Output-based Event-triggered Control

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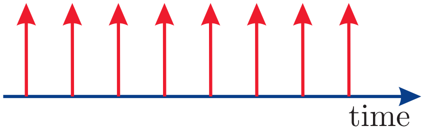
CDC Tutorial 2012

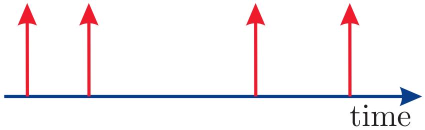
TU/e Technische Universiteit Eindhoven University of Technology

Where innovation starts

Periodic or Aperiodic: That's the question!

- Paradigm shift: Periodic control $\longrightarrow$ Aperiodic control





- Technological motivation:
  Resource-constrained large-scale cyber-physical systems
  - Computation time on embedded systems
  - Network utilisation in NCSs
  - Battery power in WCSs

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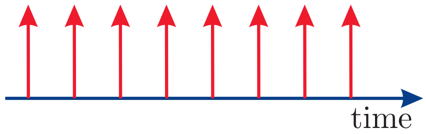
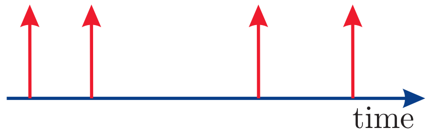
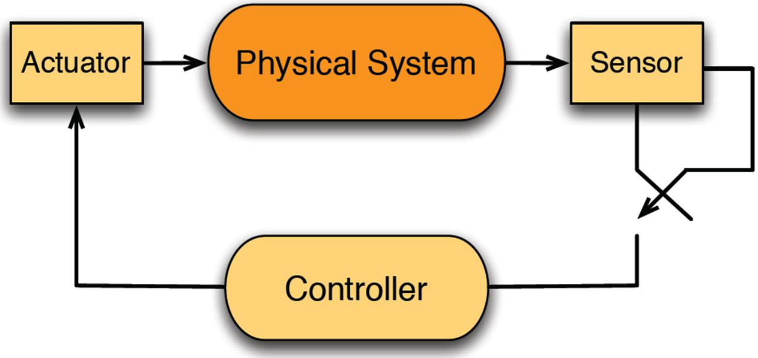




- Technological motivation:
  Resource-constrained large-scale cyber-physical systems
  - Computation time on embedded systems
  - Network utilisation in NCSs
  - Battery power in WCSs
- Fundamental motivation:
  - What is "optimal" sampling pattern for control purposes?

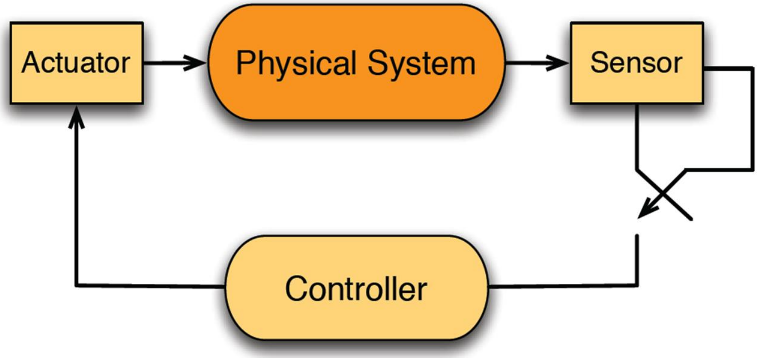
Output-based ETC



- What if full state x not available for feedback, but only output y?

Output-based ETC

Illustrative example



- Consider

$$\begin{cases} \dot{x}_p = \begin{bmatrix} 1 & -1 \\ 10 & -1 \end{bmatrix} x_p + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_p \end{cases} \qquad u = -2\hat{y}$$

- **ETM:** $\|y - \hat{y}\|^2 \geq \sigma \|y\|^2$
- **Parameter:** $\sigma = 0.5$

