Stabilization of Time-Delay Systems through Linear Differential Equations Using a Descriptor Representation

Alexandre Seuret and Karl H. Johansson

Abstract— **In this article, a new method to assess stability and to design static state feedback controller for linear time-delay systems is introduced. The method based on linear differential equations allows considering explicit Lyapunov-Krasovskii functionals with non constant matrix parameters. The stability conditions considering constant delays are delay-dependent and expressed using easy computable linear matrix inequalities. An example is introduced to show the efficiency of the stabilization criteria.**

I. Introduction

During the last decades, a great attention attracted researchers on the stability analysis of time delay systems [10], [20], [22]. The study of the time-delay systems is motivated by the apparition of delays phenomenon in many processes such as biology, chemistry, economics, as well as population dynamics [16], [20]. Moreover, processing time and propagation in actuators and sensors generally induce such delays. In particular, the recent and intense activities around networked controlled systems for which actuators and sensors exchange data through a network highly motivate this research. Many phenomena induced by the network as communication delays [12] or samplings [5], [19] can be interpreted by a delay modeling. In the case of devices which are remotely installed, delays may increase dramatically. As they usually have a disturbing effect, the problem of stability with respect to the time-delays is fundamental.

Note that there exist some systems which are not asymptotically stable for small delays but becomes for sufficiently large delays (in high-speed networks, biological systems and some examples [16]). For this class of systems, the stability cannot be performed using simple Lyapunov-Krasovskii functionals (LKF), ie. functionals parameterized with constant matrices (see, among the less conservative results, [7], [8], [11], [23]). To deal with them, the introduction of more complex and amenable LKF has been introduced to overcome these difficulties. The discretization method introduced by K. Gu [9], [10], allows constructing piecewise linear functions as the LKF parameters by dividing the delay interval into several smaller intervals on which the parameters of the LKF are linearly varying. The stability analysis leads to less conservative than in the case of constant parameters but extensions to the time-varying delay case are not straightforward. Another interesting method introduced in [21], suggested a way to build LKF with varying parameters based on sum of squares tools. Another method which takes a lot of attention nowadays is based on Integral Quadratic Constrains (IQC) [13], [14]. It allows having a better understanding of the terms of the LKF and to have a better idea of where the conservatism is introduced [2] and especially obtaining less conservative results. In [15], a complete LKF (ie. which corresponds to necessary and sufficient conditions of stability) is constructed by solving a functional differential equation. This approach is useful to derive robustness conditions with respect to delay variations [4] or parameters uncertainties [17], [18].

In this article a novel method to construct LKF with varying parameters over the delay interval is introduced. The LKF is provided by a simple and arbitrary linear differential equation and leads to sufficient stability conditions. This technic together with the descriptor representation [6] leads to suitable criteria to design static state feedback gain which ensures stability of the system with constant delay. The paper is organized as follows: Section II is devoted to the formulation of the problem. The form of the LKF is examined in Section III. Section IV and V concern the stability analysis and the stabilization of linear systems with constant delay. Section VI discuses the choice of linear differential equation. An example is provided in Section VII to show the efficiency of the method.

Notations. *Throughout the article, the function x_t corresponds to $x_t(\theta) = x(t+\theta),\ \forall\theta \in [-\tau_2,\ 0]$. The superscript 'T' stands for the matrix transposition. The notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is a symmetric and positive definite matrix. The symbol I_n represents the $n \times n$ identity. For any square matrices A and B, the notation $diag(A,B)$ denotes the block diagonal matrix where the first diagonal block is A and, B the second. Given any positive integers n,N and M, consider a matrix $A \in R^{nN\times nM}$. The notation $A_{n,ij}$ corresponds the matrix of size $n \times n$ located in the between the $in+1$ and $(i+1)n$ rows and the $jn+1$ and $(j+1)n$ columns of A. We denote $\otimes$, by the Kronecker product.*

II. Problem formulation

Consider the controlled system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + Bu(t) + B_\tau u(t-\tau) \\ x(\theta) = \phi(\theta), \quad \theta \in [-\tau,\ 0] \end{cases} \tag{1}$$

A. Seuret is with the NeCS Team, INRIA Rhne-Alpes and the Automatic Control Department GIPSA-Lab, UMR CNRS 5216, BP. 46, F-38402, Saint Martin d'Hres, France. E-mail: alexandre.seuret@gipsa-lab.inpg.fr

K.H. Johansson is with the ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm, Sweden. E-mail: kallej@kth.se

This work was supported by the European Project FeedNetBack : http://www.feednetback.eu/.

ISBN 978-963-311-369-1

where $x \in R^n$ and $u \in R^m$ represents the state and the input vectors. The function $\phi \in [-\tau\ 0] \to R^n$ corresponds to the initial conditions of the time-delay system. The matrices A, A_τ, B and B_τ are assumed to be known and constant. The delay is assumed to be constant. The control law is a linear state-feedback of the form:

$$u(t) = Kx(t) \tag{2}$$

where K is a $m \times n$ matrix. In [9], a general form of LKF's is introduced:

$$\begin{array}{rl} V(x_t) = & x^T(t)Px(t) + 2x^T(t)\int_{-\tau}^{0}\mathscr{Q}(\xi)x_t(\xi)d\xi \\ & +\int_{-\tau}^{0} x_t(T(\xi)\mathscr{S}(\xi)x_t(\xi)d\xi \\ & +\int_{-\tau}^{0}\int_{-\tau}^{0} x_t^T(s)\mathscr{R}(s,\xi)ds x_t(\xi)d\xi \end{array} \tag{3}$$

where $P > 0$, $\mathscr{Q}(\xi)$, $\mathscr{R}(s,\xi) = \mathscr{R}^T(\xi,s)$, $\mathscr{S}(\xi) \in \mathbb{R}^{n\times n}$ are continuous matrix functions. The integral terms in V represent the influence of the state x_t on the stability. In [13], it was proven those integral terms corresponds to the robustness of the system with respect to the delay operator $e^{-\tau s}$. This article introduces a new method to explicitly construct continuous functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$. These variations in the parameters allow considering more accurate LKF's and will lead to less conservative conditions.

