Abstract—In this article, a new method to assess stability and to design static state feedback controller for linear time-delay systems is introduced. The method based on linear differential equations allows considering explicit Lyapunov-Krasovskii functionals with non constant matrix parameters. The stability conditions considering constant delays are delay-dependent and expressed using easy computable linear matrix inequalities. An example is introduced to show the efficiency of the stabilization criteria.

I. INTRODUCTION

During the last decades, a great attention attracted researchers on the stability analysis of time delay systems [10], [20], [22]. The study of the time-delay systems is motivated by the apparition of delays phenomenon in many processes such as biology, chemistry, economics, as well as population dynamics [16], [20]. Moreover, processing time and propagation in actuators and sensors generally induce such delays. In particular, the recent and intense activities around networked controlled systems for which actuators and sensors exchange data through a network highly motivate this research. Many phenomena induced by the network as communication delays [12] or samplings [5], [19] can be interpreted by a delay modeling. In the case of devices which are remotely installed, delays may increase dramatically. As they usually have a disturbing effect, the problem of stability with respect to the time-delays is fundamental.

Note that there exist some systems which are not asymptotically stable for small delays but becomes for sufficiently large delays (in high-speed networks, biological systems and some examples [16]). For this class of systems, the stability cannot be performed using simple Lyapunov-Krasovskii functionals (LKF), ie. functionals parameterized with constant matrices (see, among the less conservative results, [7], [8], [11], [23]). To deal with them, the introduction of more complex and amenable LKF has been introduced to overcome these difficulties. The discretization method introduced by K. Gu [9], [10], allows constructing piecewise linear functions as the LKF parameters by dividing the delay interval into several smaller intervals on which the parameters of the LKF are linearly varying. The stability analysis leads to less conservative than in the case of constant parameters but extensions to the time-varying delay case are not straightforward. Another interesting method introduced in [21], suggested a way to build LKF with varying parameters based on sum of squares tools. Another method which takes a lot of attention nowadays is based on Integral Quadratic Constrains (IQC) [13], [14]. It allows having a better understanding of the terms of the LKF and to have a better idea of where the conservatism is introduced [2] and especially obtaining less conservative results. In [15], a complete LKF (ie. which corresponds to necessary and sufficient conditions of stability) is constructed by solving a functional differential equation. This approach is useful to derive robustness conditions with respect to delay variations [4] or parameters uncertainties [17], [18].

In this article a novel method to construct LKF with varying parameters over the delay interval is introduced. The LKF is provided by a simple and arbitrary linear differential equation and leads to sufficient stability conditions. This technic together with the descriptor representation [6] leads to suitable criteria to design static state feedback gain which ensures stability of the system with constant delay. The paper is organized as follows: Section II is devoted to the formulation of the problem. The form of the LKF is examined in Section III. Section IV and V concern the stability analysis and the stabilization of linear systems with constant delay. Section VI discusses the choice of linear differential equation. An example is provided in Section VII to show the efficiency of the method.

Notations. Throughout the article, the function \(x_t\) corresponds to \(x_t(\theta) = x(t+\theta), \forall \theta \in [-\tau,0]\). The superscript ‘T’ stands for the matrix transposition. The notation \(P > 0\) for \(P \in \mathbb{R}^{n,n}\) means that \(P\) is a symmetric and positive definite matrix. The symbol \(I_n\) represents the \(n \times n\) identity. For any square matrices \(A\) and \(B\), the notation \(\text{diag}(A,B)\) denotes the block diagonal matrix where the first diagonal block is \(A\) and, \(B\) the second. Given any positive integers \(n_N\) and \(M\), consider a matrix \(A \in \mathbb{R}^{n_N \times nM}\). The notation \(A_{n,j}\) corresponds the matrix of size \(n_N \times nM\) located in the between the \(in+1\) and \((i+1)n\) rows and the \(jn+1\) and \((j+1)n\) columns of \(A\). We denote \(\otimes\), by the Kronecker product.