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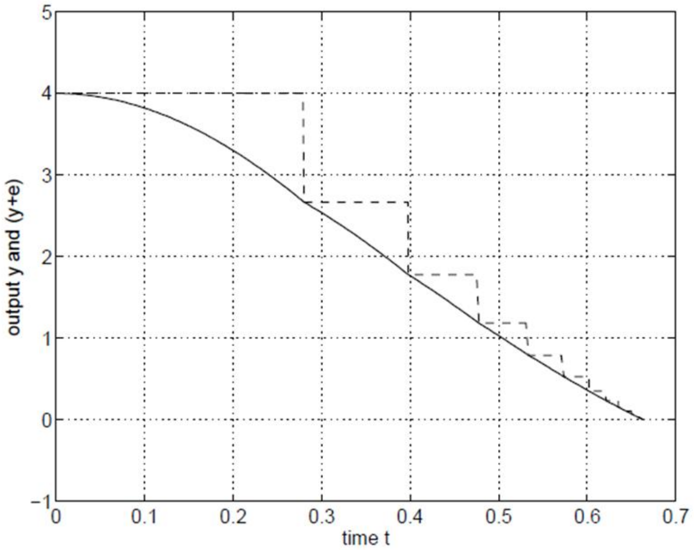
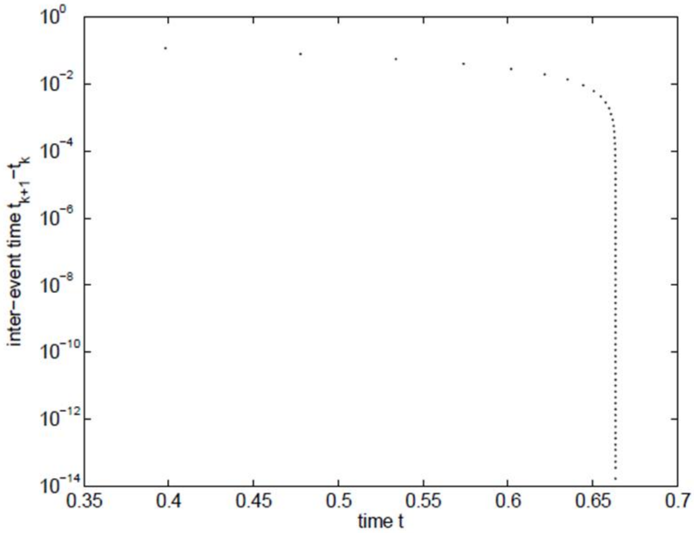
Illustrative example





Output-based ETC

Possible remedies

- Adopt alternative ETMs instead of $\|y - \hat{y}\|^2 \leq \sigma \|y\|^2$
  - Absolute: $\|y - \hat{y}\|^2 \leq \varepsilon$
  - Mixed: $\|y - \hat{y}\|^2 \leq \sigma \|y\|^2 + \varepsilon$

Output-based ETC

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  - Absolute: $\|y - \hat{y}\|^2 \leq \varepsilon$
  - Mixed: $\|y - \hat{y}\|^2 \leq \sigma \|y\|^2 + \varepsilon$
- Time regularization
  - Enforce minimal inter-event time T

$$t_{k+1} = \inf\{t > t_k + T \mid C(y(t), \hat{y}(t)) \geq 0\}$$

  - Transmission possible only at kh, $k \in \mathbb{N}$

$$t_{k+1} = \inf\{t > t_k \mid C(y(t), \hat{y}(t)) \geq 0 \wedge t = kh,\ k \in \mathbb{N}\}$$

    - Discrete-time ETC (dt plant)
    - Periodic Event-Triggered Control (PETC) (ct plant)

Output-based ETC

Possible remedies

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Output-based ETC

Possible remedies

- Adopt alternative ETMs instead of $\|y - \hat{y}\|^2 \leq \sigma\|y\|^2$
  - Mixed: $\|y - \hat{y}\|^2 \leq \sigma\|y\|^2 + \varepsilon$ [1]

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▶ Periodic Event-Triggered Control (PETC) (ct plant) [2]

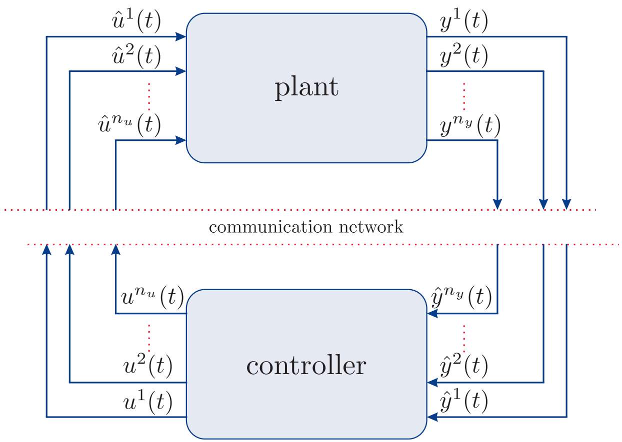
$\longrightarrow$ Tutorial paper has a categorization

[1] Donkers, Heemels, CDC 2010 & TAC 2012
[2] Heemels, Donkers, Teel, CDC 2011 & TAC 2013

System Description

Objective:

- Setup an output-based event-triggering mechanism (ETM)
- Guaranteed m.i.e.t > 0
- Mixed ETM: $\|y - \hat{y}\|^2 \leq \sigma\|y\|^2 + \varepsilon$
- General setup: decentralized ETM



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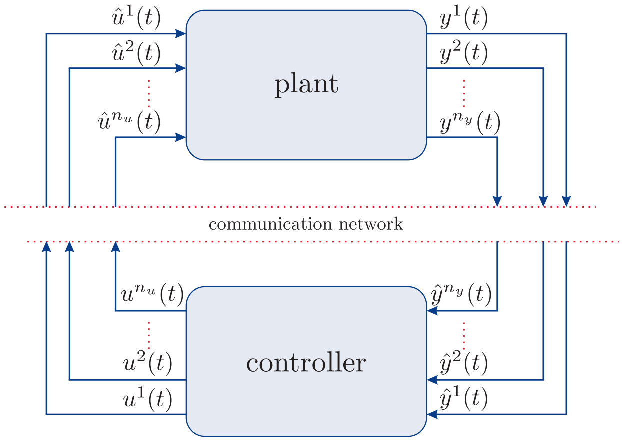

