State Estimation Over Delayed and Lossy Channels: An Encoder–Decoder Synthesis

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***Abstract*—State estimation over a communication channel, in which sensory information of a stochastic source is transmitted in real-time by an encoder to a decoder that estimates the state of the source, is one of the basic problems in networked control systems. In this article, we investigate the performance of state estimation under two primary network imperfections: packet loss and time delay. To that end, we make a causal frequency-distortion tradeoff that is defined between the packet rate and the mean square error, when the source and the channel are modeled by a partially observable Gauss–Markov process and a fixed-delay packet-erasure channel, under two distinct communication protocols: one with and one without packet-loss detection. We prove the existence of a globally optimal policy profile, and show that this policy profile is composed of a symmetric threshold scheduling policy and a non-Gaussian linear estimation policy, which are used by the encoder and the decoder, respectively. Our structural results assert that the scheduling policy is expressible in terms of $3d-1$ variables related to the source and the channel, where d is the time delay, and that the estimation policy incorporates no residuals related to signaling. The key finding is that packet-loss detection does not increase the performance of the underlying networked system in the sense of the causal frequency-distortion tradeoff. We prove this by showing that the globally optimal policy profile remains exactly the same under both of the communication protocols.**

***Index Terms*—Causal tradeoffs, linear policies, networked systems, optimal policies, packet loss, state estimators, team decision-making, threshold policies, time delay.**

Nomenclature

Symbol	Description
x_k	State of source at time k.
y_k	Output of sensor at time k.
w_k	Source noise at time k.
v_k	Sensor noise at time k.
A_k	State matrix at time k.
C_k	Output matrix at time k.
W_k	Covariance of source noise at time k.
V_k	Covariance of sensor noise at time k.
m_0	Mean of x_0.
M_0	Covariance of x_0.
N	Time horizon.
u_k	Scheduling variable at time k.
γ_k	Packet-loss variable at time k.
$\mathfrak{F}$	Packet-loss symbol.
$\mathfrak{E}$	Absence-of-transmission symbol.
$\mathfrak{D}$	Packet-loss or absence-of-transmission symbol.
z_k	Output of channel at time k.
d	Time delay.
λ_k	Packet success rate at time k.
λ'_k	Packet error rate at time k.
$\mathcal{I}^e_k$	Information set of encoder at time k.
$\mathcal{I}^d_k$	Information set of decoder at time k.
ϵ	Scheduling policy.
δ	Estimation policy.
$\epsilon^\star$	Optimal scheduling policy.
$\delta^\star$	Optimal estimation policy.
$\mathcal{E}$	Set of admissible scheduling policies.
$\mathcal{D}$	Set of admissible estimation policies.
θ_k	Weighting coefficient at time k.
Φ	Loss function.
$\check{x}_k$	State estimate at encoder at time k.
$\hat{x}_k$	State estimate at decoder at time k.
ν_k	Estimation innovation at encoder at time k.
$\hat{e}_k$	Estimation error at decoder at time k.
$\tilde{e}_k$	Estimation mismatch at time k.

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I. Introduction

STATE estimation over a communication channel, in which sensory information of a stochastic source is transmitted in real time by an encoder to a decoder that estimates the state of the source, is one of the basic problems that arises commonly in the context of networked control systems [1], [2]. Nonetheless, this problem for the joint synthesis of the encoder and the decoder when both communication and estimation costs are taken into account is a team decision-making problem, with a nonclassical information structure subject to a signaling effect, that is nonconvex and in general intractable. The information structure is nonclassical because any decision of the encoder can change the information set of the decoder while the decoder does not have access to the information used in the construction of that decision. Moreover, a signaling effect exists because implicit information can be exchanged between the encoder and the decoder even when no sensory information is successfully communicated. It is known that decision-making problems with nonclassical information structures are inherently difficult to solve [3], and that estimation problems subject to signaling effects are highly nonlinear with no analytical solutions [4]. Despite these perplexities, the aim of the present article is to determine the fundamental performance limit of state estimation of a partially observable process over a delayed and lossy channel in the sense of a causal frequency-distortion tradeoff[1] that is defined between the packet rate and the mean square error. We formulate this tradeoff as a stochastic optimization problem with an encoder and a decoder as two distributed decision makers, and seek to characterize a globally optimal policy profile composed of a scheduling policy[2] and an estimation policy to be used by the encoder and the decoder, respectively. Our study contributes to the existing literature on causal tradeoffs between communication and estimation costs, and enhances our understanding of the impacts of packet loss and time delay, as two primary network imperfections, on the performance of networked control systems.

A. Related Work

The causal frequency-distortion tradeoff between the packet rate and the mean square error has already been examined in the literature [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]. Notably, Imer and Basar [5] studied the estimation of a scalar Gauss–Markov process based on dynamic programming by considering a symmetric threshold scheduling policy, and obtained the optimal threshold value. Lipsa and Martins [6] analyzed the estimation of a scalar Gauss–Markov process based on majorization theory, and proved that the optimal scheduling policy is symmetric and the optimal estimation policy is linear. This work was extended to estimation over an independent and identically distributed (i.i.d.) packet-erasure channel in [7], where a similar structural result was found. In addition, Molin and Hirche [8] investigated the convergence properties of an iterative algorithm for the estimation of a scalar Markov process with symmetric noise distribution, and found a result coinciding with that in [6]. Chakravorty and Mahajan [9] studied the estimation of a scalar autoregressive Markov process with symmetric noise distribution based on renewal theory, and proved that the optimal scheduling policy is symmetric and the optimal estimation policy is linear. This work was generalized to estimation over an i.i.d. packet-erasure channel and a Gilbert–Elliott packet-erasure channel in [10] and [11], where a similar structural result was found. Moreover, Rabi et al. [12] formulated the estimation of the scalar Wiener process and a scalar Ornstein–Uhlenbeck process as an optimal multiple stopping time problem by discarding the signaling effect, and showed that the optimal scheduling policy is symmetric. Later, Guo and Kostina [13] made a contribution by studying the estimation of the scalar Wiener process and a scalar Ornstein–Uhlenbeck process in the presence of the signaling effect, obtained a result that matches with that in [12], and showed that, under some assumptions, a sign-of-innovation code can be used as the optimal compression policy. The authors also looked at the estimation of the scalar Wiener process over a fixed-delay lossless channel in the presence of the signaling effect in [14], and obtained a similar structural result. Finally, Sun et al. [15] studied the estimation of the scalar Wiener process over a random-delay lossless channel by discarding the signaling effect, and showed that the optimal scheduling policy is symmetric. Note that, in these studies, the scheduling policies are observation-based, as they take advantage of realized sensory information.

There are other tradeoffs in the literature that are pertinent to our study [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. On one hand, there exist a few works on the causal frequency-distortion tradeoff between the packet rate and a variance penalty, in which instead of an observation-based scheduling policy a variance-based scheduling policy is obtained. In particular, in this class, Leong et al. [20], [21] studied the estimation of a Gauss–Markov process over an i.i.d. packet-erasure channel, and showed that the optimal scheduling policy is threshold in terms of the estimation error covariance. Note that, due to a restriction on the policy structure, variance-based scheduling policies are generally outperformed by observation-based scheduling policies. On the other hand, there exist several works on the causal rate-distortion tradeoff between the bit rate and the mean square error, in which instead of a scheduling policy a compression policy is obtained. In particular, in this class, Witsenhausen [22] addressed the sequential coding of a discrete-time kth order Markov process over a finite time horizon, and showed that the optimal code depends on the last k process states and the current decoder state. Walrand and Varaiya [23] investigated the sequential coding of a discrete-time finite-state Markov process over a noisy channel with feedback, and showed that there exists a separation between the design of the encoder and the decoder through the conditional distribution. Later, Yüksel [24] extended this work to a partially observable continuous-state Markov process, and recovered the same structural result. Borkar et al. [25] studied the sequential coding of a partially observable Markov process without fixing the quantization levels, and provided a procedure based on dynamic programming for the computation of the optimal partition. Moreover, Khina et al. [26] studied the sequential coding of a Gauss–Markov process with multiple sensors over an i.i.d. packet-erasure channel, and characterized the achievable causal rate-distortion region. Note that, in these studies, compression policies rely on the fact that compressed sensory information is transmitted in a periodic way.

[1]Contrary to a conventional rate-distortion tradeoff in which the bite rate, as a communication cost, is penalized, in a frequency-distortion tradeoff the packet rate, i.e., the transmission frequency, is penalized.

[2]A scheduling policy in the context of this article refers to a communication policy that decides to transmit or not to transmit a data packet over the channel at each time.

There is also a related body of research in the literature about the effects of packet loss and time delay on stability of estimation [28], [29], [30], [31], [32]. Notably, in a seminal work, Sinopoli et al. [28] studied mean-square stability of Kalman filtering over an i.i.d. packet-erasure channel, and proved that there exists a critical point for the packet-loss probability above which the expected estimation error covariance is unbounded. Wu et al. [29] addressed peak-covariance stability of Kalman filtering over a Gilbert–Elliott packet-erasure channel, and proved that there exists a critical region defined by the recovery and failure rates outside which the expected prediction error covariance is unbounded. Quevedo et al. [30] investigated the mean-square stability of Kalman filtering over a fading packet-erasure channel with correlated gains, and established a sufficient condition that ensures the exponential boundedness of the expected estimation error covariance. Schenato [31] studied mean-square stability of Kalman filtering over a random-delay i.i.d. packet-erasure channel, obtained the optimal structure of the estimator at the decoder, and showed that there exists a critical value for the packet-loss probability, similar to what was found in [28], that is independent of the time delay. Note that, in these studies, it is assumed that raw measurements of the sensor are periodically transmitted by the encoder to the decoder. Differently, Gupta et al. [32] investigated the estimation of a Gauss–Markov process over a packet-erasure channel as a subproblem, and showed that transmitting the minimum mean-square-error state estimate at the encoder at each time leads to the maximal information set for the decoder. The authors also obtained a necessary and sufficient condition for the packet-loss probability of an i.i.d. packet-erasure channel that guarantees the boundedness of the expected estimation error covariance.