II. PROBLEM FORMULATION

Consider the controlled system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_\delta x(t - \tau) + Bu(t) + B_\delta u(t - \tau) \\
\dot{x}(\theta) &= \phi(\theta), \quad \theta \in [-\tau,0]
\end{align*}
\]  

(1)
where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represents the state and the input vectors. The function $\phi \in [-\tau \ 0] \rightarrow \mathbb{R}^m$ corresponds to the initial conditions of the time-delay system. The matrices $A, A_\tau, B$ and $B_\tau$ are assumed to be known and constant. The delay is assumed to be constant. The control law is a linear state-feedback of the form:

$$u(t) = Kx(t)$$

(2)

where $K$ is a $m \times n$ matrix. In [9], a general form of LKF's is introduced:

$$V(x_t) = x^T(t)P(t)x(t) + 2x^T(t)\int_0^{\tau} \mathcal{Q}(\xi_t)x_t(\xi_t)d\xi$$

(3)

$$+ \int_0^{\tau} x^T(t)\mathcal{R}(\xi_t)x_t(\xi_t)T$$

where $P > 0$, $\mathcal{Q}(\xi)$, $\mathcal{R}(\xi)$ are matrix functions. The integral terms in $V$ represent the influence of the state $x_t$ on the stability. In [13], it was proven that integral terms correspond to the robustness of the system with respect to the delay operator $e^{-\tau}$. This article introduces a new method to explicitly construct continuous functions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{F}$. These variations in the parameters allow considering more accurate LKF's and will lead to less conservative conditions.

### III. PARAMETRIZATION OF LYAPUNOV-KRASOVSKII FUNCTIONALS

#### A. Introduction

Consider some scalar functions $f^j$ defined on $[-\tau \ 0]$ where $i = 1, \ldots, N$ and the following functions $\mathcal{Q}$, $\mathcal{R}$ such that for all $s$ and $\xi$ in $[-\tau \ 0]$:

$$\mathcal{Q}(\xi) = \sum_{i=1}^{N} f^i(\xi)Q_i,$$

$$\mathcal{R}(\xi) = \sum_{i=1}^{N} \sum_{j=1}^{N} f^i(\xi)f^j(\xi)R_{ij},$$

$$\mathcal{F}(s, \xi) = \sum_{i=1}^{N} \sum_{j=1}^{N} f^i(s)f^j(\xi)R_{ij},$$

where $Q_i$, $R_{ij}$ and $f_{ij}$ for $i, j = 1, \ldots, N$ are constant matrices. Introducing the vector function $W(\xi) = [f^1(\xi), \ldots, f^N(\xi)]^T$, a nice expression of the functions can be derived:

$$\mathcal{Q}(\xi) = QW^T(\xi),$$

$$\mathcal{R}(\xi) = \mathcal{F}(s, \xi) = [W^T(\xi)S]W^T(\xi),$$

(4)

where $Q = W^T \otimes I_n$, $R$ and $S$ are such that $Q_i = (Q)_{n,i}$, $R_{ij} = (R)_{n,ij}$ and $S_{ij} = (S)_{n,ij}$. The functions which defined the LKF are thus expressed in a simple way. A lemma to ensure the LKF is positive definite is thus formulated:

**Lemma 1:** For given $\tau > 0$, the LKF (3) with (4) is positive definite if there exist $P > 0$ in $\mathbb{R}^{nxn}$, $S = ST > 0$ in $\mathbb{R}^{nxn}$ and $R = RT^T$ in $\mathbb{R}^{nxn}$ such that the following LMI holds:

$$\mathcal{Z} - \begin{bmatrix} P & Q \\ Q^T & R + S/\tau \end{bmatrix} > 0.$$  

(5)

**Proof:** Consider the functional (3) where the functions $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{F}$ are defined in (4). Consider the vector $\Phi_f(t) = \int_0^{\tau} W^T(\xi)x_t(\xi)d\xi$. The second and the last terms of $V$ can thus be rewritten as $2x^T(t)Q\Phi_f(t)$ and $\Phi_f^T(t)R\Phi_f(t)$. Provided that $S > 0$, the Jensen's inequality ensures that

$$\int_0^{\tau} x^T(t)W^T(\xi)S\mathcal{F}(\xi)x_t(\xi)d\xi > \Phi_f^T(t)S/\tau \Phi_f(t).$$

Denote $\xi_0(t) = [x(t) \ \Phi_f(t)]^T$, the LKF satisfies $V(x_t) \geq \xi_0^T(t)\xi_0(t)$. Then if (5) holds $V$ is positive definite.

