all \( M \in \mathbb{N}_{[1, N-1]} \) and all \( i \in \mathbb{N}_{\geq 2} \), it holds that

\[
\beta_i^{M+1} = b_i^{M+1} + \beta_i^M.
\]

**Proof.** From Lemma 3.2, \( \beta_1^1 \) can be rewritten as \( \beta_1^1 = \sum_{\ell=1}^{1} \max_{u \in \mathbb{R}^w} \gamma_i^T \Phi_i^T w + \gamma_i^1, \quad i \in \mathbb{N}_{\geq 2} \). Then, from the fact that \( \gamma_i^1 = \beta_i^M \), it follows for \( i \in \mathbb{N}_{\geq 2} \) that \( \beta_i^{M+1} = b_i^{M+1} + \sum_{\ell=1}^{M+1} \max_{u \in \mathbb{R}^w} \gamma_i^T \Phi_i^T w + \beta_i^1. \)

**3.2. Terminal set**

To ensure that constraints (12) are satisfied over an infinite prediction horizon, a terminal set is used. First, due to \( c(k_i + N + i | k_i) = 0 \) for all \( i \in \mathbb{N} \), the terminal dynamics of the nominal system can be rewritten as \( z(k_i + N + i | k_i) = \Phi z(k_i + N + i | k_i), \quad i \in \mathbb{N}. \) Define a constraint set for \( z(k_i + N | k_i) \) as

\[
[z \mid g^T \Phi z \leq h - \beta_i^{N+1}, \quad i \in \mathbb{N}].
\]

Then given some \( \bar{N} \in \mathbb{N} \), split the infinite prediction horizon in (17) into two stages \( i \in \mathbb{N}_{\geq M+\bar{N}} \) and \( i \in \mathbb{N}_{\geq M+\bar{N}+1}. \) In the second stage, an upper bound of \( \beta_i^{M+1} \) is used, which is introduced through the following lemma.

**Lemma 3.4.** For all \( M \in \mathbb{N}_{[1, N-1]} \), there exist a scalar \( 0 < \rho < 1 \) and a positive definite matrix \( S \) such that the sequence \( \beta_i^M \) for \( i \in \mathbb{N}_{M+1} \) is upper bounded by

\[
\beta_i^M \leq \bar{\beta}_i^M \leq \bar{b}_i^M + \sum_{\ell=M+2}^{i} \gamma_i^M + \frac{\rho^i}{1-\rho} \|g\|_S + \gamma_i^{M+1}.
\]

with any integer \( v \in \mathbb{N}_{\geq M+3} \) and \( \bar{b}_i^M \leq \max_{u \in \mathbb{R}^w} \max_{v=2}^{\infty} \sum_{\ell=v}^{i} A^\ell w. \)

**Proof.** In (16), the existence of the upper bound \( \bar{\beta}_i^M \) on \( b_i^M \) can be ensured by the strict stability of \( \Phi \). Furthermore, it holds that \( \sum_{\ell=M+2}^{i} d_i^M \leq \sum_{\ell=1}^{\infty} d_i^M \). From Lemma 5 of Cannon, Cheng, Kouvaritakis, and Raković (2012), the bound on \( \sum_{\ell=M+2}^{i} d_i^M \) is given by \( \sum_{\ell=M+2}^{i} d_i^M + \frac{\rho}{1-\rho} \|g\|_S \), thereby completing the proof.

The scalar \( \rho \) and matrix \( S \) can be obtained by solving semidefinite programs as (16) in Cannon et al. (2012). Replace \( \beta_i^{N+1} \) in (17) by the bound \( \bar{\beta}_i^M \) for \( i \in \mathbb{N}_{\geq M+N+1} \) and define an inner approximation of (17) as

\[
[z \mid g^T \Phi z \leq h - \rho_i^{N+1}, \quad i \in \mathbb{N}_{\geq M+N}.
\]

Finally, using Theorem 2.3 of Gilbert and Tan (1991), there exists \( n^* \in \mathbb{N}_{\geq 1} \) such that the infinite number of constraints in (19)
are ensured through the first $M + \hat{N} + n^*$ constraints. Hence, the terminal set for $z(k_j + N|k_j)$ is constructed as follows:

$$\begin{align*}
\chi_j^M &\triangleq \{ z | g^T\Phi z \leq h - \rho_i^M, i \in \mathbb{I}_{[M+1,N]}^j \}, \\
g^T\Phi z &\leq h - \beta_j^M, i \in \mathbb{I}_{[M+1,N]}^j \}.
\end{align*}$$

(20)

where the smallest allowable $n^*$ can be computed offline by solving a finite number of linear programs (see Gilbert & Tan, 1991).