B. Contributions and Outline

In this article, we synthesize an encoder and a decoder optimally by concentrating on the causal frequency-distortion tradeoff between the packet rate and the mean square error, when the source and the channel are modeled by a partially observable Gauss–Markov process and a fixed-delay packet-erasure channel, under two distinct communication protocols: one with and one without packet-loss detection. Note that, in the causal frequency-distortion tradeoff, the events related to packet loss and absence of transmission provide an opportunity for the encoder to implicitly encode extra information aside from the sensory information that is explicitly communicated through successful packet delivery. Correspondingly, depending on the structures of the encoder and the channel, the decoder could make an inference even when a packet loss occurs or no message is transmitted. This inference can, in general, be different when the channel is with and without packet-loss detection. We address all these issues. Succinctly, our major contributions are as follows.

1) We prove the existence of a globally optimal policy profile in the causal frequency-distortion tradeoff, and show that this policy profile is composed of a symmetric threshold scheduling policy and a non-Gaussian linear estimation policy. Our structural results assert that the scheduling policy is expressible in terms of $3d-1$ variables related to the source and the channel, where d is the time delay, and that the estimation policy incorporates no residuals related to signaling.
2) Apart from the characterization of a globally optimal policy profile, the key finding is that packet-loss detection does not increase the performance of the underlying networked system in the sense of the causal frequency-distortion tradeoff. We prove this by showing that the globally optimal policy profile remains exactly the same under both of the communication protocols.

We should remark that our study is different from the previous studies on state estimation of stochastic sources over communication channels. In particular, it differs from the studies in [5], [6], [8], [9], [12], and [13], which consider only ideal channels; from those in [7], [10], and [11], which are restricted to scalar sources and delay-free packet-erasure channels; from those in [14] and [15], which are restricted to scalar sources and delayed lossless channels; from those in [16], [17], [18], [19], [20], and [21], which are restricted to variance-based scheduling policies and delay-free packet-erasure channels; and from those in [22], [23], [24], and [25], which focus on compression policies rather than scheduling policies. Our results here apply to multidimensional sources and fixed-delay packet-erasure channels, without any limitations on the information structure or the policy structure, and deliver an observation-based scheduling policy and a recursive estimation policy that are jointly optimal. Moreover, in contrast to the results in [28], [29], [30], [31], and [32], which provide conditions guaranteeing stability of estimation, our results provide a basis for obtaining the minimum packet rate that guarantees a given level of estimation performance. Lastly, contrary to the studies in [28], [29], [30], and [31], where the message that is transmitted at each time is the output of the sensor at that time, in our study, the message that can be transmitted at each time, similar to what was adopted in [20] and [32], is the minimum mean-square-error state estimate at the encoder at that time. As we will see, transmitting this state estimate, which is provided by a Kalman filter,[3] yields the best performance.

The rest of this article is organized as follows: We formulate the causal frequency-distortion tradeoff in Section II, present our main theoretical results in Section III, and provide the derivation of these results in Section IV. Then, we present a numerical example in Section V. Finally, Section VI concludes the article.

C. Preliminaries

In the sequel, the sets of real numbers and nonnegative integers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. For $x, y \in \mathbb{N}$ and $x \leq y$, the set $\mathbb{N}_{[x,y]}$ denotes $\{z \in \mathbb{N} | x \leq z \leq y\}$. The sequence of all vectors x_t, $t = p, \ldots, q$, is represented by $\boldsymbol{x}_{p:q}$. For matrixes X and Y, the relations $X \succ 0$ and $Y \succeq 0$ denote that X and Y are positive definite and positive semidefinite, respectively. The indicator function of a subset $\mathcal{A}$ of a set $\mathcal{X}$ is denoted by $\mathbb{1}_{\mathcal{A}} : \mathcal{X} \to \{0, 1\}$. The symmetric decreasing rearrangement of

[3] In literature, a sensor capable of running a Kalman filter in real time is often referred to as smart sensor.

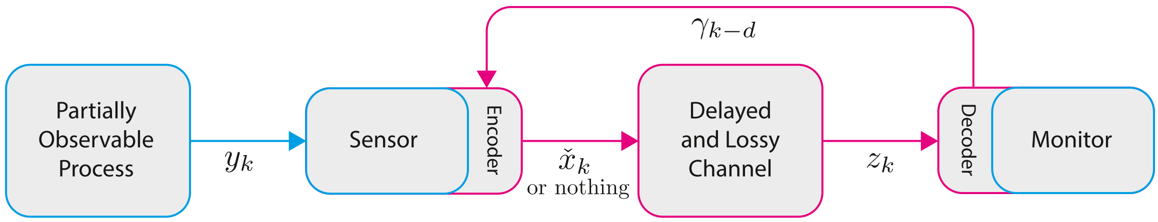

Fig. 1. State estimation of a partially observable process over a delayed and lossy channel. The signals are depicted for time k. Our objective is to synthesize an encoder and a decoder for this networked system optimally.

a Borel measurable function $f(x)$ vanishing at infinity is represented by $f^*(x)$. The product operator $\prod_{t=p}^{q} X_t$, where X_t is a matrix, is defined according to the order $X_p \cdots X_q$, and is equal to one when $q < p$. The probability measure of a random variable x is represented by $\mathsf{P}(x)$, its probability density or probability mass function by $\mathsf{p}(x)$, and its expected value and covariance by $\mathsf{E}[x]$ and $\mathrm{cov}[x]$, respectively. We will adopt stochastic kernels to represent decision policies. Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ be two measurable spaces. A Borel measurable stochastic kernel $\mathsf{P} : \mathcal{B}_{\mathcal{Y}} \times \mathcal{X} \to [0, 1]$ is a mapping such that $\mathcal{A} \mapsto \mathsf{P}(\mathcal{A}|x)$ is a probability measure on $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ for any $x \in \mathcal{X}$, and $x \mapsto \mathsf{P}(\mathcal{A}|x)$ is a Borel measurable function for any $\mathcal{A} \in \mathcal{B}_{\mathcal{Y}}$.

II. Frequency-Distortion Tradeoff in Presence of Packet Loss and Time Delay

In this section, we mathematically formulate the causal frequency-distortion tradeoff pertaining to state estimation of a partially observable process over a delayed and lossy channel. We first introduce the networked system, and then cast the tradeoff as a stochastic optimization problem with an encoder and a decoder as two distributed decision makers.

Consider a sensor observing the states of a partially observable Gauss–Markov process (see Fig. 1). The source model is expressed by the discrete-time state and output equations

$$x_{k+1} = A_k x_k + w_k \tag{1}$$

$$y_k = C_k x_k + v_k \tag{2}$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition x_0, where $x_k \in \mathbb{R}^n$ is the state of the source, $A_k \in \mathbb{R}^{n\times n}$ is the state matrix, $w_k \in \mathbb{R}^n$ is a Gaussian white noise with zero mean and covariance $W_k \succ 0$, $y_k \in \mathbb{R}^m$ is the output of the sensor, $C_k \in \mathbb{R}^{m\times n}$ is the output matrix, $v_k \in \mathbb{R}^m$ is a Gaussian white noise with zero mean and covariance $V_k \succ 0$, and N is a finite time horizon.

Assumption 1: For the source model in (1) and (2), the following assumptions are satisfied.

1) The initial condition x_0 is a Gaussian vector with mean m_0 and covariance M_0.
2) The random variables x_0, w_t, and v_s for $t, s \in \mathbb{N}_{[0,N]}$ are mutually independent, i.e., $\mathsf{p}(x_0, \boldsymbol{w}_{0:N}, \boldsymbol{v}_{0:N}) = \mathsf{p}(x_0) \prod_{k=0}^{N} \mathsf{p}(w_k) \prod_{k=0}^{N} \mathsf{p}(v_k)$.

Remark 1: The source model in (1) and (2) represents a wide class of dynamical systems. In particular, the time-varying nature of the model can lend itself to the approximation of a nonlinear system around its nominal trajectory. Moreover, the partially observable nature of the model takes into account the fact that in reality one can observe only a noisy version of the system's output. Note that the time-invariant and fully observable settings are special cases of our setting, and our results can readily be modified for them.

The sensor is connected over a fixed-delay packet-erasure channel to a monitor (see Fig. 1) where an estimate of the state of the source, represented by $\hat{x}_k$, is computed at each time k. Associated with this networked system, there exist an encoder and a decoder. Let $u_k \in \{0, 1\}$ be a binary variable such that $u_k = 1$ if a message containing sensory information, represented by $\check{x}_k$, is transmitted by the encoder to the decoder at time k, and $u_k = 0$ otherwise. A transmitted message at time k is successfully received by the decoder with probability λ_k and after fixed d-step delay, where $d \in \mathbb{N}_{[1,N]}$. Let $\gamma_k \in \{0, 1\}$ be a binary random variable modeling packet loss such that $\gamma_k = 0$ if a packet loss occurs at time k, and $\gamma_k = 1$ otherwise. Then, the probability of $\gamma_k = 0$ is $\lambda'_k := 1 - \lambda_k$. We examine two communication protocols: one with and one without packet-loss detection. Accordingly, in the case of the channel with packet-loss detection, we have the input–output relation

$$z_{k+d} = \begin{cases} \check{x}_k, & \text{if } u_k = 1 \ \wedge \ \gamma_k = 1 \\ \mathfrak{F}, & \text{if } u_k = 1 \ \wedge \ \gamma_k = 0 \\ \mathfrak{E}, & \text{otherwise} \end{cases} \tag{3}$$

for $k \in \mathbb{N}_{[0,N]}$ with $z_0, \ldots, z_{d-1} = \mathfrak{E}$ by convention, where z_k is the output of the channel, and $\mathfrak{F}$ and $\mathfrak{E}$ represent packet loss and absence of transmission, respectively. However, in the case of the channel without packet-loss detection, we have

$$z_{k+d} = \begin{cases} \check{x}_k, & \text{if } u_k = 1 \ \wedge \ \gamma_k = 1 \\ \mathfrak{D}, & \text{otherwise} \end{cases} \tag{4}$$

for $k \in \mathbb{N}_{[0,N]}$ with $z_0, \ldots, z_{d-1} = \mathfrak{D}$ by convention, where z_k is the output of the channel and $\mathfrak{D}$ represents packet loss or absence of transmission. Upon a successful delivery, the decoder transmits a packet acknowledgment to the encoder via a feedback channel.