III. Parametrization of Lyapunov-Krasovskii Functionals

A. Introduction

Consider some scalar functions f^i defined on $[-\tau\ 0]$ where $i = 1,\dots,N$ and the following functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ such that for all s and ξ in $[-\tau\ 0]$:

$$\begin{array}{rl} \mathscr{Q}(\xi) = & \sum_{i=1}^{N} f^i(\xi)Q_i, \\ \mathscr{S}(\xi) = & \sum_{i=1}^{N}\sum_{j=1}^{N} f^i(\xi)f^j(\xi)S_{ij}, \\ \mathscr{R}(s,\xi) = & \sum_{i=1}^{N}\sum_{j=1}^{N} f^i(s)f^j(\xi)R_{ij}, \end{array}$$

where Q_i, S_{ij} and R_{ij} for $i,j = 1,\dots,N$ are constant matrices. Introducing the vector function $W^f(\xi) = [f^1(\xi),\dots,f^N(\xi)]^T$, a nice expression of the functions can be derived:

$$\begin{array}{rl} \mathscr{Q}(\xi) = & Q\mathscr{W}^f(\xi), \\ \mathscr{S}(\xi) = & (\mathscr{W}^f)^T(\xi)S\mathscr{W}^f(\xi), \\ \mathscr{R}(s,\xi) = & (\mathscr{W}^f)^T(s)R\mathscr{W}^f(\xi) \end{array} \tag{4}$$

where $\mathscr{W}^f = W^f \otimes I_n$, Q, R and S are such that $Q_i = (Q)_{n,i1}$, $R_{ij} = (R)_{n,ij}$ and $S_{ij} = (S)_{n,ij}$. The functions which defined the LKF are thus expressed in a simple way. A lemma to ensure the LKF is positive definite is thus formulated:

Lemma 1: For given $\tau > 0$, the LKF (3) with (4) is positive definite if there exist $P = P^T > 0$ in $R^{n\times n}$, $S = S^T > 0$ in $R^{Nn\times Nn}$, $Q \in R^{n\times Nn}$ and $R = R^T$ in $R^{Nn\times Nn}$ such that the following LMI holds:

$$\Xi = \left[\begin{array}{cc} P & Q \\ Q^T & R + S/\tau \end{array}\right] > 0. \tag{5}$$

Proof: Consider the functional (3) where the functions $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{S}$ are defined in (4). Consider the vector $\Phi_f(t) = \int_{-\tau}^{0}\mathscr{W}^f(\xi)x_t(\xi)d\xi$. The second and the last terms of V can thus be rewritten as $2x^T(t)Q\Phi_f(t)$ and $\Phi_f^T(t)R\Phi_f^T(t)$. Provided that $S > 0$, the Jensen's inequality ensures that

$$\int_{-\tau}^{0} x_t^T(\xi)\mathscr{W}^{fT}(\xi)S\mathscr{W}^f(\xi)x_t(\xi)d\xi \ge \Phi_f^T(t)S/\tau\Phi_f(t)$$

Denote $\xi_0(t) = [x^T(t)\ \ \Phi_f^T(t)]^T$, the LKF satisfies $V(x_t) \ge \xi_0^T(t)\Xi\xi_0(t)$. Then if (5) holds V is positive definite. ∎

In [1], [2], it was proven that increasing the dimension of the state vector (considering $X_1 = [x^T, \dot{x}^T, \dots, (x^{(N)})^T]^T$ or $X_2 = [x^T(t), x^T(t-\tau/N), \dots, x^T(t-\tau)]^T$) considerably reduces the conservatism. Instead of considering these two vectors, we introduce the vector Φ_f with appropriate weighting functions f^i to design less conservative conditions.

B. Introduction of an arbitrary linear differential equation

For a given integer $N > 0$, consider a square matrix D in $R^{N\times N}$. Define the vectorial function W such that for any $\xi \in [-\tau\ 0]$:

$$\left\{\begin{array}{l} \dot{W}(\xi) = DW(\xi), \\ W(0) = W_0 \end{array}\right. \tag{6}$$

where $W_0 \in R^N$. Introduce the vector $\mathscr{W}(\xi) = W(\xi) \otimes I_n$ and the consider now the LKF (3) defined by :

$$\begin{array}{rl} \mathscr{Q}(\xi) = & Q\mathscr{W}(\xi), \\ \mathscr{S}(\xi) = & \mathscr{W}^T(\xi)S\mathscr{W}(\xi), \\ \mathscr{R}(s,\xi) = & \mathscr{W}^T(s)R\mathscr{W}(\xi) \end{array} \tag{7}$$

where Q is in $R^{n\times nN}$ and $S, R \in R^{nN\times nN}$ are symmetric constant matrices. In the latter, we will say that the pair (D, W_0) generates the LKF V if functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ are given by (6) and (7). Apparently, there is no restriction on the matrix D. Depending on the eigenvalues of D, the functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ could be polynomial, exponential and/or trigonometric. The choice of (D, W_0) is discussed in Section VI. As W is the solution of linear differential equations of the type of (6), the functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ are infinitely differentiable over the interval $[-\tau\ 0]$. Another advantage is that simple expressions of their derivative and of their value at some particular instant are simply derived from the solution of the well known linear differential equation, using $\mathscr{D} = D \otimes I_n$ and $\mathscr{W}_0 = \otimes I_n$.