3.3. Optimization problem

Given a state $x(k_j)$ and $M \in \mathbb{N}_{[1,N-1]}$, the constraints at sampling instant $k_j$ are summarized as follows.

$$z(k_j|k_j) = x(k_j)$$

(21a)

$$\forall i \in \mathbb{I}_{[2,N]} : z(k_j + i|k_j) = \Phi z(k_j + i|k_j) + Bc(k_j + i|k_j)$$

(21b)

$$\forall i \in \mathbb{I}_{[2,N]} : u(k_j + i|k_j) = Kz(k_j + i|k_j) + c(k_j + i|k_j)$$

(21c)

$$\forall i \in \mathbb{I}_{[2,N]} : g^T\Phi x(k_j) + g^T H c(k_j) \leq h - \rho_i^M$$

(21d)

$$z(k_j + N|k_j) \in \chi_j^M.$$  

(21e)

Note that there is only a finite number of deterministic constraints in [21], which can be computed for the predictions of the nominal model. Define the set of all $c(k_j)$ satisfying (21) as $\mathcal{X}^M(x(k_j)) \triangleq \{ c(k_j) | (21) \text{ holds} \}$, and the state $x(k_j)$ feasible if $\mathcal{X}^M(x(k_j)) \neq \emptyset$.

At sampling instant $k_j$, an MPC optimization problem $\mathcal{P}^M(c(k_j))$, which is the tractable version of $\mathcal{P}^M(c(k_j))$, can now be formulated as

$$V^M(k_j) \triangleq \min_{c(k_j): \mathcal{X}^M(x(k_j))} J^M(c(k_j)),$$

$$c^*(k_j) \triangleq \min_{c(k_j): \mathcal{X}^M(x(k_j))} J^M(c(k_j)),$$

where $V^M(k_j)$ denotes the optimal value function and $c^*(k_j)$ the corresponding optimal solution.

Remark 3.3. Clearly constraints (21) are all affine functions in $c(k_j)$ and $J^M(c(k_j))$ is a quadratic cost function, see Remark 3.2. Thus, $\mathcal{P}^M(c(k_j))$ is a quadratic programming problem. More importantly, the computational complexity of $\mathcal{P}^M(c(k_j))$ is not increased compared with the periodically-triggered MPC scheme in Kouvaritakis et al. (2010) regarding the number of constraints and decision variables.

4. Stochastic self-triggered MPC

Using the above MPC optimization problem as a basis, a SSMPC algorithm is designed in this section. In the self-triggered setup, the goal at each sampling instant $k_j$ is to decide not only $c(k_j)$ but also the next sampling instant $k_{j+1}$. To reduce the computation and communication, we need to find the largest $M_j$ such that $\mathcal{P}^M(c(k_j))$ is feasible for some $c(k_j) \in \mathcal{X}^M(x(k_j))$ while still maintaining certain performance of the closed-loop system. Define the self-triggered MPC problem $\mathcal{S}(x(k_j))$ as

$$M_j^* \triangleq \max \{ M \in \mathbb{N}_{[1,M_{max}]} | \mathcal{X}^M(x(k_j)) \neq \emptyset \},$$

$$V^M(k_j) \leq V^1(k_j),$$

(22)

$$c^*(k_j) \triangleq \min_{c(k_j): \mathcal{X}^M(x(k_j))} J^M(c(k_j)),$$

where $M_{max} \in \mathbb{N}_{[1,N-1]}$ is an a priori maximum of the inter-sampling time and $V^1(k_j)$ is the optimal value function corresponding to the MPC scheme in which control updates take place at every time instant.