Assumption 2: For the channel model in (3) or (4), the following assumptions are satisfied.

1) The packet-loss probabilities λ'_k for $k \in \mathbb{N}_{[0,N]}$ are random variables forming a Markov chain, i.e., $\mathsf{p}(\lambda'_k|\boldsymbol{\lambda}'_{0:k-1}) = \mathsf{p}(\lambda'_k|\lambda'_{k-1})$.
2) The packet-loss probability λ'_k is known at the encoder and the decoder at each time k.

3) The random variables γ_k for $k \in \mathbb{N}_{[0,N]}$ are mutually independent given the respective packet-loss probabilities, i.e., $\mathsf{p}(\boldsymbol{\gamma}_{0:N}|\boldsymbol{\lambda}'_{0:N}) = \prod_{k=0}^{N} \mathsf{p}(\gamma_k|\lambda'_k)$.
4) Quantization error is negligible, i.e., in a successful transmission, real-value sensory information can be conveyed from the encoder to the decoder without any error.
5) The feedback channel is ideal, i.e., packet acknowledgments are received by the encoder without any loss, delay, or error.

Remark 2: An intriguing aspect of the channel model in (3) is that the encoder can implicitly encode information through $\mathfrak{F}$ and $\mathfrak{E}$. Suppose that $u_k = 1$ if and only if $y_k \in \mathcal{A}_k$, where $\mathcal{A}_k$ can be any Borel measurable set. Then, if the decoder receives $\mathfrak{F}$ or $\mathfrak{E}$ at time $k+d$, it can infer that $y_k \in \mathcal{A}_k$ or $y_k \notin \mathcal{A}_k$, respectively. The situation is the same for the channel model in (4), but in this case the encoder has less degree of freedom as it can implicitly encode information only through $\mathfrak{D}$. In particular, under the same scheduling policy, if the decoder receives $\mathfrak{D}$ at time $k+d$, it can infer that $\{y_k \in \mathcal{A}_k \wedge \gamma_k = 0\} \vee y_k \notin \mathcal{A}_k$.

Assumption 3: The following factorization among the variables associated with the source and the channel holds:

$$\mathsf{p}(x_0, \boldsymbol{w}_{0:N}, \boldsymbol{v}_{0:N}, \boldsymbol{\gamma}_{0:N}, \boldsymbol{\lambda}'_{0:N}) = \mathsf{p}(x_0, \boldsymbol{w}_{0:N}, \boldsymbol{v}_{0:N})\, \mathsf{p}(\boldsymbol{\gamma}_{0:N}|\boldsymbol{\lambda}'_{0:N})\, \mathsf{p}(\boldsymbol{\lambda}'_{0:N}). \quad (5)$$

In our networked system, the decision variables are u_k and $\hat{x}_k$ for all $k \in \mathbb{N}_{[0,N]}$, which are decided by the encoder and the decoder, respectively, in a distributed way and based on causal information. Let the information sets of the encoder and the decoder be expressed by $\mathcal{I}_k^e = \{y_t, z_t, \lambda_t, u_s, \hat{x}_s, \gamma_r |\ t \in \mathbb{N}_{[0,k]}, s \in \mathbb{N}_{[0,k-1]}, r \in \mathbb{N}_{[0,k-d]}\}$ and $\mathcal{I}_k^d = \{z_t, \lambda_t, \hat{x}_s | t \in \mathbb{N}_{[0,k]}, s \in \mathbb{N}_{[0,k-1]}\}$, respectively. We say that a policy profile (ϵ, δ) consisting of a scheduling (i.e., encoding) policy ϵ and an estimation (i.e., decoding) policy δ is admissible if $\epsilon = \{\mathsf{P}(u_k|\mathcal{I}_k^e)\}_{k=0}^N$ and $\delta = \{\mathsf{P}(\hat{x}_k|\mathcal{I}_k^d)\}_{k=0}^N$, where $\mathsf{P}(u_k|\mathcal{I}_k^e)$ and $\mathsf{P}(\hat{x}_k|\mathcal{I}_k^d)$ are Borel measurable stochastic kernels.

Our objective is to find a globally optimal solution $(\epsilon^\star, \delta^\star)$ to the causal frequency-distortion tradeoff between the packet rate and the mean square error, which is cast by the following stochastic optimization problem:

$$\underset{\epsilon \in \mathcal{E}, \delta \in \mathcal{D}}{\text{minimize}}\ \Phi(\epsilon, \delta) \quad (6)$$

where $\mathcal{E}$ and $\mathcal{D}$ are the sets of admissible scheduling policies and admissible estimation policies, respectively, and

$$\Phi(\epsilon, \delta) := \mathsf{E}\left[\sum_{k=0}^{N} \theta_k u_k + (x_k - \hat{x}_k)^T (x_k - \hat{x}_k)\right] \quad (7)$$

for the weighting coefficient $\theta_k \geq 0$, subject to the source model in (1) and (2) and the channel model in (3) or (4). The concept of global optimality associated with this optimization problem is captured by the following definition.

Definition 1 (Global optimality): A policy profile $(\epsilon^\star, \delta^\star)$ in the casual frequency-distortion tradeoff with two distributed decision makers, i.e., the encoder and the decoder, is globally optimal if

$$\Phi(\epsilon^\star, \delta^\star) \leq \Phi(\epsilon, \delta),\ \text{for all } \epsilon \in \mathcal{E}, \delta \in \mathcal{D}.$$

Remark 3: Note that the loss function in (7), when divided by N, incorporates two criteria (see, e.g., [6], [33] for similar loss functions in the literature). The packet-rate criterion, which often appears in the analysis of packet-switching networks, evaluates the cost of communication, while the mean-square-error criterion, which often appears in the analysis of control systems, evaluates the quality of real-time estimation. Moreover, note that the causal frequency-distortion tradeoff in (6) is a team decision-making problem with a nonclassical information structure subject to a signaling effect. Although this problem is nonconvex and in general intractable, in the next section we characterize a globally optimal solution $(\epsilon^\star, \delta^\star)$, which leads to the determination of the fundamental performance limit of the underlying networked system. Our results are presented for a finite time horizon, but extension to an infinite time horizon is straightforward provided the source is time-invariant and observable, and the packet-loss probability is time-invariant.

III. Optimal Synthesis of Encoder and Decoder

In this section, we present our theoretical results on the characterization of a globally optimal policy profile. We will soon observe that the optimal design of the encoder and the decoder is generally intertwined, but we will overcome this hindrance by seeking a separation in the design. The derivation of our results is provided in Section IV.

We first show in the next lemma that the conditional means $\mathsf{E}[x_k|\mathcal{I}_k^e]$ and $\mathsf{E}[x_k|\mathcal{I}_k^d]$ are instrumental for the tasks carried out by the encoder and the decoder.

Lemma 1: Without loss of optimality, at each time k, one can adopt $\check{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^e]$ as the message that can be transmitted by the encoder, and $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^d]$ as the state estimate that can be computed by the decoder.

As a result, we can safely use $\check{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^e]$ and $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^d]$. Define the innovation at the encoder $\nu_k := y_k - C_k\,\mathsf{E}[x_k|\mathcal{I}_{k-1}^e]$, the estimation error at the decoder $\hat{e}_k := x_k - \hat{x}_k$, and the estimation mismatch $\tilde{e}_k := \check{x}_k - \hat{x}_k$. We characterize in the next two lemmas the recursive equations that the optimal estimators at the encoder and the decoder must satisfy.

Lemma 2: The optimal estimator at the encoder satisfies the recursive equations

$$\check{x}_k = m_k + K_k\left(y_k - C_k m_k\right) \quad (8)$$

$$m_k = A_{k-1}\check{x}_{k-1} \quad (9)$$

$$Q_k = \left(M_k^{-1} + C_k^T V_k^{-1} C_k\right)^{-1} \quad (10)$$

$$M_k = A_{k-1} Q_{k-1} A_{k-1}^T + W_{k-1} \quad (11)$$

for $k \in \mathbb{N}_{[1,N]}$ with initial conditions $\check{x}_0 = m_0 + K_0(y_0 - C_0 m_0)$ and $Q_0 = (M_0^{-1} + C_0^T V_0^{-1} C_0)^{-1}$, where $m_k = \mathsf{E}[x_k|\mathcal{I}_{k-1}^e]$, $Q_k = \mathsf{cov}[x_k|\mathcal{I}_k^e]$, $M_k = \mathsf{cov}[x_k|\mathcal{I}_{k-1}^e]$, and $K_k = Q_k C_k^T V_k^{-1}$.