### B. Introduction of an arbitrary linear differential equation

For a given integer $N > 0$, consider a square matrix $D$ in $\mathbb{R}^{nxn}$. Define the vectorial function $W$ such that for any $\xi \in [-\tau \ 0]$:

$$\begin{cases} W(\xi) = DW(\xi), \\ W(0) = W_0 \end{cases}$$

(6)

where $W_0 \in \mathbb{R}^n$. Introduce the vector $W(\xi) = W(\xi) \otimes I_n$, and the consider now the LKF (3) defined by:

$$\mathcal{Q}(\xi) = QW(\xi),$$

$$\mathcal{F}(s, \xi) = [W^T(s)S]W(\xi),$$

(7)

where $Q$ is in $\mathbb{R}^{nxn}$ and $S, R$ in $\mathbb{R}^{nxn}$ are symmetric constant matrices. In the latter, we will say that the pair $(D,W_0)$ generates the LKF $V$ if functions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{F}$ are given by (6) and (7). Apparently, there is no restriction on the matrix $D$. Depending on the eigenvalues of $D$, the functions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{F}$ could be polynomial, exponential and/or trigonometric. The choice of $(D,W_0)$ is discussed in Section VI. As $W$ is the solution of linear differential equations of the type of (6), the functions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{F}$ are infinitely differentiable over the interval $[-\tau \ 0]$. Another advantage is that simple expressions of their derivative and of their value at some particular instant are simply derived from the solution of the well known linear differential equation, using $\mathcal{F} = D \otimes I_n$ and $W_0 = 0 \otimes I_n$.

**Proposition 1:** The differentiation of the functions $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{F}$ is straightforwardly given by:

$$\begin{cases} \mathcal{Q}(\xi) = Q\mathcal{W}(\xi), \\ \mathcal{F}(s, \xi) = \mathcal{W}(\xi)(\mathcal{D} + 2s)\mathcal{W}(\xi), \\ \frac{d\mathcal{W}(\xi)}{ds} + \frac{d\mathcal{R}(s, \xi)}{ds} = \mathcal{W}(\xi)(\mathcal{D}^T + R)\mathcal{W}(\xi), \end{cases}$$

and their evaluation at any instant $a$ and $b$ in $[-\tau \ 0]$:

$$\mathcal{Q}(a) = Qe^{aD}W_0,$$

$$\mathcal{F}(a) = W_0^T(e^{aD})S\mathcal{W}(a),$$

$$\mathcal{R}(a, b) = W_0^T(e^{aD})R\mathcal{W}(b).$$
IV. STABILITY ANALYSIS

Consider system (1) and the controller (2) with a given state feedback gain $K$. The dynamics of the system can be rewritten as:

$$\dot{x}(t) = A_0x(t) + A_1x(t - \tau)$$ (8)

where $A_0 = A + BK$ and $A_1 = A_\tau + B_\tau K$. The following theorem holds:

**Theorem 1:** For a given $N$ and $\tau > 0$, consider a matrix $D$ in $\mathbb{R}^{N\times N}$ and a vector $W_0$. System (8) is asymptotically stable if there exist a positive symmetric definite matrix $P_1$ in $\mathbb{R}^{n\times n}$, $S_0 = S_0^T$, $P_2$, $P_3$ in $\mathbb{R}^{n\times n}$, a matrix $Q$ in $\mathbb{R}^{n\times n}$ and $R = R^T$, $S = S^T$ and $T = T^T$ in $\mathbb{R}^{N\times N}$ such that, the following LMIs hold:

$$\Pi_1 = \begin{bmatrix} P_1 & Q & R & S / \tau \\ Q^T & S / \tau & \tau \\ -P_2 & P_3 & P_3^T A_1 & -Q^T \\ P_3^T A_1 & -Q^T & -w_0^T E^T R \\ * & * & * & * \\ \end{bmatrix} > 0,\quad \Pi_2 = \begin{bmatrix} \mathcal{D}(S + T) & (S + T) \mathcal{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{bmatrix} > 0,\quad \Pi_4 = S_0 + w_0^T E^T T W_0 > 0$$

and

$$\Pi_5 = \mathcal{D}(T + T) \mathcal{D} > 0$$

where $E = e^{-\tau \mathcal{D}}$ and where $\pi_1$, $\pi_2$ and $\pi_3$ are given by:

$$\pi_1 = P_1^T A_0 + A_0^T P_1 + S_0 + w_0^T (S + T) W_0 + Q W_0 + (Q W_0)^T$$

$$\pi_2 = -S_0 - w_0^T (e^{-\tau \mathcal{D}})(S + T)(e^{-\tau \mathcal{D}}) W_0$$

$$\pi_3 = -\mathcal{D}(R + 1/\tau(S + T)) - (R + 1/\tau(S + T)) \mathcal{D}$$

Proof: Consider the candidate LKF:

$$V_1(x_t) = V(x_t) + \int_{-\xi}^{0} x_t^T(\xi) (S_0 + w_0^T (\xi) T W_0(\xi)) x_t(\xi) d\xi$$

where $V$ is given by (3) and the functions $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{S}$ are given by (7). Compared to (3), an integral term is added to take into account to reduce the conservatism.

The proof is divided into two parts. The first ensures the LKF is positive definite. According to Lemma 1 and introducing the vector $\Phi(t) = \int_0^t W(\xi) x_t(\xi) d\xi$, $V$ is positive definite if $\Pi_1 > 0$. A careful attention is now required on the second term of $V_1$. The function $S_0 + w_0^T (\xi) T W_0(\xi)$ must be positive whatever $\xi \in [-\tau 0]$. Consider a non zero vector $v$ in $\mathbb{R}^n$ and the associated function $f_\xi$ such that:

$$\forall \xi \in [-\tau 0], \quad f_\xi(\xi) = v^T (S_0 + w_0^T (\xi) T W_0(\xi)) v$$

the function $f_\xi$ is continuous and differentiable over $[-\tau 0]$. Using the definition of $W$, its derivative with respect to $\xi$ is $f_\xi(\xi) = v^T W^T(\xi) \Pi_2 W(\xi) v$. From Theorem 1, it means that $f_\xi > 0$ and, consequently, that $f_\xi$ is an increasing function of $\xi$. Then the following inequality holds:

$$v^T \Pi_2 v < v^T (S_0 + W^T(\xi) T W(\xi)) v.$$

As this inequality holds for any vector $v$, it means that $\Pi_4 > 0$, the matrix function $S_0 + W^T(\xi) T W(\xi)$ is positive definite over $[-\tau 0]$. Finally, the functional $V_1$ is positive definite. Note that the matrix $S_0$ or $T$ are not necessary positive definite as it is usually required in the literature.

The second part of the proof consists in ensuring the negativeness of the derivative of $V_1$ along the trajectories of (8). Differentiating $V_1$ leads to:

$$\dot{V}_1(x_t) = 2x_t^T(t) \left[ P x(t) + \int_{-\xi}^{0} \mathcal{D}(\xi) x_t(\xi) d\xi \right] + \int_{-\xi}^{0} \mathcal{D}(\xi) x_t(\xi) d\xi$$

Finally, the functional $V_1$ is negative definite.