- Outputs and inputs grouped into sensor (and actuator) nodes
- y and $\hat{y}$ networked version: $\hat{y} \neq y$
- Signals in y corresponding to node i given by y^i
- Node i communicates at time $t^i_{k_i}$ for k_i-th time

$$\hat{y}^i(t) = y^i(t^i_{k_i}) \text{ for all } t \in (t^i_{k_i}, t^i_{k_i+1}]$$

The Event-Triggered Control System

System Description

- Node i communicates at time $t = t^i_{k_i}$ for k_i-th time

$$\hat{y}^j(t^+) = \begin{cases} y^i(t), \text{ when } j = i \\ \hat{y}^i(t), \text{ when } j \neq i \end{cases}$$

- Compact notation using ETM-induced error $e = \hat{y} - y$ at $t = t^i_{k_i}$

$$e^j(t^+) = \begin{cases} 0, \text{ when } j = i \\ e^j(t), \text{ when } j \neq i \end{cases} \qquad e(t^+) = \Lambda_i e(t)$$

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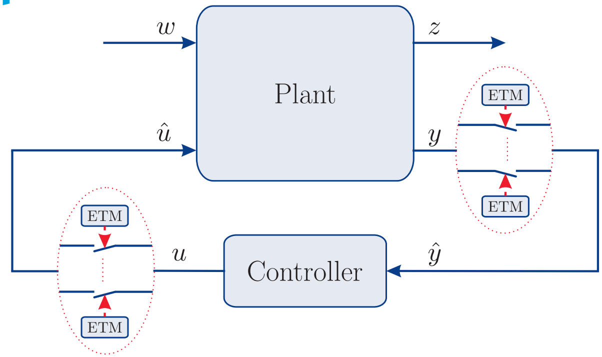

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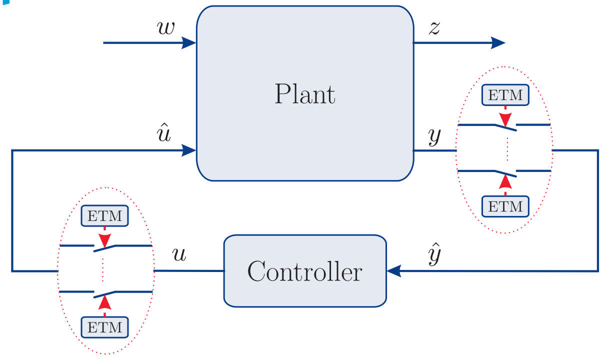

- $e(t^+) = \Lambda_i e(t)$ at $t = t^i_{k_i}$
- Event time $t^i_{k_i+1}$ is determined by

$$t^i_{k_i+1} = \inf\{t > t^i_{k_i} \mid \| \underbrace{\hat{v}^i - v^i}_{=e^i} \|^2 \geq \sigma_i \|v^i\|^2 + \varepsilon_i\}$$

Towards a complete model

- Combining the plant, the controller ...

$$\begin{cases} \dot{x}_p &= A_p x_p + B_p u + B_w w \\ y &= C_p x_p \end{cases} \qquad \begin{cases} \dot{x}_c &= A_c x_c + B_c \hat{y} \\ u &= C_c x_c \end{cases}$$

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- ... the update at event times ...

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- ... the update at event times ...

$$e(t^+) = \Lambda_i e(t)$$

- .... and the event-triggering mechanism

$$\|e^i\|^2 = \sigma_i \|y^i\|^2 + \varepsilon_i \text{ then event "}i\text{"}$$

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- ... using state variable $\bar{x} = (x_p, x_c, e)$ with $e = \hat{y} - y$ yields

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}w, & \text{for } \bar{x} \in \mathcal{C} \\ \bar{x}^+ = \bar{G}_i \bar{x}, & \text{for } \bar{x} \in \mathcal{D}_i \end{cases}$$

Closed-loop model

Impulsive System

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}w, & \text{for } \bar{x} \in \mathcal{C} \\ \bar{x}^{+} = \bar{G}_i \bar{x}, & \text{for } \bar{x} \in \mathcal{D}_i \\ z = \bar{C}\bar{x} + \bar{D}w \end{cases}$$

[1] Goebel, Sanfelice, Teel, CSM'09

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- Connection to Part I ?