Lemma 3: The optimal estimator at the decoder when the channel is with packet-loss detection satisfies the recursive equation

$$\hat{x}_k = u_{k-d}\gamma_{k-d}\left(\prod_{t=1}^{d} A_{k-t}\right)\check{x}_{k-d} + (1-u_{k-d})\imath_{k-1} + u_{k-d}(1-\gamma_{k-d})\jmath_{k-1} + (1-u_{k-d}\gamma_{k-d})A_{k-1}\hat{x}_{k-1} \quad (12)$$

for $k \in \mathbb{N}_{[d,N]}$ with initial conditions $\hat{x}_\tau = (\prod_{t=1}^{\tau} A_{\tau-t})m_0$ for $\tau \in \mathbb{N}_{[0,d-1]}$, where $\imath_{k-1} = A_{k-1}\,\mathsf{E}[\hat{e}_{k-1}|\mathcal{I}^d_{k-1}, u_{k-d}=0]$ and $\jmath_{k-1} = A_{k-1}\,\mathsf{E}[\hat{e}_{k-1}|\mathcal{I}^d_{k-1}, u_{k-d}=1]$ are signaling residuals. The optimal estimator at the decoder when the channel is without packet-loss detection satisfies the recursive equation

$$\hat{x}_k = u_{k-d}\gamma_{k-d}\left(\prod_{t=1}^{d} A_{k-t}\right)\check{x}_{k-d} + (1-u_{k-d}\gamma_{k-d})(A_{k-1}\hat{x}_{k-1} + \varsigma_{k-1}) \quad (13)$$

for $k \in \mathbb{N}_{[d,N]}$ with initial conditions $\hat{x}_\tau = (\prod_{t=1}^{\tau} A_{\tau-t})m_0$ for $\tau \in \mathbb{N}_{[0,d-1]}$, where $\varsigma_{k-1} = A_{k-1}\,\mathsf{E}[\hat{e}_{k-1}|\mathcal{I}^d_{k-1}, u_{k-d}\gamma_{k-d}=0]$ is a signaling residual.

Remark 4: The results of Lemmas 2 and 3 show that the conditional distribution $\mathsf{P}(x_k|\mathcal{I}^e_k)$ is Gaussian and the optimal estimator at the encoder is linear, while the conditional distribution $\mathsf{P}(x_k|\mathcal{I}^d_k)$ is in general non-Gaussian and the optimal estimator at the decoder is in general nonlinear. The nonlinearity of the optimal estimator at the decoder is in fact caused by the signaling residuals $\imath_k$, $\jmath_k$, and ς_k, which are defined when z_k is equal to $\mathfrak{E}$, $\mathfrak{F}$, and $\mathfrak{D}$, respectively. These variables can be calculated numerically by techniques from nonlinear filtering. The existence of these terms implies that the decoder might be able to decrease its uncertainty even when no message is transmitted or when a packet loss occurs. According to the above results, such an inference can be different when the channel is with and without packet-loss detection. Finally, it is worth mentioning that $\mathcal{I}^d_k \subset \mathcal{I}^e_k$. Therefore, both $\check{x}_k$ and $\hat{x}_k$ can be obtained at the encoder at each time k.

In the following definition, we introduce a value function by adopting the estimation policy $\delta^\star$ that is constructed based on $\mathsf{E}[x_k|\mathcal{I}^d_k]$. Note that this definition takes into account the fact that the scheduling variable u_k can affect the mean square error only from time $k+d$ onward, implying that $\boldsymbol{u}_{N-d+1:N}$ should be zero in the scheduling policy $\epsilon^\star$.

Definition 2 (Value function): The value function $V_k(\mathcal{I}^e_k)$ associated with the loss function $\Phi(\epsilon,\delta)$ and the information set $\mathcal{I}^e_k$ is defined as

$$V_k(\mathcal{I}^e_k) := \min_{\epsilon\in\mathcal{E}:\delta=\delta^\star} \mathsf{E}\left[\sum_{t=k}^{N-d} \theta_t u_t + \hat{e}^T_{t+d}\hat{e}_{t+d}\middle|\mathcal{I}^e_k\right] \quad (14)$$

for $k \in \mathbb{N}_{[0,N]}$.

We now present the primary theorem of this article, which characterizes a globally optimal solution in the causal frequency-distortion tradeoff.

Theorem 1: The causal frequency-distortion tradeoff in (6) pertaining to state estimation of a partially observable Gauss–Markov process modeled by (1) and (2) over a fixed-delay packet-erasure channel modeled by (3) or (4) admits a globally optimal solution $(\epsilon^\star, \delta^\star)$ such that $\epsilon^\star$ is a symmetric threshold scheduling policy given by

$$u_k = \mathbb{1}_{\chi_k(\tilde{e}_k, \boldsymbol{\nu}_{k-d+2:k}, \boldsymbol{\lambda}_{k-d+1:k}, \boldsymbol{u}_{k-d+1:k-1}) - \theta_k \geq 0} \quad (15)$$

for $k \in \mathbb{N}_{[0,N-d]}$, where $\chi_k = \mathsf{E}[\tilde{e}^T_{k+d}\tilde{e}_{k+d} + V_{k+1}(\mathcal{I}^e_{k+1}) |\mathcal{I}^e_k, u_k = 0] - \mathsf{E}[\tilde{e}^T_{k+d}\tilde{e}_{k+d} + V_{k+1}(\mathcal{I}^e_{k+1})|\mathcal{I}^e_k, u_k = 1]$ is a symmetric function of $\boldsymbol{\nu}_{0:k}$ and expressible in terms of $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k}$, $\boldsymbol{\lambda}_{k-d+1:k}$, and $\boldsymbol{u}_{k-d+1:k-1}$; and $\delta^\star$ is a non-Gaussian linear estimation policy given by

$$\hat{x}_k = u_{k-d}\gamma_{k-d}\left(\prod_{t=1}^{d} A_{k-t}\right)\check{x}_{k-d} + (1-u_{k-d}\gamma_{k-d})A_{k-1}\hat{x}_{k-1} \quad (16)$$

for $k \in \mathbb{N}_{[d,N]}$ without being influenced by the signaling residuals $\imath_{k-1}$, $\jmath_{k-1}$, and ς_{k-1}, with initial conditions $\hat{x}_\tau = (\prod_{t=1}^{\tau} A_{\tau-t})m_0$ for $\tau \in \mathbb{N}_{[0,d-1]}$.

Remark 5: The results of Theorem 1 show the existence of a globally optimal solution that is composed of a symmetric threshold scheduling policy and a non-Gaussian linear estimation policy. First, it is important to note that this policy profile is exactly the same whether the channel is with or without packet-loss detection. This in fact certifies that packet-loss detection cannot enhance the performance of the underlying networked system. Second, note that the scheduling policy is expressible in terms of $3d-1$ variables related to the source and the channel (i.e., $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k}$, $\boldsymbol{\lambda}_{k-d+1:k}$, and $\boldsymbol{u}_{k-d+1:k-1}$ at each time k), and that the estimation policy incorporates no signaling residuals (i.e., $\imath_k$, $\jmath_k$, and ς_k become equal to zero for all $k \in \mathbb{N}_{[d-1,N-1]}$). The former implies that the encoder does not require a large memory to save the whole history, and the latter implies that the decoder need not perform complex computations. Third, note that these scheduling and estimation policies can be designed separately, while we have already observed from (12) to (14) that the optimal design of the encoder and the decoder is generally intertwined as ϵ depends on δ and vice versa. Finally, note that, at the characterized policy profile, the benefit in transmitting a message is equal to χ_k and its cost is equal to θ_k. Accordingly, a message is transmitted from the encoder to the decoder at each time only when its benefit surpasses its cost. This event-triggered property of the networked system based on the value of information has directly emerged from our cost-benefit analysis (see [34], [35], [36], [37] for more discussion).

The results of Theorem 1 depend on the value function $V_k(\mathcal{I}^e_k)$ in (14), which should be solved backward in time. We end this section by presenting a corollary, which specializes the results of Theorem 1 for the time horizon $N=1$ and the time delay $d=1$, and thereby provides a closer look into the structure of the globally optimal solution in an analytical way. Note that, in this case, one only needs to determine u_0 for the encoder and $\hat{x}_1$ for the decoder.

Corollary 1: The causal frequency-distortion tradeoff in (6) pertaining to state estimation of a partially observable Gauss–Markov process modeled by (1) and (2) over a fixed-delay packet-erasure channel modeled by (3) or (4) for the time horizon $N=1$ and the time delay $d=1$ admits a globally optimal solution $(\epsilon^\star, \delta^\star)$ such that $\epsilon^\star$ is a symmetric threshold scheduling policy given by

$$u_0 = \mathbb{1}_{\lambda_0 \nu_0^T K_0^T A_0^T A_0 K_0 \nu_0 - \theta_0 \geq 0} \tag{17}$$

and $\delta^\star$ is a non-Gaussian linear estimation policy given by

$$\hat{x}_1 = A_0 m_0 + u_0 \gamma_0 A_0 K_0 \nu_0 \tag{18}$$

where $\nu_0 = y_0 - C_0 m_0$.

Remark 6: The results of Corollary 1 for the time horizon $N = 1$ and the time delay $d = 1$ are particularly intriguing as they offer an analytical depiction of the globally optimal solution and provide insights into its underlying structure. Note that in this case, the scheduling and estimation policies are expressed in terms of ν_0 (or $\tilde{e}_0$) and λ_0. Moreover, note that, for any fixed ν_0 and θ_0, there exists a cutoff value for λ_0 below which u_0 becomes zero. This means that when the channel condition is poor no message should be transmitted over the channel. It is worth mentioning that this adaptiveness of the optimal scheduling policy to the channel condition in our study in fact resembles that of the optimal transmit power policy to the channel condition in [38] and [39]. These works examined a tradeoff between the average transmit power and the fading channel capacity, and aimed to identify the optimal transmit power policy in such a tradeoff. It was concluded in [38] and [39] that no data should be transmitted over the channel when the channel condition is below a cutoff value. Despite these shared findings, we should point out that our study is different, as we consider a tradeoff between the packet rate and the mean square error.

IV. Derivation of Main Results

This section is dedicated to the derivation of the main results. We first present the statements of a few results that are necessary for our subsequent analysis.

The next two lemmas characterize the recursive equations that the estimation mismatch must satisfy.