Proposition 1: The differentiation of the functions $\mathscr{Q}, \mathscr{S}, \mathscr{R}$ is straightforwardly given by:

$$\begin{array}{l} \dot{\mathscr{Q}}(\xi) = Q\mathscr{D}\mathscr{W}(\xi), \\ \dot{\mathscr{S}}(\xi) = \mathscr{W}^T(\xi)[\mathscr{D}^T S + S\mathscr{D}]\mathscr{W}(\xi), \\ \frac{\partial \mathscr{R}(s,\xi)}{\partial s} + \frac{\partial R^T(s,\xi)}{\partial \xi} = \mathscr{W}^T(s)(\mathscr{D}^T R + R\mathscr{D})\mathscr{W}(\xi), \end{array}$$

and their evaluation at any instant a and b in $[-\tau\ 0]]$

$$\begin{array}{l} \mathscr{Q}(a) = Qe^{a\mathscr{D}}\mathscr{W}_0, \\ \mathscr{S}(a) = \mathscr{W}_0^T\left(e^{a\mathscr{D}}\right)^T Se^{a\mathscr{D}}\mathscr{W}_0, \\ \mathscr{R}(a,b) = \mathscr{W}_0^T\left(e^{a\mathscr{D}}\right)^T Re^{b\mathscr{D}}\mathscr{W}_0. \end{array}$$

IV. Stability analysis

Consider system (1) and the controller (2) with a given state feedback gain K. The dynamics of the system can be rewritten as:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) \tag{8}$$

where $A_0 = A + BK$ and $A_1 = A_\tau + B_\tau K$. The following theorem holds:

Theorem 1: For a given N and $\tau > 0$, consider a matrix D in $R^{N\times N}$ and a vector W_0. System (8) is asymptotically stable if there exist a positive symmetric definite matrix P_1 in $R^{n\times n}$, $S_0 = S_0^T$, P_2, P_3 in $R^{n\times n}$, a matrix Q in $R^{n\times nN}$ and $R = R^T$, $S = S^T$ and $T = T^T$ in $R^{nN\times nN}$ such that, the following LMIs hold:

$$\Pi_1 = \begin{bmatrix} P_1 & Q \\ Q^T & R + S/\tau \end{bmatrix} > 0,$$

$$\Pi_2 = \mathscr{D}^T(S+T) + (S+T)\mathscr{D} > 0,$$

$$\begin{bmatrix} \pi_3^1 & P_1 - P_2^T + A_0^T P_3 & P_2^T A_1 - QE\mathscr{W}_0 & -Q\mathscr{D} + \mathscr{W}_0^T R \\ * & -P_3 - P_3^T & P_3^T A_1 & Q \\ * & * & \pi_3^3 & -\mathscr{W}_0^T E^T R \\ * & * & * & \pi_3^5 \end{bmatrix} = \Pi_3 < 0$$

$$\Pi_4 = S_0 + \mathscr{W}_0^T E^T T E \mathscr{W}_0 > 0$$

and

$$\Pi_5 = \mathscr{D}^T T + T\mathscr{D} > 0$$

where $E = e^{-\tau\mathscr{D}}$ and where π_3^1, π_3^3 and π_3^4 are given by:

$$\begin{aligned} \pi_3^1 &= P_2^T A_0 + A_0^T P_2 + S_0 + \mathscr{W}_0^T(S+T)\mathscr{W}_0 \\ &\quad + Q\mathscr{W}_0 + (Q\mathscr{W}_0)^T \\ \pi_3^4 &= -S_0 - \mathscr{W}_0^T (e^{-\tau\mathscr{D}})^T (S+T)(e^{-\tau\mathscr{D}})\mathscr{W}_0 \\ \pi_3^5 &= -\mathscr{D}^T(R + 1/\tau(S+T)) - (R + 1/\tau(S+T))\mathscr{D} \end{aligned}$$

Proof: Consider the candidate LKF:

$$V_1(x_t) = V(x_t) + \int_{-\tau}^{0} x_t^T(\xi)(S_0 + \mathscr{W}^T(\xi) T \mathscr{W}(\xi)) x_t(\xi) d\xi \tag{9}$$

where V is given by (3) and the functions $\mathscr{Q}$, $\mathscr{Q}$ and $\mathscr{S}$ are given by (7). Compared to (3), an integral term is added to take into account to reduce the conservatism.