[1] Goebel, Sanfelice, Teel, CSM'09

Perturbed system model

- The LTI plant and controller are given by

$$\begin{cases} \dot{x}_p = A_p x_p + B_p u + B_w w \\ y = C_p x_p \end{cases} \qquad \begin{cases} \dot{x}_c = A_c x_c + B_c \hat{y} \\ u = C_c x_c \end{cases}$$

  - $\hat{y} = y + e$ with e ETM-induced error
  - $x = (x_p, x_c)$

- Perturbed system model:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A_p & B_p C_c \\ B_c C_p & A_c \end{pmatrix} x + \begin{pmatrix} B_p \\ 0 \end{pmatrix} e + \begin{pmatrix} B_w \\ 0 \end{pmatrix} w \\ y &= C_p x \end{aligned}$$

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- Bounds on e $\qquad \|e\|^2 \leq \sigma \|y\|^2 + \varepsilon$

$$t^i_{k_i+1} = \inf\{t > t^i_{k_i} \mid \| \underbrace{\hat{y}^i - y^i}_{=e^i} \|^2 = \sigma_i \|y^i\|^2 + \varepsilon_i\}$$

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- Bounds on e $\qquad \|e\|^2 \leq \sigma \|y\|^2 + \varepsilon$
- Observation: $\dot{e}$ not included in PS
- $x = (x_p, x_c)$ in PS and $\bar{x} = (x_p, x_c, e)$ in IS

Stability and $\mathcal{L}_\infty$-gain Analysis

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}w, & \text{for } \bar{x} \in \mathcal{C} \\ \bar{x}^+ = \bar{G}_i\bar{x}, & \text{for } \bar{x} \in \mathcal{D}_i \\ z = \bar{C}\bar{x} + \bar{D}w \end{cases}$$

- The compact set $\mathcal{A}$ is globally asymptotically stable (GAS) if
  - the set $\mathcal{A}$ is Lyapunov stable, and
  - $\bar{x}(t) \to \mathcal{A}$ when $t \to \infty$
- ‘ultimate boundedness’ or ‘practical stability’
- $\mathcal{L}_\infty$-gain smallest γ such that $\|z\|_{\mathcal{L}_\infty} \leqslant \gamma\|w\|_{\mathcal{L}_\infty} + \beta(\bar{x}(0))$

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- $\mathcal{L}_\infty$-gain smallest γ such that $\|z\|_{\mathcal{L}_\infty} \leqslant \gamma\|w\|_{\mathcal{L}_\infty} + \beta(\bar{x}(0))$
- Two approaches
  - Perturbed system (PS) approach
  - Impulsive system (IS) approach

Stability Analysis via PS

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A_p & B_pC_c \\ B_cC_p & A_c \end{pmatrix} x + \begin{pmatrix} B_p \\ 0 \end{pmatrix} e + \begin{pmatrix} B_w \\ 0 \end{pmatrix} w \\ y &= C_p x \end{aligned}$$

- **Bounds on** e $\qquad \|e\|^2 \leq \sigma \|y\|^2 + \varepsilon$

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- **Bounds on** e $\qquad \|e\|^2 \leq \sigma \|y\|^2 + \varepsilon$
- **Perturbed system GES (**$w = 0$**) and** $H_\infty / \mathcal{L}_2$**-gain from** e **to** $y \leq \beta$

$$\dot{V} \leq -\alpha V(x) - \|y\|^2 + \beta \|e\|^2$$

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$$\dot{V} \le -\alpha V(x) - \|y\|^2 + \beta \|e\|^2$$

- Small gain: If $\beta\sigma < 1$ then ultimate boundedness

$$\dot{V} \le -\alpha V(x) - \|y\|^2 + \beta(\sigma\|y\|^2 + \varepsilon) \le -\alpha V(x) + \beta\varepsilon$$

- GAS of set $\mathcal{A} = \{x \in \mathbb{R}^n \mid V(x) \le \frac{\beta\varepsilon}{\alpha}\}$

Stability Analysis via IS

- Model: $$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}w, & \text{for } \bar{x} \in \mathcal{C} \\ \bar{x}^{+} = \bar{G}_i\bar{x}, & \text{for } \bar{x} \in \mathcal{D}_i \\ z = \bar{C}\bar{x} + \bar{D}w \end{cases}$$
- Tools from hybrid system theory!

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- Tools from hybrid system theory!
- Stability: Lyapunov function $W(\bar{x})$
- $\mathcal{A} = \{\bar{x} \mid W(\bar{x}) = 0\}$
  - $W(\bar{x}) > 0$ when $\bar{x} \notin \mathcal{A}$ (positive definite)
  - $\dot{W}(\bar{x}) < 0$ when $\bar{x} \notin \mathcal{A}$ and $\bar{x} \in \mathcal{C}$
  - $W(G_i\bar{x}) \leq W(\bar{x})$ when $\bar{x} \in \mathcal{D}_i$
- proves GAS of $\mathcal{A}$

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Stability Analysis via IS

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  - $W(G_i\bar{x}) \leq W(\bar{x})$ when $\bar{x} \in \mathcal{D}_i$
- proves GAS of $\mathcal{A}$
- Constructive LMI conditions based on

$$W(\bar{x}) = \max\{0, \bar{x}^\top P\bar{x} - \sum_{i=1}^{N} \tfrac{\mu_i \varepsilon_i}{\alpha}\}$$

Comparison and observations

- Comparison [1]
  - IS framework less conservative than PS approach
  - IS describes ETC closed loop more accurately than PS
  - PS framework (numerically) simpler to use and more insightful