Lemma 4: The estimation mismatch when the channel is with packet-loss detection satisfies the recursive equation

$$\begin{aligned}\tilde{e}_k &= u_{k-d}\gamma_{k-d} \sum_{t=1}^{d} \left(\prod_{t'=1}^{t-1} A_{k-t'} \right) K_{k-t+1}\nu_{k-t+1} \\ &\quad - u_{k-d}(1-\gamma_{k-d})\jmath_{k-1} - (1-u_{k-d})\imath_{k-1} \\ &\quad + (1-u_{k-d}\gamma_{k-d})(A_{k-1}\tilde{e}_{k-1} + K_k\nu_k) \end{aligned} \tag{19}$$

for $k \in \mathbb{N}_{[d,N]}$ with initial conditions $\tilde{e}_\tau = \sum_{t=1}^{\tau+1}(\prod_{t'=1}^{t-1} A_{\tau-t'})K_{\tau-t+1}\nu_{\tau-t+1}$ for $\tau \in \mathbb{N}_{[0,d-1]}$. The estimation mismatch when the channel is without packet-loss detection satisfies the recursive equation

$$\begin{aligned}\tilde{e}_k &= u_{k-d}\gamma_{k-d} \sum_{t=1}^{d} \left(\prod_{t'=1}^{t-1} A_{k-t'} \right) K_{k-t+1}\nu_{k-t+1} \\ &\quad + (1-u_{k-d}\gamma_{k-d})(A_{k-1}\tilde{e}_{k-1} + K_k\nu_k - \varsigma_{k-1}) \end{aligned} \tag{20}$$

for $k \in \mathbb{N}_{[d,N]}$ with initial conditions $\tilde{e}_\tau = \sum_{t=1}^{\tau+1}(\prod_{t'=1}^{t-1} A_{\tau-t'})K_{\tau-t+1}\nu_{\tau-t+1}$ for $\tau \in \mathbb{N}_{[0,d-1]}$.

Lemma 5: Let $\mathsf{p}(u_k|\boldsymbol{\nu}_{0:k}, \boldsymbol{u}_{0:k-1})$ be a symmetric function of $\boldsymbol{\nu}_{0:k}$ for all $k \in \mathbb{N}_{[0,N]}$. Then, $\imath_k$, $\jmath_k$, and ς_k defined in Lemma 3 are equal to zero for all $k \in \mathbb{N}_{[d-1,N-1]}$, and the estimation mismatch, regardless of the existence of packet-loss detection, satisfies the recursive equation

$$\begin{aligned}\tilde{e}_k &= u_{k-d}\gamma_{k-d} \sum_{t=1}^{d} \left(\prod_{t'=1}^{t-1} A_{k-t'} \right) K_{k-t+1}\nu_{k-t+1} \\ &\quad + (1-u_{k-d}\gamma_{k-d})(A_{k-1}\tilde{e}_{k-1} + K_k\nu_k) \end{aligned} \tag{21}$$

for $k \in \mathbb{N}_{[d,N]}$ with initial conditions $\tilde{e}_\tau = \sum_{t=1}^{\tau+1}(\prod_{t'=1}^{t-1} A_{\tau-t'})K_{\tau-t+1}\nu_{\tau-t+1}$ for $\tau \in \mathbb{N}_{[0,d-1]}$.

The next two technical lemmas are related to symmetric decreasing rearrangements of nonnegative functions (see, e.g., [40] for the formal definition of symmetric decreasing rearrangements). We will use these lemmas later when we introduce a symmetric scheduling policy. We will not present the proofs of these lemmas (see, e.g., [41] and [42] for the proofs).

Lemma 6 (Hardy-Littlewood inequality): Let f and g be nonnegative functions defined on $\mathbb{R}^n$ that vanish at infinity. Then,

$$\int_{\mathbb{R}^n} f(x)g(x)dx \leq \int_{\mathbb{R}^n} f^*(x)g^*(x)dx. \tag{22}$$

Lemma 7: Let $\mathcal{B}(r) \subseteq \mathbb{R}^n$ be a ball of radius r centered at the origin, and f and g be nonnegative functions defined on $\mathbb{R}^n$ that vanish at infinity and satisfy

$$\int_{\mathcal{B}(r)} f^*(x)dx \leq \int_{\mathcal{B}(r)} g^*(x)dx \tag{23}$$

for all $r \geq 0$. Then,

$$\int_{\mathcal{B}(r)} h(x)f^*(x)dx \leq \int_{\mathcal{B}(r)} h(x)g^*(x)dx \tag{24}$$

for all $r \geq 0$ and any nonnegative nonincreasing function h.

The next lemma provides an equivalent loss function in the sense that it yields the same optimal solution.

Lemma 8: Let $\tilde{e}_\tau$ be given according to $\sum_{t=1}^{\tau+1}(\prod_{t'=1}^{t-1} A_{\tau-t'})K_{\tau-t+1}\nu_{\tau-t+1}$ for $\tau \in \mathbb{N}_{[0,d-1]}$, and δ be constructed based on $\mathsf{E}[x_k|\mathcal{I}_k^d]$. Optimizing the loss function $\Phi(\epsilon,\delta)$ over $\epsilon \in \mathcal{E}$ is equivalent to optimizing the following loss function $\Omega_\epsilon^N(\tilde{e}_{d-1})$ over $\epsilon \in \mathcal{E}$:

$$\begin{aligned}&\Omega_\epsilon^N(\tilde{e}_{d-1}) \\ &= \sum_{k=d-1}^{N} \{\theta_{k-d+1}\, \mathsf{p}_\epsilon(\boldsymbol{\sigma}_{0:k-d} = 0) \\ &\quad \times \mathsf{E}_\epsilon\left[u_{k-d+1} \middle| \boldsymbol{\sigma}_{0:k-d} = 0\right] \\ &\quad + \mathsf{p}_\epsilon(\boldsymbol{\sigma}_{0:k-d} = 0)\, \mathsf{E}_\epsilon\left[\tilde{e}_k^T\tilde{e}_k \middle| \boldsymbol{\sigma}_{0:k-d} = 0\right] \\ &\quad + \mathsf{p}_\epsilon(\boldsymbol{\sigma}_{0:k-d} = 0, \sigma_{k-d+1} = 1) \\ &\quad \times \mathsf{E}_\epsilon\left[\Omega_\epsilon^{k+1,N}(\tilde{e}_{k+1}) \middle| \boldsymbol{\sigma}_{0:k-d} = 0, \sigma_{k-d+1} = 1\right]\} \end{aligned} \tag{25}$$

where $\sigma_k = u_k\gamma_k$ and

$$\Omega_\epsilon^{k,N}(\tilde{e}_k) = \sum_{t=k}^{N} \mathsf{E}\left[\theta_{t-d+1}u_{t-d+1} + \tilde{e}_t^T\tilde{e}_t\right]$$

for $k \in \mathbb{N}_{[d,N]}$ when $\tilde{e}_k$ is given according to $\sum_{t=1}^{d}(\prod_{t'=1}^{t-1} A_{k-t'})K_{k-t+1}\nu_{k-t+1}$.

A. Proof of Lemma 1

Proof: Let ϵ^o be a globally optimal scheduling policy that is implemented. Given all the information available at the decoder, $\mathsf{E}[x_k|\mathcal{I}_k^d]$ is the optimal value that minimizes the mean square error at the decoder at time k. Moreover, $\mathsf{E}[x_k|\mathcal{I}_k^e]$ fuses all the previous and current outputs of the sensor that are available at the encoder at time k. This implies that if this message is transmitted, from the minimum mean-square-error perspective, the decoder is able to develop a state estimate upon the successful receipt of the message, say at time k', that would be the same if it had all the previous outputs of the sensor until time $k'-d$, which is the best possible case for the decoder. Hence, both claims hold. ■

B. Proof of Lemma 2

Proof: Given the information set $\mathcal{I}_k^e$, we can easily verify that the optimal estimator at the encoder should satisfy the Kalman filter equations for the conditional mean and the conditional covariance (see, e.g., [43]). ■

C. Proof of Lemma 3

Proof: Writing x_k in terms of x_{k-d} based on (1), and taking expectation given the information set $\mathcal{I}_k^d$, we obtain

$$\mathsf{E}\left[x_k\middle|\mathcal{I}_k^d\right] = \left(\prod_{t=1}^{d} A_{k-t}\right)\mathsf{E}\left[x_{k-d}\middle|\mathcal{I}_k^d\right] \tag{26}$$

for $k\in\mathbb{N}_{[d,N]}$ as $\sum_{t=1}^{d}(\prod_{t'=1}^{t-1}A_{k-t'})\,\mathsf{E}[w_{k-t}|\mathcal{I}_k^d]=0$. When a successful delivery occurs at time k, under either channel model, we have $z_k=\check{x}_{k-d}$. In this case, we get $\mathsf{E}[x_{k-d}|\mathcal{I}_k^d]=\mathsf{E}[x_{k-d}|\mathcal{I}_{k-1}^d, z_k=\check{x}_{k-d},\lambda_k,\hat{x}_{k-1}]=\mathsf{E}[x_{k-d}|\check{x}_{k-d},Q_{k-d}]=\check{x}_{k-d}$ as $\{\check{x}_{k-d},Q_{k-d}\}$ is a sufficient statistic of $\mathcal{I}_k^d$ with respect to x_{k-d}. Hence, using (26), if $u_{k-d}\gamma_{k-d}=1$, we get

$$\mathsf{E}\left[x_k\middle|\mathcal{I}_k^d\right] = \left(\prod_{t=1}^{d} A_{k-t}\right)\check{x}_{k-d} \tag{27}$$

for $k\in\mathbb{N}_{[d,N]}$. Furthermore, writing x_k in terms of x_{k-1} based on (1), and taking expectation given the information set $\mathcal{I}_k^d$, we obtain

$$\mathsf{E}\left[x_k\middle|\mathcal{I}_k^d\right] = A_{k-1}\,\mathsf{E}\left[x_{k-1}\middle|\mathcal{I}_k^d\right] \tag{28}$$