The proof is divided into two parts. The first ensures the LKF is positive definite. According to Lemma 1 and introducing the vector $\Phi(t) = \int_\tau^0 \mathscr{W}(\xi)x_t(\xi)d\xi$, V is positive definite if $\Pi_1 > 0$. A careful attention is now required on the second term of V_1. The matrix function $S_0 + \mathscr{W}^T(\xi)T\mathscr{W}(\xi)$ must be positive whatever $\xi \in [-\tau\ 0]$. Consider a non zero vector v in R^n and the associated function f_v such that:

$$\forall \xi \in [-\tau, \quad 0], \quad f_v(\xi) = v^T(S_0 + \mathscr{W}^T(\xi)T\mathscr{W}(\xi))v.$$

the function f_v is continuous and differentiable over $[-\tau\ 0]$. Using the definition of $\mathscr{W}$, its derivative with respect to ξ is $\dot{f}_v(\xi) = v^T \mathscr{W}^T(\xi)\Pi_5\mathscr{W}(\xi))v$. From Theorem 1, it means that $\dot{f}_v > 0$ and, consequently, that f_v is an increasing function of ξ. Then the following inequality holds:

$$v^T \Pi_4 v < v^T(S_0 + \mathscr{W}^T(\xi)T\mathscr{W}(\xi))v.$$

As this inequality holds for any vector v, it means that if $\Pi_4 > 0$, the matrix function $S_0 + \mathscr{W}^T(\xi)T\mathscr{W}(\xi)$ is positive definite over $[-\tau\ 0]$. Finally, the functional V_1 is positive definite. Note that the matrix S_0 or T are not necessary positive definite as it is usually required in the literature.

The second part of the proof consists in ensuring the negativeness of the derivative of V_1 along the trajectories of (8). Differentiating V_1 leads to:

$$\begin{aligned} \dot{V}_1(x_t) &= 2\dot{x}^T(t)\left[Px(t) + \int_{-\tau}^0 \mathscr{Q}(\xi)x_t(\xi)d\xi\right] \\ &\quad + 2x^T(t)\int_{-\tau}^0 \mathscr{Q}(\xi)\dot{x}_t(\xi)d\xi \\ &\quad + 2\int_{-\tau}^0\int_{-\tau}^0 x_t^T(s)\mathscr{R}(s,\xi)\dot{x}_t(\xi)d\xi ds \\ &\quad + 2\int_{-\tau}^0 \dot{x}_t^T(\xi)(\mathscr{S}(\xi) + \mathscr{W}^T(\xi)T\mathscr{W}(\xi))x_t(\xi)d\xi \\ &\quad + x^T(t)S_0x(t) - x^T(t-\tau)S_0x(t-\tau) \end{aligned} \tag{10}$$

Integrating by parts the terms with $\dot{x}(\xi)$ in the previous equation and introducing the term $2\left[P_2x(t) + P_3\dot{x}(t)\right]^T\left[-\dot{x}(t) + A_0x(t) + A_1x(t-\tau)\right] = 0$, equivalent to the descriptor representation of [6], the following equality is established:

$$\begin{aligned} \dot{V}_1(x_t) &= 2\dot{x}^T(t)P_1x(t) + x^T(t)(S_0 + \mathscr{Q}(0) + \mathscr{Q}^T(0) \\ &\quad + \mathscr{S}(0) + \mathscr{W}_0^T T\mathscr{W}_0)x(t) - 2x^T(t)\mathscr{Q}(-\tau)x(t-\tau) \\ &\quad + 2\dot{x}^T(t)\int_{-\tau}^0 \mathscr{Q}(\xi)x_t(\xi)d\xi + x^T(t-\tau)S_0x(t-\tau) \\ &\quad + 2\left[P_2x(t) + P_3\dot{x}(t)\right]^T\left[-\dot{x}(t) + A_0x(t) + A_1x(t-\tau)\right] \\ &\quad - \int_{-\tau}^0 x_t^T(\xi)(\dot{\mathscr{S}}(\xi) + 2\dot{\mathscr{W}}^T(\xi)T\mathscr{W}(\xi))x_t(\xi)d\xi \\ &\quad - x^T(t-\tau)(\mathscr{S}(-\tau) + 2\mathscr{W}_0^T E^T T E\mathscr{W}_0)x(t-\tau) \\ &\quad + 2x^T(t)\int_{-\tau}^0[-\dot{\mathscr{Q}}(\xi) + \mathscr{R}(0,\xi)]x_t(\xi)d\xi \\ &\quad - \int_{-\tau}^0\int_{-\tau}^0 x_t^T(s)\left(\tfrac{\partial}{\partial s}\mathscr{R}(s,\xi) + \tfrac{\partial}{\partial \xi}\mathscr{R}(s,\xi)\right)ds\, x_t(\xi)d\xi \\ &\quad - 2x^T(t-\tau)\int_{-\tau}^0 \mathscr{R}(-\tau,\xi)x_t(\xi)d\xi \end{aligned} \tag{11}$$

Replacing the functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ by their expression using Proposition 1, (11) can be rewritten as:

$$\begin{aligned} \dot{V}_1(x_t) &= 2\dot{x}^T(t)P_1x(t) \\ &\quad + x^T(t)(S_0 + Q\mathscr{W}_0 + \mathscr{W}_0^T Q^T + \mathscr{W}_0^T(S+T)\mathscr{W}_0)x(t) \\ &\quad + 2\left[P_2x(t) + P_3\dot{x}(t)\right]^T\left[-\dot{x}(t) + A_0x(t) + A_1x(t-\tau)\right] \\ &\quad - 2x^T(t)QE\mathscr{W}_0x(t-\tau) + 2\dot{x}^T(t)Q\Phi(t) \\ &\quad - \int_{-\tau}^0 x_t^T(\xi)\mathscr{W}^T(\xi)[\mathscr{D}^T(S+T) + (S+T)\mathscr{D}] \\ &\quad \times \mathscr{W}(\xi)x_t(\xi)d\xi + 2x^T(t)[-Q\mathscr{D} + \mathscr{W}_0^T R]\Phi(t) \\ &\quad - 2x^T(t-\tau)\mathscr{W}_0^T E^T R\Phi(t) - \Phi^T(t)[\mathscr{D}^T R + R\mathscr{D}]\Phi(t) \\ &\quad - x^T(t-\tau)(S_0 + \mathscr{W}_0^T E^T(S+T)E\mathscr{W}_0)x(t-\tau) \end{aligned}$$