[1] Donkers, Heemels, TAC'12

Comparison and observations

- Comparison [1]
  - IS framework less conservative than PS approach
  - IS describes ETC closed loop more accurately than PS
  - PS framework (numerically) simpler to use and more insightful
- Observations for ETM $\|e^i\|^2 \leq \sigma_i \|v^i\|^2 + \varepsilon_i$
  - σ_i affect the ultimate boundedness
  - ε_i only influence the 'size' of $\mathcal{A}$
  - $\varepsilon_i = 0$ all i yields $\mathcal{A} = \{0\}$, but also zero inter-event times
  - Tradeoff between number of events and ultimate bound through σ_i and ε_i

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[1] Donkers, Heemels, TAC'12

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Guaranteeing a non-zero lower bound

$$t^i_{k_i+1} = \inf\{t > t^i_{k_i} \mid \|\hat{y}^i - y^i\|^2 = \sigma_i \|y^i\|^2 + \varepsilon_i\}$$

- Lower bound $h^i_{\min} > 0$ for node i for $\bar{x}(0)$ in bounded set, i.e.

$$t^i_{k_i+1} - t^i_{k_i} \geq h^i_{\min}$$

- Only when $\varepsilon_i > 0$, we have $h^i_{\min} > 0$
- More transmissions if $\|\bar{x}(0)\|$ or $\|w\|_{\mathcal{L}_\infty}$ are larger
- More transmissions if ε_i are smaller

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Example 1: Comparison to existing method

- Consider $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(t)$ and $\hat{u}(t) = \begin{bmatrix} 1 & -4 \end{bmatrix} \hat{x}(t)$
- Centralised state feedback as considered in [1]
- We look for largest σ for which ETC system is stable: $\|e\|^2 \leq \sigma \|x\|^2$

	σ	$h_{\min}$
Results from [1]	0.0030	0.0318
By minimising the $\mathcal{L}_2$-gain	0.0273	0.0840
Impulsive System	0.0588	0.1136

[1] Tabuada, TAC ’07

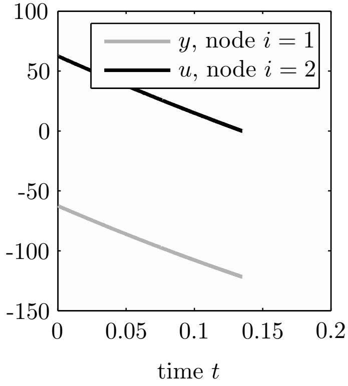
Illustrative Examples

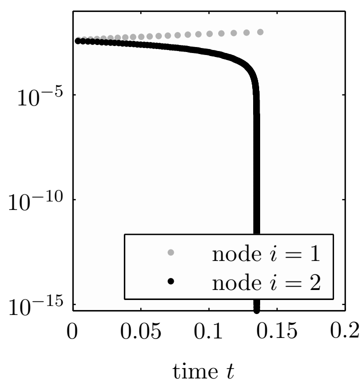
Example 2: $\varepsilon_i = 0$ zero inter-event times!

- Consider

$$\begin{cases} \dot{x}_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u} \\ y = \begin{bmatrix} -1 & 4 \end{bmatrix} x_p \end{cases} \qquad \begin{cases} \dot{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{y} \\ u = \begin{bmatrix} 1 & -4 \end{bmatrix} x_c \end{cases}$$

- ETM: $\|e^i\|^2 = \sigma_i \|v^i\|^2 + \varepsilon_i$
- Parameters: $\sigma_1 = \sigma_2 = 10^{-3}$ and $\varepsilon_1 = \varepsilon_2 = 0$





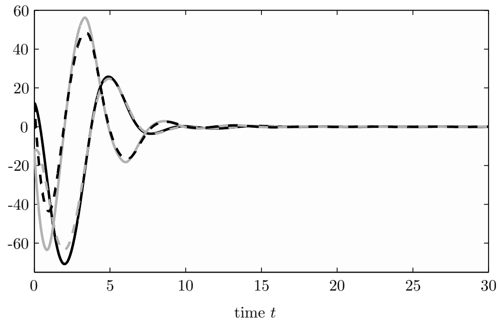
Illustrative Examples

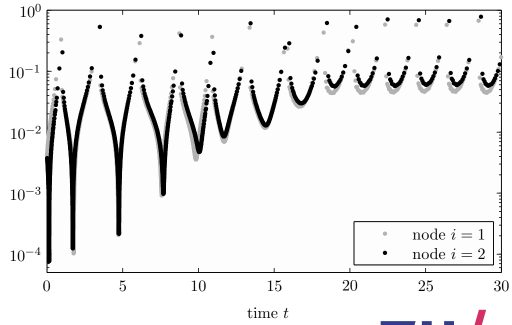
Example 2: Need for extending ETM including ε_i