Integrating with parts the terms with $\dot{x}(\xi)$ in the previous equation and introducing the term $\int_{-\xi}^{0} [x(t) + \mathcal{A}(x(t)) + \mathcal{A}_1 x(t - \tau)] = 0$, equivalent to the descriptor representation of (6), the following equality is established:

$$\dot{V}_1(x_t) = 2x_t^T(t) [P x(t) + x^T(t) (S_0 + \mathcal{D}(\xi) x_t(\xi) d\xi$$

$$+ \int_{-\xi}^{0} \mathcal{D}(\xi) x_t(\xi) d\xi + 2x_t^T(t) \int_{-\xi}^{0} \mathcal{D}(\xi) x_t(\xi) d\xi + \int_{-\xi}^{0} \mathcal{D}(\xi) x_t(\xi) d\xi$$

$$= \Pi_1 < 0$$

Replacing the functions $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{S}$ by their expression using Proposition 1, (11) can be rewritten as:

$$\dot{V}_1(x_t) = 2x_t^T(t) P x(t)$$

$$+ x^T(t) (S_0 + \mathcal{D}(\xi) x_t(\xi) d\xi$$

$$+ \int_{-\xi}^{0} \mathcal{D}(\xi) x_t(\xi) d\xi$$

Since $\Pi_2 = \mathcal{D}(S + T) + (S + T) \mathcal{D} > 0$, the Jensen’s inequality ensures that the integral term of the equation above is bounded by $-1/\tau \Phi(t) (\mathcal{D}(S + T) \mathcal{D}) \Phi(t)$. Introducing the vector $\phi(t) = [x^T(t), x^T(t), \mathcal{D}(x(t)), \mathcal{D}(x(t))]^T$, the following inequality holds:

$$\dot{V}_1(x_t) \leq \xi^T(t) \Pi_3 \xi(t)$$

Then provided that the conditions of Theorem 1 hold, the derivative of the LKF $V_1$ is negative definite and the system is asymptotically stable.
A. Seuret and K. H. Johansson: Stabilization of Time-Delay Systems through Linear Differential Equations

Remark 1: An extension to the case of uncertainties in the system parameters can be dealt by considering system (8) and with \( A_0 \) and \( A_1 \) from the uncertain polytope:

\[
\forall t \in \mathbb{R}^+, \quad \Omega(t) = \sum_{k=1}^{M} \lambda_k(t) \Omega_k,
\]

where \( \forall t \in \mathbb{R}^+, \sum_{k=1}^{M} \lambda_k(t) = 1, \quad \forall k = 1, \ldots, M, \quad 0 \leq \lambda_k(t). \)

The \( \Omega \) vertices of the polytope are described by \( \Omega_k = [A_0(k) \quad A_1(k)] \). As the conditions based in Theorem 1 are linear with respect to the matrices \( A_0 \) and \( A_1 \), one has to solve those LMIs simultaneously for all the \( \Omega \) vertices.

Remark 2: The use of the descriptor representation does not reduce the conservatism. However its interest will be exposed in the following section.

V. Stabilization of Time-Delay Systems

Consider system (1). The objective is now to design the gain \( K \) of the control law (2) such that the closed-loop system is asymptotically stable. The following result holds:

Theorem 2: For a given \( N \), consider a matrix \( D \) in \( R^{N \times N} \) and a vector \( W_0 \). If for a given \( \epsilon > 0 \), there exist \( P_1 = P_1^T > 0 \), \( S_0 = S_0^T \), \( \bar{P} \), \( R \) in \( R^{n \times n} \), a matrix \( Y \) in \( R^{m \times n} \), a matrix \( \bar{Q} \) in \( R^{n \times n} \) and \( R = RT^T \), \( \bar{S} = \bar{S}_T \), \( 0 < T < \infty \) such that:

\[
Y_1 = \begin{bmatrix} P_1 & Q \bar{Q} & R + S/\tau \end{bmatrix} > 0,
\]

\[
Y_2 = \bar{Q}^T (\bar{S} + T) + (\bar{S} + T) \bar{Q} > 0,
\]