Since $\Pi_2 = \mathscr{D}^T(S+T) + (S+T)\mathscr{D} > 0$, the Jensen's inequality ensures that the integral term of the equation above is bounded by $-1/\tau\Phi^T(t)[\mathscr{D}^T S + S\mathscr{D}]\Phi(t)$. Introducing the vector $\zeta(t) = [x^T(t), \dot{x}^T(t), x^T(t-\tau), \Phi^T(t)]^T$, the following inequality holds:

$$\dot{V}_1(x_t) \le \zeta^T(t)\Pi_3\zeta(t)$$

Then provided that the conditions of Theorem 1 hold, the derivative of the LKF V_1 is negative definite and the system is asymptotically stable. ■

Remark 1: An extension to the case of uncertainties in the system parameters can be dealt by considering system (8) and with A_0 and A_1 from the uncertain polytope:

$$\forall t \in R^+, \quad \Omega(t) = \sum_{k=1}^{M} \lambda_i(t)\Omega_i,$$

where $\forall t \in R^+$, $\sum_{k=1}^{M} \lambda_i(t) = 1, \quad \forall k = 1,..,M, \quad 0 \leq \lambda_i(t)$. The Ω vertices of the polytope are described by $\Omega_k = [A_0(k) \quad A_1(k)]$. As the conditions based in Theorem 1 are linear with respect to the matrices A_0 and A_1, one has to solve those LMIs simultaneously for all the Ω vertices.

Remark 2: The use of the descriptor representation does not reduce the conservatism. However its interest will be exposed in the following section.

V. Stabilization of time-delay systems

Consider system (1). The objective is now to design the gain K of the control law (2) such that the closed-loop system is asymptotically stable. The following result holds:

Theorem 2: For a given N, consider a matrix D in $R^{N\times N}$ and a vector W_0. If for a given $\varepsilon > 0$, there exist $\bar{P}_1 = \bar{P}_1^T > 0$, $\bar{S}_0 = \bar{S}_0^T$, $\bar{P}$ in $R^{n\times n}$, a matrix Y in $R^{m\times n}$, a matrix $\bar{Q}$ in $R^{n\times nN}$ and $\bar{R} = \bar{R}^T$, $\bar{S} = \bar{S}^T > 0$ and $\bar{T} = \bar{T}^T$ in $R^{nN\times nN}$ such that, the following LMIs hold:

$$\Upsilon_1 = \begin{bmatrix} \bar{P}_1 & \bar{Q} \\ \bar{Q}^T & \bar{R} + \bar{S}/\tau \end{bmatrix} > 0,$$

$$\Upsilon_2 = \mathscr{D}^T(\bar{S}+\bar{T}) + (\bar{S}+\bar{T})\mathscr{D} > 0,$$

$$\begin{bmatrix} \upsilon_3^1 & \upsilon_3^{12} & \upsilon_3^{13} & -\bar{Q}\mathscr{D} + \mathscr{W}_0^T\bar{R} \\ * & -\varepsilon(\bar{P}+\bar{P}^T) & \varepsilon(A_\tau\bar{P} + B_\tau Y) & \bar{Q} \\ * & * & \upsilon_3^3 & -\mathscr{W}_0^T E^T \bar{R} \\ * & * & * & \upsilon_3^5 \end{bmatrix} = \Upsilon_3 < 0$$

and

$$\Upsilon_4 = \bar{S}_0 + \mathscr{W}_0^T E^T \bar{T} E \mathscr{W}_0 > 0,$$

$$\Upsilon_5 = \mathscr{D}^T\bar{T} + \bar{T}\mathscr{D} > 0$$

where $E = e^{-\tau\mathscr{D}}$ and where υ_3^1, υ_3^3 and υ_3^4 are given by:

$$\begin{aligned}
\upsilon_3^1 &= A\bar{P} + \bar{P}^T A^T + BY + Y^T B^T + \bar{S}_0 + \bar{S}_1 \\
&\quad + \mathscr{W}_0^T(\bar{S}+\bar{T})\mathscr{W}_0 + \bar{Q}\mathscr{W}_0 + (\bar{Q}\mathscr{W}_0)^T \\
\upsilon_3^{12} &= \bar{P}_1 - \bar{P} + \varepsilon\bar{P}^T A^T + \varepsilon Y^T B^T \\
\upsilon_3^{13} &= A_\tau\bar{P} + B_\tau Y - \bar{Q}E\mathscr{W}_0 \\
\upsilon_3^3 &= -S_0 - \mathscr{W}_0^T E^T(\bar{S}+\bar{T})E\mathscr{W}_0 \\
\upsilon_3^4 &= -\mathscr{D}^T(\bar{R} + (\bar{S}+\bar{T})/\tau) - (\bar{R} + (\bar{S}+\bar{T})/\tau)\mathscr{D}
\end{aligned}$$

Then the system (1) with the state feedback control law (2) with $K = Y\bar{P}^{-1}$ is asymptotically stable.