- Consider

$$\begin{cases} \dot{x}_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u} \\ y = \begin{bmatrix} -1 & 4 \end{bmatrix} x_p \end{cases} \qquad \begin{cases} \dot{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{y} \\ u = \begin{bmatrix} 1 & -4 \end{bmatrix} x_c \end{cases}$$

- ETM: $\|e^i\|^2 = \sigma_i \|v^i\|^2 + \varepsilon_i$
- Parameters: $\sigma_1 = \sigma_2 = 10^{-3}$ and $\varepsilon_1 = \varepsilon_2 = 10^{-3}$





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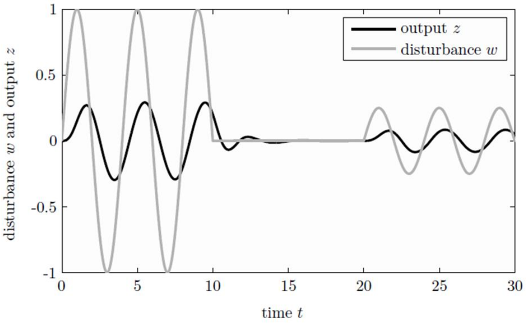
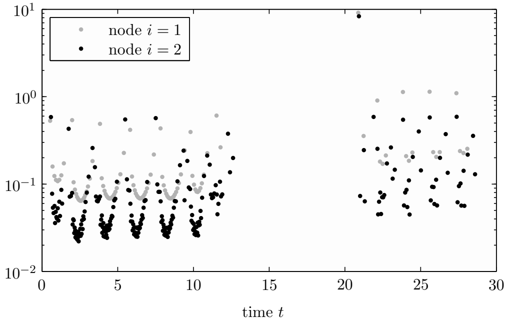
Example 3: What ETC is all about!

- Consider

$$\begin{cases} \dot{x}_p = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_p \end{cases} \qquad \begin{cases} \dot{x}_c = \begin{bmatrix} -2 & 1 \\ -13 & -3 \end{bmatrix} x_c + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \hat{y} \\ u = \begin{bmatrix} 5 & 2 \end{bmatrix} x_c \end{cases}$$

- Taking $\sigma_1 = \sigma_2 = 10^{-3}$ and $\varepsilon_1 = \varepsilon_2 = 10^{-3}$, yields $\mathcal{L}_\infty$-gain of 0.46





Example 3: What ETC is all about!

- Consider

$$\begin{cases} \dot{x}_p = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_p \end{cases} \quad \begin{cases} \dot{x}_c = \begin{bmatrix} -2 & 1 \\ -13 & -3 \end{bmatrix} x_c + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \hat{y} \\ u = \begin{bmatrix} 5 & 2 \end{bmatrix} x_c \end{cases}$$

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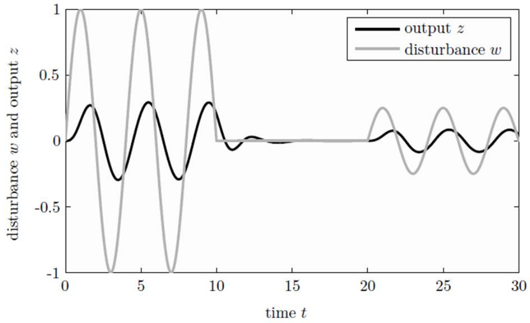

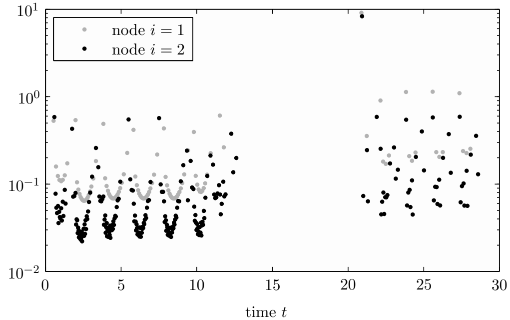

- Act when needed!

Part IIB: Periodic ETC

- Adopt alternative ETMs instead of $\|y - \hat{y}\|^2 \leq \sigma \|y\|^2$
  - **Mixed:** $\|y - \hat{y}\| \leq \sigma \|y\| + \varepsilon$

- Time regularization
  - Transmission possible only at kh, $k \in \mathbb{N}$

$$t_{k+1} = \inf\{t > t_k \mid C(y(t), \hat{y}(t)) \geq 0 \wedge t = kh,\ k \in \mathbb{N}\}$$

  - ▶ Discrete-time ETC (dt plant)
  - ▶ Periodic Event-Triggered Control (PETC) (ct plant) [1,2]

$\longrightarrow$ Combining the best of two worlds !?