Remark 4.1. The idea adopted in (22) is similar to Aydiner et al. (2015), Barradas Berglind et al. (2012) and Brunner et al. (2016), in which, by introducing a tuning parameter $\alpha$ as in (7), the optimal value function of the $M$-step open-loop MPC scheme is required to be not worse than that of a periodically-triggered MPC scheme. The inherent differences are that in Barradas Berglind et al. (2012) the system is undisturbed and in Aydiner et al. (2015) and Brunner et al. (2016) the system is subject to bounded disturbances and hard constraints. As a result, the choice of the cost function and the design of the tightened constraint sets are significantly different from that in (22).

Remark 4.2. By employing the triggering mechanism in (22), parameter $\alpha$ may be used to trade off the control performance and the frequency of control updates. If we choose $\alpha = 1$, the optimal value function $V^M(k_j)$ for $M \in \mathbb{N}_{[1,M_{max}]}$ might be in general no smaller than $V^1(k_j)$, thereby leading to a small inter-sampling time $M_j^*$. To obtain a larger one, we can increase the value of $\alpha$ to counter the effect of the open-loop control, while possibly sacrificing slightly the control performance.

The resulting SSMPC algorithm is summarized below.

Algorithm 1 SSMPC

Offline:

Determine $\alpha, Q, R, K, N, \hat{N}, M_{max}$, and $n^*$. For all $M \in \mathbb{N}_{[1,M_{max}]}$, compute $\gamma_i^M, i \in \mathbb{I}_{[1,M]}$, $b_i^M, i \in \mathbb{I}_{[1,N+\hat{N}]}$, $d_i^M, i \in \mathbb{I}_{[N+\hat{N}]}$, $k_{M+1}^*$, and $\beta_j^M$.

Online:

1: Initialize $k = 0$, measure the initial state $x(k)$ and obtain $M^*$ and $c^*(k)$ by solving $\mathcal{S}(x(k))$.

2: For all $i \in \mathbb{I}_{[1,M_{max}]}$, apply the input $u(k + i) = Kz(k + i|k) + c^*(k + i|k)$ to the system.

3: Take the next sampling instant as $k + M^*$ and set $k = k + M^*$.

4: Measure the current state $x(k)$.

5: Solve $\mathcal{S}(x(k))$ to obtain $M^*$ and $c^*(k)$.

6: Go to step 2.

Under Algorithm 1, the resulting closed-loop system is

$$\begin{align*}
x(k + 1) &= Ax(k) + Bu(k) + w(k), \\
u(k) &= Kz(k|k) + c^*(k|k), k \in \mathbb{I}_{[k_0,k_{j+1}]}.
\end{align*}$$

(24)

$$k_{j+1} = k_j + M_j^*,$$

(25)

for $j \in \mathbb{N}$ and $k_0 = 0$, which has the following properties.

Theorem 4.1 (Recursive Feasibility and Constraint Satisfaction). If $\mathcal{S}(x(k_j))$ is feasible at time $k_j$, then the feasibility of $\mathcal{S}(x(k_j))$ can be ensured at every sampling instant $k_j$, $j \in \mathbb{N}$, for (24)-(25) under Algorithm 1. Furthermore, for all $k \in \mathbb{N}$, constraints (2) are satisfied.

Proof. Consider two successive sampling instants $k_j$ and $k_{j+1}$. Let $M_j$ and $c(k_j)$ be solutions of $\mathcal{S}(x(k_j))$ at time $k_j$. Define $c(k_{j+1}) = I^M(c(k_j))$. We will show that $c(k_{j+1})$ and $M_{j+1} = 1$ are the feasible solutions of $\mathcal{S}(x(k_{j+1}))$ at time $k_{j+1}$, i.e., $c(k_{j+1}) \in \mathcal{X}^1(x(k_{j+1}))$.