Proof: Consider a state feedback gain K. Assume the conditions from Theorem 1 are satisfied with $A_0 = A + BK$ and $A_1 = A_\tau + B_\tau K$. Then the controlled system (1) is asymptotically stable for a delay τ. Assume that:

$$P_2 = \varepsilon P_3$$

Noting that a necessary conditions for Π_3 to be negative definite is that the matrix P_2 is non singular. It is thus possible to define $\bar{P} = P_2^{-1}$ and $\bar{\mathscr{P}} = I_N \otimes \bar{P}$. From the definition of the Kroenecker product, it is easy to see that:

$$\bar{P}\mathscr{W}_0 = \mathscr{W}_0\bar{\mathscr{P}}, \quad \bar{\mathscr{P}}\mathscr{D} = \mathscr{D}\bar{\mathscr{P}}$$

Introducing the variables $\bar{P}_1 = \bar{P}^T P_1 \bar{P}$, $\bar{S}_0 = \bar{P}^T S_0 \bar{P}$, $\bar{S}_1 = \bar{P}^T S_1 \bar{P}$, $\bar{Q} = \bar{P}^T Q \bar{\mathscr{P}}$, $\{\bar{R},\ \bar{S},\ \bar{T}\} = \bar{\mathscr{P}}^T\{R,\ S,\ T\}\bar{\mathscr{P}}$, and $Y = K\bar{P}$, it is straightforward to see that $\Upsilon_1 = (I_{N+1}\otimes\bar{P})^T\Theta_1\ (I_{N+1}\otimes\bar{P})$, $\Upsilon_2 = \bar{\mathscr{P}}^T\Theta_2\ \bar{\mathscr{P}}$, $\Upsilon_3 = (I_{N+4}\otimes\ \bar{P})^T\Theta_3\ (I_{N+4}\otimes\bar{P})$, $\Upsilon_4 = \bar{P}^T\Theta_4\bar{P}$ and $\Upsilon_5 = \bar{\mathscr{P}}^T\Theta_5\ \bar{\mathscr{P}}$ and the proof is completed. ■

The interest of the descriptor representation appears in the stabilization problem. Obtaining constructive LMI conditions which design the controller gain K not based on descriptor representation is not as straightforward as the one based on Theorem 1.

In Theorems 1 and 2, the conservatism is considerably reduced by the introduction of the matrix T. First the conditions $\Pi_4 > 0$ and $\Pi_5 > 0$ allows the matrices T and S_0 to be not necessarily positive definite. It is also possible to consider matrices D which only have non zero eigenvalues since T is not necessarily positive definite. However it is still required the D does not have zero eigenvalues.

Despite the reduction of the conservatism, the price to pay concerns the number of variables to solve. As in the discretization method, it would be possible to reduce the number of variable by imposing restrictions on the form of S and T (for instance to consider block diagonal matrices).

A large degree of freedom in the stability conditions from Theorem 1 are provided since the matrix D (ie. its dimension, N, and its eigenvalues) and the initial condition of (6), W_0, are arbitrarily chosen. One can see that behind the general presentation of Theorems 1 and 2, there exists an infinite number of different stability conditions which lead to different results in terms of conservatism. The following section proposes some basic restrictions on the choice of (D, W_0).

VI. Discussions on the choice of (D, W_0)

This section deals with the influence on the matrix D and on the initial conditions W_0. The following gives a method to choose (D, W_0) in an efficient way. The idea behind the introduction of the matrix D is to make appear variations in the functions $\mathscr{Q}, \mathscr{R}, \mathscr{S}$ using the vector function W.

A first constraint comes from the conditions from Theorems 1 and 2. Since Π_2, Π_3 and Π_5 (and Υ_2, Υ_3 and Υ_5) are negative definite, it is thus required that D is non singular. Then possible functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ generated by Theorems 1 or 2 do not contain constant or polynomial components.

Focuss now on the pair (D, W_0). For a given matrix D, consider the set defined by:

$$L_D = \{W_0 \mid \forall X \in R^N \backslash \{0\},\ \forall \xi \in [-\tau\ 0],\ X^T e^{\xi D} W_0 \neq 0\}$$

The set L_D represents the set of initial conditions of (6) such that all the components of the solutions W are linearly independent. Then L_D has the following properties.

Proposition 2: Consider a matrix D. If there exists a change of coordinates such that the D is expressed a block diagonal matrix and such that one of this block is of the form λI where $\lambda \in C$, then $L_D = \emptyset$.

Proof: If such a change of coordinates exists, it means that several components of the solutions $W(t)$ will be proportional, which makes that the set L_D is empty. ■

For instance, it is easy to see that $L_{I_N} = \emptyset$. Coming back to the definition of the functions $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$, the choice of the identity matrix does not give so much interest since all the components of W will be proportional. Finally the size of the LMIs would have been increased to solve several times the same problem which corresponds to Q_0e^t, R_0e^t, S_0e^t.

On the other side, the set $L_{d\{1,2\}}$ is not an empty set. However if, for instance, we consider $W_0^T = [1 \quad 0]$ (which does not belong to $L_{diag(1,2)} = \{W_0 \in R^2 \quad | \quad \forall i = 1,2, \quad (W_0)_{1,i,1}\}$), the functions generated by $(diag(1,2), W_0 \neq 0)$ will only be Q_0e^t, R_0e^t, S_0e^t. One more the size of the LMIs would have been increasing without reducing the conservatism. Then it appears that taking $W_0 \in L_D$ (which also ensures that $L_D \neq$) provides an efficient choice of the pair (D, W_0).