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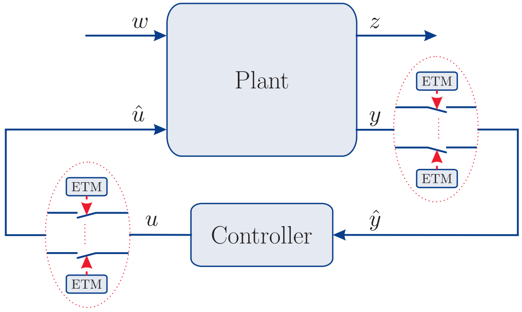
[1] Heemels, Donkers & Teel, CDC/ECC'11 [2] Heemels, Donkers & Teel , TAC 13

Output-Based Decentralised PETC

Plant: $\begin{cases} \dot{x}^p = A^p x^p + B^p \hat{u} + B^w w \\ y = C^p x^p \end{cases}$



Controller: $\begin{cases} x^c_{k+1} = A^c x^c_k + B^c \hat{y}_k \\ u(t_k) = u_k = C^c x^c_k + D^c \hat{y}(t_k) \end{cases}$

- For $t_k = kh$, $k \in \mathbb{N}$, and h sampling period: For $t \in (t_k, t_{k+1}]$

$$\hat{y}^j(t) = \begin{cases} y^j(t_k), & \text{if } \|y^j(t_k) - \hat{y}^j(t_k)\| > \sigma_j \|y^j(t_k)\| \\ \hat{y}^j(t_k), & \text{if } \|y^j(t_k) - \hat{y}^j(t_k)\| \leqslant \sigma_j \|y^j(t_k)\| \end{cases}$$

- Continuous Event-Triggered Control (CETC):

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}w, & \text{for } \bar{x} \in \mathcal{C} \\ \bar{x}^+ = \bar{G}_i\bar{x}, & \text{for } \bar{x} \in \mathcal{D}_i \end{cases} \qquad (\text{i.e. } \|y^i - \hat{y}^i\| > \sigma_i\|y^i\|)$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} \bar{A}\xi + \bar{B}w \\ 1 \end{bmatrix} \qquad \text{when } \tau \in [0, h]$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} \bar{G}_{\mathcal{J}(\xi)}\xi \\ 0 \end{bmatrix} \qquad \text{when } \tau = h$$

where

$$\mathcal{J}(\xi) = \{i \mid \xi \in \mathcal{D}_i\} = \{i \mid \|y^i - \hat{y}^i\| > \sigma_i\|y^i\|\}$$

and

$$\bar{G}_\emptyset = I$$

Output-Based Decentralised PETC

- Impulsive model
- $\xi = (x^p, x^c, \hat{y})$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} \bar{A}\xi + \bar{B}w \\ 1 \end{bmatrix} \quad \text{when } \tau \in [0, h]$$

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Output-Based Decentralised PETC

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- $\xi = (x^p, x^c, \hat{y})$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} \bar{A}\xi + \bar{B}w \\ 1 \end{bmatrix} \quad \text{when } \tau \in [0, h]$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} \bar{G}_{\mathcal{J}(\xi)}\xi \\ 0 \end{bmatrix} \quad \text{when } \tau = h$$

- Stability and $\mathcal{L}_2$-gain analysis
  - Discrete-time perturbed linear system approach
  - Discrete-time PWL system approach
  - Impulsive system approach: Riccati differential equation leading to Lyapunov/storage function $\xi^\top P(\tau)\xi$

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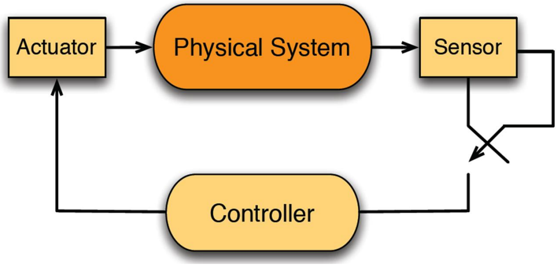
Numerical example



$$\frac{\mathrm{d}}{\mathrm{d}t}x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \qquad u = \hat{y}$$

$$y = [1 \ -4]x$$

- Sampling times: $t_k = kh$ with $h = 0.05$. For $t \in (t_k, t_{k+1}]$

$$\hat{y}(t) = \begin{cases} y(t_k), & \text{when } \|\hat{y}(t_k) - y(t_k)\| > \sigma \|y(t_k)\| \\ \hat{y}(t_k), & \text{when } \|\hat{y}(t_k) - y(t_k)\| \leq \sigma \|y(t_k)\| \end{cases}$$

Numerical example

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \qquad u = \hat{y}$$

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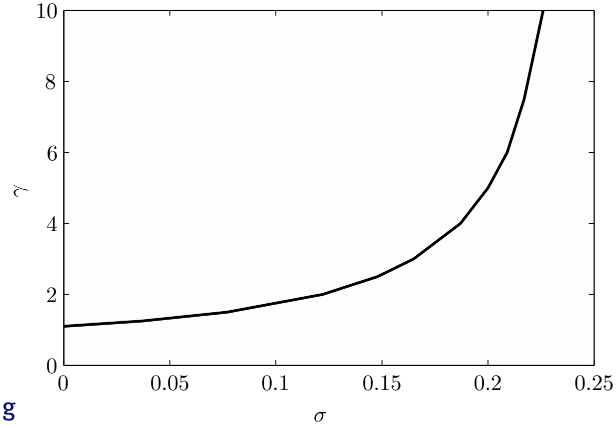
- $\mathcal{L}_2$ gain analysis $z = [0\ 1]x$