Constraints (21a)-(21c) in $\mathcal{X}^1(x(k_{j+1}))$ are trivially satisfied. The satisfaction of (21d) by $c(k_{j+1})$ is ensured from Theorem 3.1.
Further, for all \( i \in \mathbb{N} \), we have
\[
g^T \Phi \vec{z}(k_i + M_j + N|k_i + M_j) = g^T \Phi^{M_j+N} z(k_j + N|k_j) + g^T \Phi^N \sum_{\ell=0}^{M_j-1} A^\ell w
\]
\[
\leq h - \beta_{N+i+M_j}^{M_j} + h_{N+i+M_j}^{M_j} \leq h - \beta_{N+i}^1,
\]
where the last inequality follows from Lemma 3.3. Therefore, we immediately obtain that \( z(k_{j+1} + N|k_{j+1}) \in \chi^1 \), i.e., (21e) in \( \mathcal{F}^1(\chi(k_{j+1})) \) is satisfied. From all of the above, it can be concluded that at time \( k_{j+1} \), \( S(\chi(k_{j+1})) \) is feasible and further by induction, \( S(\chi(k_j)) \) are feasible at all sampling instants \( k_j, j \in \mathbb{N} \).

For all \( k \in \mathbb{N} \), the satisfaction of probabilistic constraints (2) is guaranteed directly from Theorem 3.1.

**Theorem 4.2 (Quadratic Stability Property).** The closed-loop system (24)–(25) under Algorithm 1 satisfies the quadratic stability condition
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{k=k_0}^{k-1} \mathbb{E}[\|x(k)\|_Q^2 + \|u(k)\|_R^2] \leq \epsilon_{ss}.
\]

**Proof.** For any \( j \in \mathbb{N} \), let \( M_j \) and \( \overline{c}(k_j) \) be the optimal solutions at sampling instant \( k_j \) and the corresponding optimal value function \( V^{M_j}(k_j) \) be a Lyapunov function candidate. As in the proof of Theorem 4.1, \( \overline{c}(k_{j+1}) = T^{M_j} \overline{c}(k_j) \) together with \( M_{j+1} = 1 \) is a feasible solution at sampling instant \( k_{j+1} \). Define \( \overline{V}^1(k_{j+1}) \) as the value function associated with this feasible solution. Using the fact that \( \alpha \geq 1 \), it holds for the closed-loop system that
\[
\mathbb{E}_k[\overline{V}^1(k_{j+1})] \leq V^{M_j}(k_j) - \frac{1}{\alpha} \sum_{i=0}^{M_j-1} \mathbb{E}_k[\|x(k_j + i|k_j)\|_Q^2 + \|u(k_j + i|k_j)\|_R^2] + \epsilon_{ss}
\]
\[
\leq V^1(k_j) - \frac{1}{\alpha} \sum_{i=0}^{M_j-1} \mathbb{E}_k[\|x(k_j + i|k_j)\|_Q^2]
\]
\[
+ \|u(k_j + i|k_j)\|_R^2 - \epsilon_{ss},
\]
where the second inequality follows from (22). The optimality of the solution leads to
\[
\mathbb{E}_k[\overline{V}^1(k_{j+1})] \leq V^1(k_j) - \frac{1}{\alpha} \sum_{i=0}^{M_j-1} \mathbb{E}_k[\|x(k_j + i|k_j)\|_Q^2 + \|u(k_j + i|k_j)\|_R^2] + \|u(k_j + i|k_j)\|_R^2 - \epsilon_{ss},
\]
Fig. 2. Closed-loop trajectories under four different control schemes for 100 realizations of the uncertainty sequence.
Summing the inequality for $j \in \mathbb{N}_{[0, r-1]}$ and taking expectation on both sides,
\[ \sum_{j=0}^{r-1} \frac{1}{\alpha} \sum_{i=0}^{M_j-1} \mathbb{E}[\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 - \ell_{ss}] \leq \mathbb{E}[V^1(k_0)] - \mathbb{E}[V^1(k_r)]. \]
Since $\mathbb{E}[V^1(k_0)]$ is finite by assumption and $\mathbb{E}[V^1(k_r)]$ is lower bounded due to Remark 3.2, it holds that
\[ \lim_{r \rightarrow \infty} \sum_{j=0}^{r-1} \frac{1}{\alpha} \sum_{i=0}^{M_j-1} \mathbb{E}[\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2] \leq \ell_{ss}, \]
which implies the quadratic stability condition (26).