for $k\in\mathbb{N}_{[1,N]}$ as $\mathsf{E}[w_{k-1}|\mathcal{I}_k^d]=0$. When the channel is with packet-loss detection and no delivery occurs at time k, we have $z_k=\mathfrak{E}$ or $z_k=\mathfrak{F}$. Note that $z_k=\mathfrak{E}$ and $z_k=\mathfrak{F}$ are equivalent to $u_{k-d}=0$ and $\{u_{k-d}=1,\gamma_{k-d}=0\}$, respectively. Define $p_{k-1}:=\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^d,u_{k-d}=0]-\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^d]$ when $z_k=\mathfrak{E}$, and $q_{k-1}:=\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^d,u_{k-d}=1]-\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^d]$ when $z_k=\mathfrak{F}$. Then, using (28), if $u_{k-d}=0$, we get

$$\mathsf{E}\left[x_k\middle|\mathcal{I}_k^d\right] = A_{k-1}\,\mathsf{E}\left[x_{k-1}\middle|\mathcal{I}_{k-1}^d\right] + A_{k-1}p_{k-1} \tag{29}$$

for $k\in\mathbb{N}_{[d,N]}$, where we used $\mathcal{I}_k^d=\{\mathcal{I}_{k-1}^d,z_k=\mathfrak{E},\lambda_k,\hat{x}_{k-1}\}$ and the fact that x_{k-1} is independent of λ_k; and if $u_{k-d}=1$ and $\gamma_{k-d}=0$, we get

$$\mathsf{E}\left[x_k\middle|\mathcal{I}_k^d\right] = A_{k-1}\,\mathsf{E}\left[x_{k-1}\middle|\mathcal{I}_{k-1}^d\right] + A_{k-1}q_{k-1} \tag{30}$$

for $k\in\mathbb{N}_{[d,N]}$, where we used $\mathcal{I}_k^d=\{\mathcal{I}_{k-1}^d,z_k=\mathfrak{F},\lambda_k,\hat{x}_{k-1}\}$ and the fact that x_{k-1} is independent of γ_{k-d} and λ_k. However, when the channel is without packet-loss detection and no delivery occurs at time k, we have $z_k=\mathfrak{D}$. Note that $z_k=\mathfrak{D}$ is equivalent to $u_{k-d}\gamma_{k-d}=0$. Define $r_{k-1}:=\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^d,u_{k-d}\gamma_{k-d}=0]-\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^d]$ when $z_k=\mathfrak{D}$. Then, using (28), if $u_{k-d}\gamma_{k-d}=0$, we get

$$\mathsf{E}\left[x_k\middle|\mathcal{I}_k^d\right] = A_{k-1}\,\mathsf{E}\left[x_{k-d}\middle|\mathcal{I}_{k-d}^d\right] + A_{k-1}r_{k-1} \tag{31}$$

for $k\in\mathbb{N}_{[d,N]}$, where we used $\mathcal{I}_k^d=\{\mathcal{I}_{k-1}^d,z_k=\mathfrak{D},\lambda_k,\hat{x}_{k-1}\}$ and the fact that x_{k-1} is independent of λ_k. Now, define $\imath_{k-1}:=A_{k-1}p_{k-1}$, $\jmath_{k-1}:=A_{k-1}q_{k-1}$, and $\varsigma_{k-1}:=A_{k-1}r_{k-1}$. We obtain the recursive equation (12) by combining (27), (29), and (30), and the recursive equation (13) by combining (27) and (31). Finally, since nothing is transmitted in the time interval $\tau\in\mathbb{N}_{[0,d-1]}$, the initial conditions are obtained by writing x_τ in terms of x_0 and taking expectation, i.e., $\mathsf{E}[x_\tau]=(\prod_{t=1}^{\tau}A_{\tau-t})\,\mathsf{E}[x_0]$ for $\tau\in\mathbb{N}_{[0,d-1]}$. ■

D. Proof of Lemma 4

Proof: From the definition of the innovation ν_k and the variable m_k, we obtain

$$\nu_k = y_k - C_k m_k \tag{32}$$

for $k\in\mathbb{N}_{[1,N]}$. Note that ν_k is a white Gaussian noise with zero mean and covariance $N_k=C_kM_kC_k^T+V_k$. Using (8), (9), and (32), we can write $\check{x}_k$ in terms of $\check{x}_{k-d}$ as

$$\check{x}_k = \left(\prod_{t=1}^{d} A_{k-t}\right)\check{x}_{k-d} + \sum_{t=1}^{d}\left(\prod_{t'=1}^{t-1} A_{k-t'}\right)K_{k-t+1}\nu_{k-t+1} \tag{33}$$

for $k\in\mathbb{N}_{[d,N]}$ with initial conditions $\check{x}_\tau=(\prod_{t=1}^{\tau}A_{\tau-t})m_0+\sum_{t=1}^{\tau+1}(\prod_{t'=1}^{t-1}A_{\tau-t'})K_{\tau-t+1}\nu_{\tau-t+1}$ for $\tau\in\mathbb{N}_{[0,d-1]}$, and in terms of $\check{x}_{k-1}$ as

$$\check{x}_k = A_{k-1}\check{x}_{k-1} + K_k\nu_k \tag{34}$$

for $k\in\mathbb{N}_{[1,N]}$ with initial condition $\check{x}_0=m_0+K_0\nu_0$. Since (33) and (34) are equivalent and $u_{k-d}\gamma_{k-d}+(1-u_{k-d}\gamma_{k-d})=1$, we can also write

$$\begin{aligned}\check{x}_k &= u_{k-d}\gamma_{k-d}\left(\prod_{t=1}^{d} A_{k-t}\right)\check{x}_{k-d}\\ &\quad + u_{k-d}\gamma_{k-d}\sum_{t=1}^{d}\left(\prod_{t'=1}^{t-1} A_{k-t'}\right)K_{k-t+1}\nu_{k-t+1}\\ &\quad + (1-u_{k-d}\gamma_{k-d})(A_{k-1}\check{x}_{k-1}+K_k\nu_k)\end{aligned} \tag{35}$$

for $k \in \mathbb{N}_{[d,N]}$ with initial conditions $\check{x}_\tau = (\prod_{t=1}^{\tau} A_{\tau-t})m_0 + \sum_{t=1}^{\tau+1}(\prod_{t'=1}^{t-1} A_{\tau-t'})K_{\tau-t+1}\nu_{\tau-t+1}$ for $\tau \in \mathbb{N}_{[0,d-1]}$. By definition, $\tilde{e}_k = \check{x}_k - \hat{x}_k$. Accordingly, we find (19) and (20) by subtracting (12) and (13) from (35), respectively. The initial conditions $\tilde{e}_\tau$ for $\tau \in \mathbb{N}_{[0,d-1]}$ are also obtained by subtracting $\hat{x}_\tau$ associated with (12) or (13) from $\check{x}_\tau$ associated with (35). ∎

E. Proof of Lemma 5

Proof: Observe that the initial conditions $\tilde{e}_\tau$ for $\tau \in \mathbb{N}_{[0,d-1]}$ are independent of the scheduling policy. We assume that $\imath_t = \jmath_t = \varsigma_t = 0$ for all $t \in \mathbb{N}_{[d-1,k-1]}$, and will show that $\imath_k = \jmath_k = \varsigma_k = 0$. We can write

$$\mathsf{p}\left(\boldsymbol{u}_{0:t} \middle| \boldsymbol{\nu}_{0:t}\right) = \prod_{t'=0}^{t} \mathsf{p}\left(u_{t'} \middle| \boldsymbol{\nu}_{0:t'}, \boldsymbol{u}_{0:t'-1}\right)$$

where we used the fact that $u_{t'}$ is independent of $\boldsymbol{\nu}_{t'+1:t}$ given $\boldsymbol{\nu}_{0:t'}$ and $\boldsymbol{u}_{0:t'-1}$. Therefore, by the hypothesis, we can deduce that $\mathsf{p}(\boldsymbol{u}_{0:t}|\boldsymbol{\nu}_{0:t})$ is symmetric with respect to $\boldsymbol{\nu}_{0:t}$. It follows that

$$\mathsf{p}\left(\boldsymbol{\nu}_{0:t} \middle| \boldsymbol{u}_{0:t}\right) \propto \mathsf{p}\left(\boldsymbol{u}_{0:t} \middle| \boldsymbol{\nu}_{0:t}\right) \mathsf{p}\left(\boldsymbol{\nu}_{0:t}\right).$$

Since $\mathsf{p}(\boldsymbol{\nu}_{0:t})$ is a symmetric distribution, we deduce that $\mathsf{p}(\boldsymbol{\nu}_{0:t}|\boldsymbol{u}_{0:t})$ is also symmetric with respect to $\boldsymbol{\nu}_{0:t}$. Moreover, for any $s \in \mathbb{N}_{[0,t]}$, by marginalization and the fact that $\boldsymbol{\nu}_{s:t}$ is independent of $\boldsymbol{u}_{0:s-1}$, we can find that $\mathsf{p}(\boldsymbol{\nu}_{s:t}|\boldsymbol{u}_{s:t})$ is symmetric with respect to $\boldsymbol{\nu}_{s:t}$. Furthermore, for any $k \in \mathbb{N}_{[t+1,N]}$, we have

$$\mathsf{p}\left(\boldsymbol{\nu}_{s:k} \middle| \boldsymbol{u}_{s:t}\right) = \mathsf{p}\left(\boldsymbol{\nu}_{s:t} \middle| \boldsymbol{u}_{s:t}\right) \mathsf{p}\left(\boldsymbol{\nu}_{t+1:k}\right).$$

Therefore, we can see that $\mathsf{p}(\boldsymbol{\nu}_{s:k}|\boldsymbol{u}_{s:t})$ is also symmetric with respect to $\boldsymbol{\nu}_{s:k}$.