\[
\begin{bmatrix}
    u^1_1 & u^1_2 & u^3_3 & -\bar{Q} \bar{Q}^T + \nu^T \bar{Q} & * & * & * & \nu^5_5 \\
    * & -\epsilon (\bar{P} + \bar{Q}^T) & \epsilon (A_1 \bar{P} + B_1 \nu) & \nu^3_3 & * & * & -\nu^T \bar{Q} T \bar{P} \bar{Q}^T + \nu^T \bar{Q} T \bar{Q} & * & * & * & \nu^5_5 \\
    \end{bmatrix}
\]

and

\[
Y_4 = \bar{S}_0 + \nu^T \bar{Q} T \bar{P} \bar{Q}^T \bar{Q} > 0,
\]

\[
Y_5 = Y_2^T \bar{Q}^T \bar{Q} \bar{P} \nu^T \bar{Q} \nu^T \bar{Q} > 0
\]

where \( E = e^{-\tau \bar{Q} \bar{P} \nu} \) and where \( u^1_1, u^3_3 \) and \( u^4_4 \) are given by:

\[
\begin{align*}
u^1_1 &= A_0 \bar{P} + \bar{P}^T A^T + BY + Y^T B^T + \bar{S}_0 + \bar{S}_1 + \bar{Q} \nu^T \bar{Q} \\
u^1_2 &= P_1 - \bar{P} + \bar{P}^T A^T + e^{Y^T B^T} \\
u^3_3 &= A_1 \bar{P} + B_1 \nu - \bar{Q} \nu^T \bar{Q} \\
u^4_4 &= -e^{Y^T B^T} (\bar{S} + T) + \nu^T \bar{Q} \nu^T \bar{Q} \\
u^5_5 &= -e^{Y^T B^T} (\bar{S} + T) / \tau - (\bar{S} + T) / \tau \bar{Q} \bar{P} \nu^T \bar{Q} \nu^T \bar{Q}
\end{align*}
\]

Then the system (1) with the state feedback control law (2) with \( K = Y \bar{P}^{-1} \) is asymptotically stable.

Proof: Consider a state feedback gain \( K \). Assume the conditions from Theorem 1 are satisfied with \( A_0 = A + BK \) and \( A_1 = A_T + B_1 K \). Then the controlled system (1) is asymptotically stable for a delay \( \tau \). Assume that:

\[
P_2 = \epsilon P_3
\]

Noting that a necessary conditions for \( \Pi_3 \) to be negative definite is that the matrix \( P_3 \) is non singular. It is thus possible to define \( \bar{P} = P_3^{-1} \) and \( \bar{Q} = I_n \otimes \bar{P} \). From the definition of the Kronecker product, it is easy to see that:

\[
P \bar{W}_0 = \bar{W}_0 \bar{P}, \quad \bar{Q} \bar{P} = \bar{Q} \bar{P}
\]

Introducing the variables \( \bar{P}_1 = \bar{P}^T P_1 \), \( \bar{S}_0 = \bar{P}^T S_0 \), \( \bar{S}_1 = \bar{P}^T S_1 \), \( \bar{Q} = \bar{P}^T \bar{Q} \), \( \bar{R} = \bar{S} = \bar{T} \), \( \bar{P}_3 = \bar{P}^T \bar{P}_3 \bar{P} \), \( \bar{R}_3 = \bar{P}^T \bar{R}_3 \bar{P} \), \( \bar{R}_4 = \bar{P}^T \bar{R}_4 \bar{P} \), \( \bar{T}_3 = \bar{P}^T \bar{T}_3 \bar{P} \) and the proof is completed.
The set $L_D$ represents the set of initial conditions of (6) such that all the components of the solutions $W$ are linearly independent. Then $L_D$ has the following properties.

**Proposition 2:** Consider a matrix $D$. If there exists a change of coordinates such that the $D$ is expressed a block diagonal matrix and such that one of this block is of the form $\lambda I$ where $\lambda \in \mathbb{C}$, then $L_D = \emptyset$.

**Proof:** If such a change of coordinates exists, it means that several components of the solutions $W(t)$ will be proportional, which makes that the set $L_D$ is empty.