Numerical example

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \qquad u = \hat{y}$$

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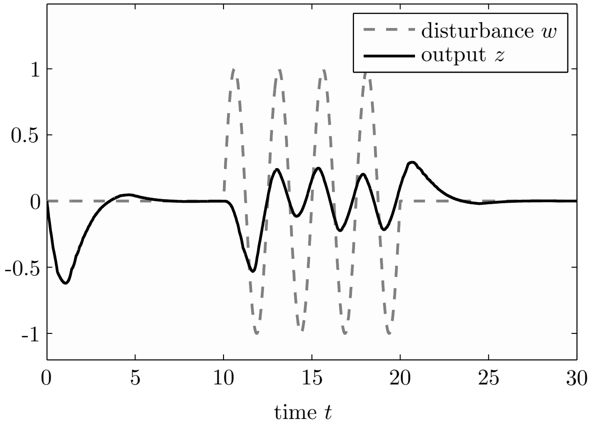
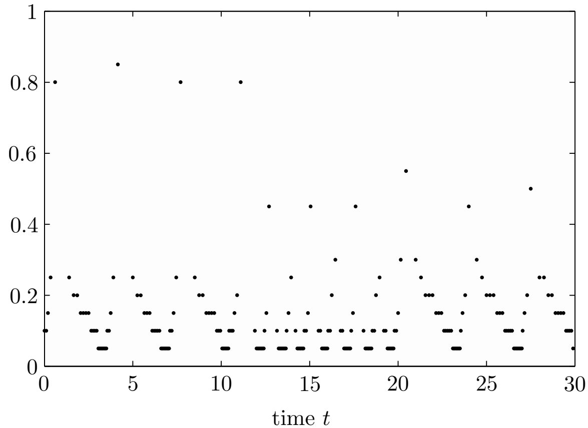
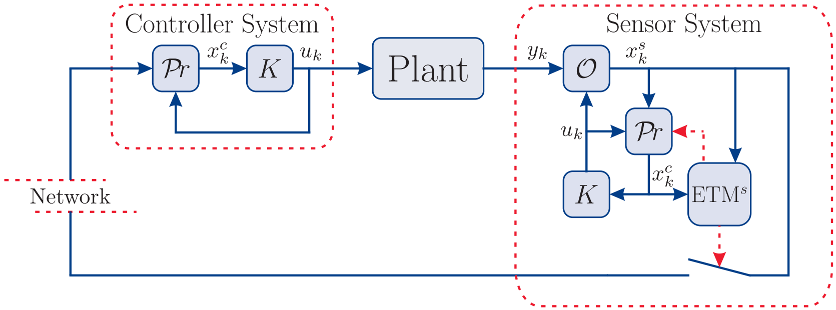




- $\sigma = 0.2$
- **Inter-event times:** 0.05 **to** 0.85 **(17 times h)!!!!**

Extension: Model-based PETC



$$x_{k+1} = Ax_k + Bu_k + Ew_k$$

$$\mathcal{O}: \quad x^s_{k+1} = Ax^s_k + Bu_k + L(y_k - Cx^s_k)$$

$$\mathcal{P}r: \quad x^c_{k+1} = \begin{cases} Ax^c_k + Bu_k, \text{ when } x^s_k \text{ is not sent} \\ Ax^s_k + Bu_k, \text{ when } x^s_k \text{ is sent} \end{cases}$$

$$ETM^s: \quad x^s_k \text{ is sent} \quad \Leftrightarrow \quad \|x^s_k - x^c_k\| > \sigma_s \|x^s_k\|$$

$$u_k = \begin{cases} Kx^c_k, \text{ when } x^s_k \text{ is not sent} \\ Kx^s_k, \text{ when } x^s_k \text{ is sent} \end{cases}$$

Conclusions

- THM1: Output-based $+$ relative triggering: Zero m.i.e.t (Zeno)
- Adopt alternative event-triggering mechanisms
  - **Mixed:** $\|y - \hat{y}\| \leq \sigma\|y\| + \varepsilon$
- Time regularization
  - Periodic Event-Triggered Control (PETC)
- Impulsive system models
- THM2: IS less conservative than PS
- Capturing sensitivity properties in $\mathcal{L}_\infty$ and $\mathcal{L}_2$-gains (intersample)
- THM3: Model-based predictions further enhance comm. savings

Final comments

- Open issues:
  - Demonstrating improvement beyond periodic control quantitatively (J_{cont}, J_{comm}) – [1,2,3]
  - Impulsive: Improving performance estimates
  - Practical deployment
- More information:
  - Invited sessions on Thursday (ThB01, ThC01)
  - Tutorial paper providing overview
  - Homepage

    `http://www.dct.tue.nl/heemels`

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[1] Antunes, Heemels, Tabuada, CDC12 ThC01.4
[2] Gommans, Heemels, Donkers, Tabuada, submitted
[3] Barradas-Berglind, Gommans, Heemels, NMPC12