5. Numerical example

Simulation studies are provided to show the effectiveness and the advantages of the proposed SSMPC in comparison with PSMPC (by setting $M_j = 1$), RSMPC (by setting $p = 1$), and the unconstrained LQR control. Consider a linearized DC–DC converter system as in Lorenzen, Dabbene, Tempo, and Allgöwer (2017),
\[ x(k+1) = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix} x(k) + \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix} u(k) + w(k) \]
subject to $Pr\{\|x(k)\| \leq 2\} \geq 0.8$. Elements of $w(k)$ are assumed to be i.i.d. truncated Gaussian random variables with zero mean, variance 0.04, and bounded by $|w_i(k)| \leq 0.1$ for $i = 1, 2$. In (7), $Q = \text{diag}(1.35), R = 0.1, \alpha = 1.2$, and $\ell_{ss} = 0.37$. $K = [0.263 - 0.329]$ is chosen as the unconstrained LQR gain. The prediction horizon and horizons in (21e) are $N = 8, \tilde{N} = 12$, and $n^* = 1$. The maximal open-loop length is $M_{\max} = 8$. To obtain $\beta^M$ in (21d), $\gamma^M, i \in [1, M_j]$, and $e^M_{0:1}$ are calculated according to Remark 3.2 of Dai et al. (2017). In (21e), we choose $\nu = 13$ and calculate $\rho$ and $\delta$ of $\beta^M$ by (16) in Cannon et al. (2012). Simulations for four control schemes are performed with 1000 realizations of the uncertainty sequence, initial condition [2.5 2.8]T, and a simulation length of $T_{\text{run}} = 18$ steps. The simulations are implemented in Matlab R2012b with Yalmip and SeDuMi solver.

Stability and constraint violation: The state trajectories $(x(k), k = 0, 1, \ldots)$ for 100 realizations of the uncertainty sequence are depicted in Fig. 2 with the black dotted lines being the constraint bounds. The right plots of subfigure (a)-(c) in Fig. 2 enlarge the region of constraint bound to show the constraint violation. As it turns out, with SSMPC, the observed probabilities of constraint violation in the first 5 steps are 19.7%, 20.4%, 19.8%, 20.2%, and 16.3%, while by PSMPC, the violation rates are 19.8%, 20.1%, 19.9%, 16.8%, and 9% for the same 1000 realizations. Furthermore, as expected, RSMPC achieves no constraint violations, whereas violation rate is 100% in the first 3 steps under the unconstrained LQR control. The simulation results indicate that by the proposed SSMPC, the closed-loop state converges to a neighborhood of the origin and the constraint violation is tight to the specified violation value 20%.

Average inter-sampling time and performance: To illustrate the decreased communication achieved by SSMPC, Fig. 3 shows the state trajectories under SSMPC and PSMPC for 1 realization of the uncertainty sequence. The sampling instants are highlighted by red solid circles. It can be observed that the number of the sampling instants is significantly reduced. The associated average number of steps between sampling instants is compared. For each scheme, the same uncertainty sequences are used and the average is taken over 1000 realizations and 18 steps. Under the self-triggered scheme, the average inter-sampling time is $\hat{M} = 2.9$, which leads to an average reduction in communication by 65.5% compared to the scheme with updates at every time instant. In addition, let us compare the performance measure
\[ J_{\text{perf}} = \frac{1}{T_{\text{run}}} \sum_{k=0}^{T_{\text{run}}-1} (\|x(k)\|_Q^2 + \|u(k)\|_R^2 - \ell_{ss}). \]
It is 6.82 for SSMPC, as compared with 6.75 for PSMPC. It can be concluded that by the proposed SSMPC, communication is decreased without much loss in performance, which can also be illustrated by Fig. 3.

6. Conclusion

We proposed a SSMPC strategy for the stabilization of systems with additive disturbances and probabilistic constraints. It was shown that the required amount of communication was reduced while simultaneously guaranteeing a specific performance loss when compared with a periodically-triggered scheme. By taking the disturbances occurring during the inter-sampling period into account and making use of their probability distribution, a set of deterministic constraints and terminal sets were constructed to formulate a computationally tractable MPC optimization problem.
Probabilistic constraints were ensured at each time instant despite the open-loop operation between any two sampling instants. Moreover, recursive feasibility and stability were proved for the closed-loop system. The results were compared in simulations with other MPC methods from the literature.

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References


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