For instance, it is easy to see that $L_{K_0} = \emptyset$. Coming back to the definition of the functions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{F}$, the choice of the identity matrix does not give so much interest since all the components of $W$ will be proportional. Finally the size of the LMIs would have been increased to solve several times the same problem which corresponds to $Q_0\mathcal{R}_0, R_0, S_0\mathcal{R}_0$.

On the other side, the set $L_{d(1,2)}$ is not an empty set. However if, for instance, we consider $W_{0} = [1 \ 0]$ (which does not belong to $L_{d(1,2)} = \{W_{0} \in \mathbb{R}^2 \mid \forall i = 1, 2, (W_{0})_{1,i,1}\}$), the functions generated by $\mathcal{Q}(1,2), W_{0} \neq 0$ will only be $Q_0\mathcal{R}_0, R_0, S_0\mathcal{R}_0$. One more the size of the LMIs would have been increasing without reducing the conservativeness. Then it appears that taking $W_{0} \in L_D$ (which also ensures that $L_D \neq \emptyset$) provides an efficient choice of the pair $(D, W_{0})$.

The following gives some properties on the functions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{F}$ with respect of the choice of the pair $(D, W_{0})$.

**Consider the following definition:**

**Definition 1:** Two pairs $(D, W_{0})$ and $(D^{'}, W_{0}^{'})$ are said equivalent if there exists a non singular matrix $M$ such that:

$$\forall \xi \in [-\tau, 0], \ W(\xi) = MW^{'}(\xi).$$

The following propositions hold:

**Proposition 3:** Assume that the pairs $(D, W_{0})$ and $(D^{'}, W_{0}^{'})$ are equivalent. If there exist $Q$, $R$ and $S$ of appropriate dimension such that $V$ generated by $(D, W_{0})$ is a LKF for system (8), then there also exists a LKF $V^{'}$ generated by $(D^{'}, W_{0}^{'})$ for system (8).

**Proof:** Assume that $V$ generated by $(D, W_{0})$ is a LKF for system (8) with some constant matrices $Q, R$ and $S$. Then consider

$$V^{'}(x_{i}) = x_{i}^{T}(t)Px_{i}(t) + 2x_{i}^{T}(t)\int_{-\tau}^{0}Q_{M}^{-1}W^{'}(\xi)x_{i}(\xi)d\xi$$

$$+ \int_{-\tau}^{0}T_{2}(\xi)W^{'}(\xi)x_{i}(\xi)d\xi + \int_{-\tau}^{0}T_{1}(\xi)x_{i}(\xi)d\xi,$$

where $M = M \otimes I_{n}$, it is easy to see that $V^{'} = V$. Then $V^{'}$ is also a LKF for system (8).

**Proposition 4:** Consider a matrix $D$ in $\mathbb{R}^{N \times N}$ such that $L_D \neq \emptyset$ and a non singular matrix $M$. Then the two following statements hold:

- For any $W_{0} \in L_D$, the vector $MW_{0}$ is in $L_{\text{MDM}^{-1}}$ and the two pairs $(D, W_{0})$ and $(D^{'}, W_{0}^{'}) = (\text{MDM}^{-1}, MW_{0})$ are equivalent;
- For any $W_{0}$ and $W_{0}^{'} \in L_D$, the two pairs $(D, W_{0})$ and $(D^{'}, W_{0}^{'})$ are equivalent.

**Proof:** The proof is straightforward since $W$ and $W^{'}$ are the solutions of the same differential equations expressed over a different system of coordinates. Noting that $W_{0}$ and $W_{0}^{'}$ belongs to $L_D$ (and $L_{D'}$), the vectorial set of functions of $W$ and $W^{'}$ both are of dimension $N$.

Then for a given matrix $D$, it is sufficiently to test if $V$ is a LKF for (8) for any vector $W_{0} \in L_D$ (if there exists one).