The following gives some properties on the functions $\mathscr{Q}, \mathscr{R}$ and $\mathscr{S}$ with respect of the choice of the pair (D, W_0). Consider the following definition:

Definition 1: Two pairs (D, W_0) and (D', W_0') are said equivalent if there exists a non singular matrix M such that: $\forall \xi \in [-\tau \; 0], \quad W(\xi) = MW'(\xi)$.

The following propositions hold:

Proposition 3: Assume that the pairs (D, W_0) and (D', W_0') are equivalent. If there exist Q, R and S of appropriate dimension such that V generated by (D, W_0) is a LKF for system (8), then there also exists a LKF V' generated by (D', W_0') for system (8).

Proof: Assume that V generated by (D, W_0) is a LKF for system (8) with some constant matrices Q, R and S. Then consider

$$\begin{aligned} V'(x_t) &= x^T(t)P_1x(t) + 2x^T(t)\int_{-\tau}^0 Q\mathscr{M}^{-1}\mathscr{W}'(\xi)x_t(\xi)d\xi \\ &+ \int_{-\tau}^0 x_t^T(\xi)\mathscr{W}'^T\mathscr{M}^{-T}S\mathscr{M}^{-1}\mathscr{W}'(\xi)x_t(\xi)d\xi \\ &+ \int_{-\tau}^0\int_{-\tau}^0 x_t^T(s)\mathscr{W}'^T(s)\mathscr{M}^{-T}R\mathscr{M}^{-1}\mathscr{W}'(\xi)ds\, x_t(\xi)d\xi, \end{aligned}$$

where $\mathscr{M} = M \otimes I_n$. It is easy to see that $V' = V$. Then V' is also a LKF for system (8). ■

Proposition 4: Consider a matrix D in $R^{N\times N}$ such that $L_D \neq \emptyset$ and a non singular matrix M. Then the two following statement hold:

- For any $W_0 \in L_D$, the vector MW_0 is in $L_{MDM^{-1}}$ and the two pairs (D, W_0) and $(D', W_0') = (MDM^{-1}, MW_0)$ are equivalent;
- For any W_0 and $W_0' \in L_D$, the two pairs (D, W_0) and (D, W_0') are equivalent.

Proof: The proof is straightforward since W and W' are the solutions of the same differential equations expressed over a different system of coordinates. Noting that W_0 (and W_0') belongs to L_D (and $L_{D'}$), the vectorial set of functions of W and W' both are of dimension N. ■

Theorems	τ_{max}	Theorem 1	τ_{max}
Fridman et al [7]	4.47	D^6, $N=2$	5.06
Fridman et al [8]	1.63	D^6, $N=4$	5.98
Wu et al [23]	4.47	D^6, $N=6$	6.14
Kao et al [13]	4.47	D^1, $N=6$	4.00
He et al [11]	4.47	D^2, $N=6$	5.69
Ariba et al [1]	5.12	D^3, $N=6$	3.35
Papachristodoulou et al [21]	-	D^4, $N=6$	5.38
Kao et al [14]	6.11	D^5, $N=6$	**6.15**
Ariba et al (N=6) [3]	6.12	D^6, $N=6$	6.14
Gu (N=6) [9]	**6.17**	D^7, $N=6$	5.71
-		D^8, $N=6$	6.13

TABLE I
THE MAXIMAL ALLOWABLE DELAY τ

Then for a given matrix D, it is sufficiently to test if V is a LKF for (8) for any vector $W_0 \in L_D$ (if there exists one).

VII. EXAMPLES

The stability is analyzed using LKF generated by $W = [\,1 \quad 1 \quad \dots \quad 1\,]^T$ and the matrices:

$$D^1 = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & N \end{bmatrix},$$
$$D^2 = I_N + \begin{bmatrix} 0_{1\times N-1} & 0 \\ diag(1,\dots,N-1) & 0_{N-1\times 1} \end{bmatrix},$$
$$D^3 = diag(1,-1,\dots,N/2,-N/2),$$
$$D^4 = -D^2,$$
$$D^5 = 0.05D^1,\; D^6 = 0.05D^2,$$
$$D^7 = 0.05D^3,\; D^8 = 0.05D^4,$$

In D_3, the sign of the term in the last row and last column depends on wether N is pair or not. The matrices D_k for $k = 1,3,4,5,7,8$ are diagonal matrices. It thus allows producing solutions of the form $W^T(\xi) = [e^{D^k_{1,11}\xi}, \dots, e^{D^k_{1,NN}\xi}]$. The matrices D_k for $k = 2,6$ produce $W^T(\xi) = [e^{d^k\xi}, \dots, \xi^N e^{d^k\xi}]$ where $d^2 = 1$ and $d^6 = 0.05$.

Consider the following example [7], [9], [13] or [23]

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-\tau)$$

with a constant delay. The upper bounds of the delay τ_{max} delivered by stability criteria from the literature and by Theorem 1 for several matrices D are given in Table I.

As a first remark, the results delivered by Theorem 1 and several matrices D are not equivalent. This also means that behind the general stability conditions of Theorem 1, an infinite number of criteria can be considered by changing D. As Theorem 1 is employed for only few matrices, it would be possible that one can find greater upper-bound of the delay using another matrix.