Note that $z_{k+1} = \mathfrak{E}$, $z_{k+1} = \mathfrak{F}$, and $z_{k+1} = \mathfrak{D}$ are equivalent to $u_{k-d+1} = 0$, $\{u_{k-d+1} = 1, \gamma_{k-d+1} = 0\}$, and $u_{k-d+1}\gamma_{k-d+1} = 0$, respectively. Accordingly, when $z_{k+1} = \mathfrak{E}$, $z_{k+1} = \mathfrak{F}$, or $z_{k+1} = \mathfrak{D}$, we can write

$$\begin{aligned}
&\mathsf{E}\left[\hat{e}_k \middle| \mathcal{I}_k^d, z_{k+1}\right] \\
&= \mathsf{E}\left[\mathsf{E}\left[\hat{e}_k \middle| \mathcal{I}_k^e, z_{k+1}\right] \middle| \mathcal{I}_k^d, z_{k+1}\right] \\
&= \mathsf{E}\left[\mathsf{E}\left[\hat{e}_k \middle| \mathcal{I}_k^e\right] \middle| \mathcal{I}_k^d, z_{k+1}\right] \\
&= \mathsf{E}\left[\tilde{e}_k \middle| \mathcal{I}_k^d, z_{k+1}\right]
\end{aligned} \tag{36}$$

where in the first equality we used the tower property of conditional expectations, and in the second equality the facts that $\hat{e}_k$ is independent of γ_{k-d+1}, and that $u_{k-d+1} \in \mathcal{I}_k^e$. In addition, from (19) and (20), when $\imath_t = \jmath_t = \varsigma_t = 0$ for all $t \in \mathbb{N}_{[d-1,k-1]}$, we can write

$$\begin{aligned}
\tilde{e}_k &= \sum_{t=1}^{\eta_k+d} \left(\prod_{t'=1}^{t-1} A_{k-t'}\right) K_{k-t+1}\nu_{k-t+1} \\
&= D_k \boldsymbol{\nu}_{k-\eta_k-d+1:k}
\end{aligned} \tag{37}$$

for $k \in \mathbb{N}_{[0,N]}$, where $\eta_k \in \mathbb{N}_{[0,k]}$ denotes the time elapsed since the last successful delivery when we are at time k with the convention $\eta_k = k$ if no successful delivery has occurred, and D_k is a matrix of proper dimension.

Clearly, at time k, the decoder knows exactly when the last successful delivery occurred, i.e., it knows η_k. Moreover, from the definition of η_k, when we are at time k, no successful delivery occurred from $k - \eta_k + 1$ to k.

Let us define $a_t, b_t \in \{0, 1\}$ such that $a_t b_t = 0$ for $t \in \mathbb{N}_{[0,N]}$. In the case of the channel with packet-loss detection, by (36), (37), and the definitions of $\imath_k$ and $\jmath_k$, we obtain

$$\begin{aligned}
\imath_k &= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \mathcal{I}_k^d, z_{k+1} = \mathfrak{E}\right] \\
&= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \mathcal{I}_{k+1}^1\right] \\
\jmath_k &= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \mathcal{I}_k^d, z_{k+1} = \mathfrak{F}\right] \\
&= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \mathcal{I}_{k+1}^2\right]
\end{aligned}$$

where $\mathcal{I}_{k+1}^1 = \{\mathcal{I}_{k-\eta_k}^d,\ \boldsymbol{u}_{k-\eta_k-d+1:k-d} = \boldsymbol{a}_{k-\eta_k-d+1:k-d}, u_{k-d+1} = 0,\ \boldsymbol{\gamma}_{k-\eta_k-d+1:k-d} = \boldsymbol{b}_{k-\eta_k-d+1:k-d}\}$ and $\mathcal{I}_{k+1}^2 = \{\mathcal{I}_{k-\eta_k}^d,\ \boldsymbol{u}_{k-\eta_k-d+1:k-d} = \boldsymbol{a}_{k-\eta_k-d+1:k-d},\ u_{k-d+1} = 1, \boldsymbol{\gamma}_{k-\eta_k-d+1:k-d} = \boldsymbol{b}_{k-\eta_k-d+1:k-d}\}$, and we used the fact that $\boldsymbol{\nu}_{k-\eta_k-d+1:k}$ is independent of γ_{k-d+1}. Equivalently, we can write

$$\begin{aligned}
\imath_k &= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \bar{\mathcal{I}}_{k+1}^1\right] \\
\jmath_k &= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \bar{\mathcal{I}}_{k+1}^2\right]
\end{aligned}$$

where $\bar{\mathcal{I}}_{k+1}^1 = \{\boldsymbol{u}_{k-\eta_k-d+1:k-d} = \boldsymbol{a}_{k-\eta_k-d+1:k-d}, u_{k-d+1} = 0\}$ and $\bar{\mathcal{I}}_{k+1}^2 = \{\boldsymbol{u}_{k-\eta_k-d+1:k-d} = \boldsymbol{a}_{k-\eta_k-d+1:k-d}, u_{k-d+1} = 1\}$, and we used the fact that $\boldsymbol{\nu}_{k-\eta_k-d+1:k}$ is independent of $\mathcal{I}_{k-\eta_k}^d$ and $\boldsymbol{\gamma}_{k-\eta_k-d+1:k-d}$. We have already shown that $\mathsf{p}(\boldsymbol{\nu}_{k-\eta_k-d+1:k}|\boldsymbol{u}_{k-\eta_k-d+1:k-d+1})$ is symmetric with respect to $\boldsymbol{\nu}_{k-\eta_k-d+1:k}$. This implies that $\imath_k = 0$ and $\jmath_k = 0$, and that (19) yields (21).

Let us define $\sigma_t := u_t\gamma_t$ for $t \in \mathbb{N}_{[0,N]}$. In the case of the channel without packet-loss detection, by (36), (37), and the definition of ς_k, we obtain

$$\begin{aligned}
\varsigma_k &= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \mathcal{I}_k^d, z_{k+1} = \mathfrak{D}\right] \\
&= A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \mathcal{I}_{k+1}^3\right]
\end{aligned}$$

where $\mathcal{I}_{k+1}^3 = \{\mathcal{I}_{k-\eta_k}^d, \boldsymbol{\sigma}_{k-\eta_k-d+1:k-d+1} = 0\}$. Equivalently, we can write

$$\varsigma_k = A_k D_k \,\mathsf{E}\left[\boldsymbol{\nu}_{k-\eta_k-d+1:k} \middle| \bar{\mathcal{I}}_{k+1}^3\right]$$

where $\bar{\mathcal{I}}_{k+1}^3 = \{\boldsymbol{\sigma}_{k-\eta_k-d+1:k-d+1} = 0\}$, and we used the fact that $\boldsymbol{\nu}_{k-\eta_k-d+1:k}$ is independent of $\mathcal{I}_{k-\eta_k}^d$. It is easy to see that $\mathsf{p}(\boldsymbol{\nu}_{k-\eta_k-d+1:k}|\boldsymbol{\sigma}_{k-\eta_k-d+1:k-d+1} = 0)$ can be expressed as a linear combination of $\mathsf{p}(\boldsymbol{\nu}_{k-\eta_k-d+1:k}|\boldsymbol{u}_{k-\eta_k-d+1:k-d+1})$ for different values of $\boldsymbol{u}_{k-\eta_k-d+1:k-d+1}$ as $\boldsymbol{\nu}_{k-\eta_k-d+1:k}$ is independent of $\boldsymbol{\gamma}_{k-\eta_k-d+1:k-d+1}$. Again, we have already shown that $\mathsf{p}(\boldsymbol{\nu}_{k-\eta_k-d+1:k}|\boldsymbol{u}_{k-\eta_k-d+1:k-d+1})$ is symmetric with respect to $\boldsymbol{\nu}_{k-\eta_k-d+1:k}$. This implies that $\varsigma_k = 0$, and that (20) yields (21). ∎

F. Proof of Lemma 8

Proof: From the definition of the loss function $\Phi(\epsilon,\delta)$, we can write

$$\begin{aligned}\Phi(\epsilon,\delta) &= \sum_{k=0}^{N} \mathsf{E}\left[\theta_k u_k + \hat{e}_k^T \hat{e}_k\right]\\ &= \sum_{k=0}^{N} \mathsf{E}\left[\mathsf{E}\left[\theta_k u_k + \hat{e}_k^T \hat{e}_k \middle| \mathcal{I}_k^e\right]\right]\\ &= \sum_{k=0}^{N} \mathsf{E}\left[\theta_k u_k + \tilde{e}_k^T \tilde{e}_k + \mathrm{tr} Q_k\right] \end{aligned} \tag{38}$$

where in the second equality we used the tower property of conditional expectations. Observe that, by Lemma 4, regardless of the existence of packet-loss detection, $\tilde{e}_{k+1}$ satisfies

$$\tilde{e}_{k+1} = \sum_{t=1}^{d}\left(\prod_{t'=1}^{t-1} A_{k-t'+1}\right) K_{k-t+2}\nu_{k-t+2} \tag{39}$$

only when $\sigma_{k-d+1}=1$. We show in two steps that, without loss of optimality, $\boldsymbol{u}_{k-d+2:N}$ and $\tilde{\boldsymbol{e}}_{k+1:N}$ are independent of $\boldsymbol{\sigma}_{0:k-d}$ when $\tilde{e}_{k+1}$ is given according to (39), i.e., when $\sigma_{k-d+1}=1$. *Step (I)*. Note that, by Lemma 4, regardless of the existence of packet-loss detection, we can deduce that $\tilde{e}_t$ for $t\in\mathbb{N}_{[k+1,N]}$ are independent of $\boldsymbol{u}_{0:k-d}$ and $\gamma_{0:k-d}$ when $\sigma_{k-d+1}=1$ and $\boldsymbol{u}_{k-d+2:N-d+1}$ is fixed. This clearly means that $\tilde{\boldsymbol{e}}_{k+1:N}$ is independent of $\boldsymbol{\sigma}_{0:k-d}$ when $\sigma_{k-d+1}=1$ and $\boldsymbol{u}_{k-d+2:N-d+1}$ is fixed. *Step (II)*. Moreover, by Lemma 4, we can deduce that u_t for $t\in\mathbb{N}_{[k-d+2,N-d+1]}$ only affect $\tilde{e}_t$ for $t\in\mathbb{N}_{[k+2,N]}$. Hence, given the loss function in (38) and from the fact that $\mathrm{tr}Q_t$ for $t\in\mathbb{N}_{[0,N]}$ are independent of the scheduling policy, we can write $\boldsymbol{u}^\star_{k-d+2:N-d+1} = \mathrm{argmin}_{\boldsymbol{u}_{k-d+2:N-d+1}} \mathsf{E}[\sum_{t=k-d+2}^{N-d+1}\theta_t u_t + \sum_{t=k+2}^{N}\tilde{e}_t^T\tilde{e}_t]$. Following what we obtained in Step (I), this implies that $\boldsymbol{u}^\star_{k-d+2:N-d+1}$ and the corresponding $\tilde{\boldsymbol{e}}_{k+1:N}$ are independent of $\boldsymbol{\sigma}_{0:k-d}$ when $\sigma_{k-d+1}=1$.