**VII. EXAMPLES**

The stability is analyzed using LKF generated by $W = [1 \ 1 \ 1 \ 1 \ 1 \ 1]^{T}$ and the matrices:

$$D^{1} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N \end{bmatrix},$$

$$D^{2} = N + \begin{bmatrix} 0_{1,1-N} & 0 \\ \begin{bmatrix} 1,2,\ldots,\text{N} \end{bmatrix} & 0_{\text{N},\text{N}} \end{bmatrix},$$

$$D^{3} = \text{diag}(1, -1, \ldots, N/2, -N/2),$$

$$D^{4} = D^{2},$$

$$D^{5} = 0.05D^{1}, D^{6} = 0.05D^{2},$$

$$D^{7} = 0.05D^{3}, D^{8} = 0.05D^{4}.$$  

In $D_{3}$, the sign of the term in the last row and last column depends on wether $N$ is pair or not. The matrices $D_{k}$ for $k = 1, 3, 4, 5, 7, 8$ are diagonal matrices. It thus allows producing solutions of the form $W_{T}(\xi) = [e^{D_{1}}, e^{D_{2}}, \ldots, e^{D_{N}N\xi}]$. The $D_{k}$ for $k = 2, 6$ produce $W_{T}(\xi) = [e^{D_{1}}, e^{D_{2}}, \ldots, e^{D_{N}N\xi}]$ where $d^{2} = 1$ and $d^{6} = 0.05$.

Consider the following example [7], [9], [13] or [23]

$$x(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-\tau)$$

with a constant delay. The upper bounds of the delay $\tau_{\text{max}}$ delivered by stability criteria from the literature and by Theorem 1 for several matrices $D$ are given in Table I.

As a first remark, the results delivered by Theorem 1 and several matrices $D$ are not equivalent. This also means that behind the general stability conditions of Theorem 1, an infinite number of criteria can be considered by changing $D$. As Theorem 1 is employed for only few matrices, it would be

| Theorems | $\tau_{\text{max}}$ | Theorem 1 | $\tau_{\text{max}}$ |
|----------|----------------------|--------------------|
| Fridman et al [7] | 4.47 | $D^{1}$, $N = 2$ | 5.06 |
| Fridman et al [8] | 1.63 | $D^{4}$, $N = 4$ | 5.98 |
| Wu et al [23] | 4.47 | $D^{6}$, $N = 6$ | 6.14 |
| Kao et al [13] | 4.47 | $D^{7}$, $N = 6$ | 6.00 |
| He et al [11] | 4.47 | $D^{8}$, $N = 6$ | 6.59 |
| Ariba et al [1] | 5.12 | $D^{3}$, $N = 6$ | 3.35 |
| Papachristodoulou et al [21] | - | $D^{2}$, $N = 6$ | 5.38 |
| Kao et al [14] | 6.11 | $D^{6}$, $N = 6$ | 6.15 |
| Gu (N=6) [9] | 6.17 | $D^{7}$, $N = 6$ | 5.71 |
| - | | $D^{8}$, $N = 6$ | 6.13 |

**TABLE I**

**The maximal allowable delay** $\tau$
possible that one can find greater upper-bound of the delay using another matrix.

Compare to the literature, the conservatism of Theorem 1 is better than the other results except the discretization method by [9]. In this method, the parameters of the LKF are piecewise linear functions. This allows thinking that investigating into extensions of Theorem 1 to a discretization method or to polynomial parameters (which corresponds to a nilpotent matrix D) could reduce the conservatism.

VIII. CONCLUSION

In this article, a novel approach to construct Lyapunov-Krasovskii functional using descriptor representation and linear differential equations is introduced. The proposed method allows considering continuous varying functions in the parameters in the LKF. This method leads to less conservative results than most of the existing results on the stability of linear systems with constant delays.

An important issue to improve the present method would be first to extend this approach to time-varying delays. Another one consists in enlarging the class of functions $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$. A restriction on the matrices which has to be non singular, still remains. It thus does not allow considering constant or polynomial functions. Another possibility would be to consider time-varying matrices D and thus obtaining a more general class of LKF. Another interesting issue concerns the number of variable to solve. It is clear that at this level of development, this method requires a larger number of variables to determine especially compare to the discretization method. It would be interesting to reduce them.

REFERENCES