Compare to the literature, the conservatism of Theorem 1 is better than the other results except the discretization method by [9]. In this method, the parameters of the LKF are piecewise linear functions. This allows thinking that investigating into extensions of Theorem 1 to a discretization method or to polynomial parameters (which corresponds to a nilpotent matrix D) could reduce the conservatism.

VIII. Conclusion

In this article, a novel approach to construct Lyapunov-Krasovskii functional using descriptor representation and linear differential equations is introduced. The proposed method allows considering continuous varying functions in the parameters in the LKF. This method leads to less conservative results than most of the existing results on the stability of linear systems with constant delays.

An important issue to improve the present method would be first to extend this approach to time-varying delays. Another one consists in enlarging the class of functions $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{S}$. A restriction on the matrices which has to be non singular, still remains. It thus does not allow considering constant or polynomial functions. Another possibility would be to consider time-varying matrices D and thus obtaining a more general class of LKF. Another interesting issue concerns the number of variable to solve. It is clear that at this level of development, this method requires a larger number of variables to determine especially compare to the discretization method. It would be interesting to reduce them.

References

[1] Y. Ariba and F. Gouaisbaut, *Delay-dependent stability analysis of linear systems with time-varying delay*, Proc. of the 46^{th} IEEE Conference on Decision and Control (New Orleans, LA, USA), 2007.

[2] ______, *Robust stability of time-delay systems with interval delays*, Proc. of the 46^{th} IEEE Conference on Decision and Control (New Orleans, LA, USA), 2007.

[3] ______, *Construction of Lyapunov-Krasovskii functional for time-varying delay systems analysis*, Proc. of the 47^{th} IEEE Conference on Decision and Control (Cancun, Mexico), 2008.

[4] E. Fridman and S.-I. Niculescu, *On complete Lyapunov-Krasovskii functional techniques for uncertain systems with fast-varying delays*, Int. J. Robust Nonlinear Control **8** (2008), no. 2, 364–374.

[5] E. Fridman, A. Seuret, and J.-P. Richard, *Robust sampled-data stabilization of linear systems: An input delay approach*, Automatica **40** (2004), no. 8, 1141–1446.

[6] E. Fridman and U. Shaked, *A descriptor system approach to H^∞ control of linear time-delay systems*, IEEE Trans. on Automatic Control **47** (2002), no. 2, 253–270.

[7] ______, *An improved stabilization method for linear time-delay systems*, IEEE Trans. on Automatic Control **47** (2002), no. 11, 1931–1937.

[8] ______, *Input-output approach to stability and L_2-gain analysis of systems with time-varying delays*, Systems and Control Letters **55** (2006), 10411053.

[9] K. Gu, *Discretized LMI set in the stability problem of linear uncertain time-delay systems*, Int. J. of Control **68** (1997), 923–934.

[10] K. Gu, V.-L. Kharitonov, and J. Chen, *Stability of time-delay systems*, Birkhauser, 2003.

[11] Y. He, Q. G. Wang, C. Lin, and M. Wu, *Further improvement of freeweighting matrices technique for systems with time-varying delay*, IEEE Trans. on Automatic Control **52** (2007), no. 2, 293–299.

[12] J.P. Hespanha, P. Naghshtabrizi, and Y. Xu, *A survey of recent results in networked control systems*, Proceedings of the IEEE **95** (2007), no. 1, 138–162.

[13] C. Y. Kao and A. Rantzer, *Robust stability analysis of linear systems with time-varying delays*, 16th IFAC World Congress, IFAC WC'05 (Prague, Czech Republic), 2005.

[14] ______, *Stability analysis of systems with uncertain time-varying delays*, Automatica **43** (2007), 959–970.

[15] V.L. Kharitonov and A.P. Zhabko, *Lyapunov-Krasovskii approach to robust stability of time delay systems*, In First IFAC/IEEE symposium on system structure and control (Prague, Czech Rep.), 2001.

[16] V.B. Kolmanovskii and A.D. Myshkis, *Applied theory of functional differential equations*, Kluwer, 1999.

[17] V.B. Kolmanovskii and A. Zhabko, *Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems*, Automatica **39** (2003), no. 1, 15–20.

[18] S. Mondie, V.L. Kharitonov, and O. Santos, *Complete Lyapunov-Krasovskii functional with a given cross term in the time derivative*, Proc. of the 44^{th} IEEE Conference on Decision and Control (Sevilla, Spain), 2005.

[19] P. Naghshtabrizi, J.P. Hespanha, and A.R. Teel, *On the robust stability and stabilization of sampled-data systems: A hybrid system approach*, Proceedings of the 45^{th} IEEE Conference on Decision and Control (San Diego, CA, USA), 2006.

[20] S.-I. Niculescu, *Delay effects on stability. a robust control approach*, Springer-Verlag, 2001.

[21] A. Papachristodoulou, M. M. Peet, and S.-I. Niculescu, *Stability analysis of linear systems with time-varying delays: Delay uncertainty and quenching*, Proc. of the 46^{th} IEEE Conference on Decision and Control (New Orleans, LA, USA), 2007.

[22] J.-P. Richard, *Time delay systems: an overview of some recent advances and open problems*, Automatica **39** (2003), no. 10, 1667–1694.

[23] M. Wu, Y. He, J. H. She, , and G. P. Liu, *Delay-dependent criteria for robust stability of time-varying delay systems*, Automatica **40** (2004), no. 8, 1435–1439.