Consequently, associated with the optimal scheduling policy, the following relation is satisfied:

$$\begin{aligned}&\mathsf{E}\left[\theta_{t-d+1}u_{t-d+1} + \tilde{e}_t^T\tilde{e}_t \middle| \sigma_{k-d+1}=1\right]\\ &\quad= \mathsf{E}\left[\theta_{t-d+1}u_{t-d+1} + \tilde{e}_t^T\tilde{e}_t \middle| \boldsymbol{\sigma}_{0:k-d}, \sigma_{k-d+1}=1\right]\end{aligned} \tag{40}$$

for any $t\in\mathbb{N}_{[k+1,N]}$, where we used the fact that, without loss of optimality, $\boldsymbol{u}_{k-d+2:N-d+1}$ and $\tilde{\boldsymbol{e}}_{k+1:N}$ are independent of $\boldsymbol{\sigma}_{0:k-d}$ given $\sigma_{k-d+1}=1$.

Now, define the loss function $\Omega_\epsilon^N(\tilde{e}_{d-1})$ as

$$\Omega_\epsilon^N(\tilde{e}_{d-1}) := \sum_{k=d-1}^{N} \mathsf{E}\left[\theta_{k-d+1}u_{k-d+1} + \tilde{e}_k^T\tilde{e}_k\right]. \tag{41}$$

Following the facts that $\tilde{e}_\tau$ for $\tau\in\mathbb{N}_{[0,d-2]}$ and $\mathrm{tr}Q_t$ for $t\in\mathbb{N}_{[0,N]}$ are independent of the scheduling policy, and that u_t for $t\in\mathbb{N}_{[N-d+2,N]}$ have no effects on $\tilde{e}_t$ for $t\in\mathbb{N}_{[0,N]}$, to optimize $\Phi(\epsilon,\delta)$ in (38) over $\epsilon\in\mathcal{E}$, it suffices to optimize $\Omega_\epsilon^N(\tilde{e}_{d-1})$ in (41) over $\epsilon\in\mathcal{E}$. In addition, observe that from the law of total probability, we have

$$\begin{aligned}&\mathsf{p}_\epsilon(\sigma_0=1) + \mathsf{p}_\epsilon(\boldsymbol{\sigma}_{0:t}=0)\\ &\qquad + \sum_{t'=1}^{t}\mathsf{p}_\epsilon(\boldsymbol{\sigma}_{0:t'-1}=0, \sigma_{t'}=1) = 1\end{aligned} \tag{42}$$

for any $t\in\mathbb{N}_{[0,N-d+1]}$. Applying the law of total expectation for the terms $\mathsf{E}[\theta_{k-d+1}u_{k-d+1}]$ and $\mathsf{E}[\tilde{e}_k^T\tilde{e}_k]$ in $\Omega_\epsilon^N(\tilde{e}_{d-1})$ on a partition provided by the identity (42) for $t=k-1$, repeating this procedure for $k\in\mathbb{N}_{[d,N]}$, and using the definition of $\Omega_\epsilon^{k+1,N}(\tilde{e}_{k+1})$ along with the relation in (40), we obtain (25). This completes the proof. ■

G. Proof of Theorem 1

Proof: Let (ϵ^o,δ^o) denote a policy profile in the set of globally optimal solutions. We prove global optimality of the policy profile $(\epsilon^\star,\delta^\star)$ in the claim by showing that $\Phi(\epsilon^\star,\delta^\star)$ cannot be greater than $\Phi^o := \Phi(\epsilon^o,\delta^o)$. Succinctly, our proof is structured as follows:

$$\Phi(\epsilon^o,\delta^o) = \Phi(\epsilon^n,\delta^o) \geq \Phi(\epsilon^s,\delta^o) \geq \Phi(\epsilon^\star,\delta^\star).$$

In particular, in Step (I), we will find an equivalent innovation-based scheduling policy ϵ^n such that $\Phi(\epsilon^n,\delta^o)=\Phi(\epsilon^o,\delta^o)$. Then, in Step (II), we will derive a symmetric scheduling policy ϵ^s such that $\Phi(\epsilon^s,\delta^o)\leq\Phi(\epsilon^n,\delta^o)$. Finally, in Step (III), we will show that for the policy profile in the claim we have $\Phi(\epsilon^\star,\delta^\star)\leq\Phi(\epsilon^s,\delta^o)$. Without loss of generality, we assume that $m_0=0$. Similar arguments can be made for $m_0\neq 0$ following a coordinate transformation.

Step (I): We say a scheduling policy is innovation-based if, at each time k, it depends on $\boldsymbol{\nu}_{0:k}$ instead of $\boldsymbol{y}_{0:k}$ and $\boldsymbol{z}_{0:k}$, i.e., if it can be expressed in the form $\mathsf{p}(u_k|\boldsymbol{\nu}_{0:k},\boldsymbol{\lambda}_{0:k},\boldsymbol{u}_{0:k-1},\boldsymbol{\gamma}_{0:k-d})$. In the first step, we find an equivalent innovation-based scheduling policy ϵ^n such that $\Phi(\epsilon^n,\delta^o)=\Phi(\epsilon^o,\delta^o)$. From (32) and (34), given the fact that $m_0=0$, we obtain $\boldsymbol{y}_{0:k}=\boldsymbol{\nu}_{0:k}+G_{k-1}\check{\boldsymbol{x}}_{0:k-1}$ and $\check{\boldsymbol{x}}_{0:k}=H_k\boldsymbol{\nu}_{0:k}$, respectively, where G_k and H_k are matrixes of proper dimensions. Hence, we have $\boldsymbol{y}_{0:k}=\boldsymbol{\nu}_{0:k}+G_{k-1}H_{k-1}\boldsymbol{\nu}_{0:k-1}$. Moreover, from (3) and (4), we see that $\boldsymbol{z}_{0:k}$ depends on $\check{\boldsymbol{x}}_{0:k-d}$, $\boldsymbol{u}_{0:k-d}$, and $\boldsymbol{\gamma}_{0:k-d}$. Therefore, the stochastic kernel $\mathsf{p}_{\epsilon^o}(u_k|\boldsymbol{y}_{0:k},\boldsymbol{z}_{0:k},\boldsymbol{\lambda}_{0:k},\boldsymbol{u}_{0:k-1},\hat{\boldsymbol{x}}_{0:k-1},\boldsymbol{\gamma}_{0:k-d})$ can equivalently be written as $\mathsf{p}_{\epsilon^n}(u_k|\boldsymbol{\nu}_{0:k},\boldsymbol{\lambda}_{0:k},\boldsymbol{u}_{0:k-1},\boldsymbol{\gamma}_{0:k-d})$. This establishes that $\Phi(\epsilon^n,\delta^o)=\Phi(\epsilon^o,\delta^o)$. As our subsequent analysis is valid for any value of $\boldsymbol{\lambda}_{0:k}$, for brevity, hereafter we omit the dependency of ϵ^n on this variable.

Step (II): Let $\mathcal{B}(r)$ be a ball of radius r centered at the origin and of proper dimension. Define $\varpi_k := T_k\boldsymbol{\nu}_{0:k}$ for a given matrix T_k, which will be introduced in Step (II.C), $\sigma_k := u_k\gamma_k$, $l_{\epsilon^n}(\varpi_k) := (\lambda'_k + \lambda_k\,\mathsf{p}_{\epsilon^n}(u_k = 0|\varpi_k,\boldsymbol{\sigma}_{0:k-1}=0))\,\mathsf{p}_{\epsilon^n}(\varpi_k|\boldsymbol{\sigma}_{0:k-1}=0)$, and $l_{\epsilon^s}(\varpi_k) := (\lambda'_k + \lambda_k\,\mathsf{p}_{\epsilon^s}(u_k = 0|\varpi_k,\boldsymbol{\sigma}_{0:k-1}=0))\,\mathsf{p}_{\epsilon^s}(\varpi_k|\boldsymbol{\sigma}_{0:k-1}=0)$. In the second step, we prove that $\Phi(\epsilon^s,\delta^o)\leq\Phi(\epsilon^n,\delta^o)$, where ϵ^s is a special form of ϵ^n that is symmetric with respect to $\boldsymbol{\nu}_{0:k}$ at each time k and such that

$$\int_{\mathcal{B}(r)} l_{\epsilon^s}(\varpi_k) d\varpi_k = \int_{\mathcal{B}(r)} l^*_{\epsilon^n}(\varpi_k) d\varpi_k \quad (43)$$

for all $r \geq 0$ with $l_{\epsilon^s}(\varpi_k)$ as a radially symmetric function of ϖ_k. Note that $\sigma_k = 0$ either when $u_k = 1$ and $\gamma_k = 0$ or when $u_k = 0$ regardless of the value of γ_k. Accordingly, we can write

$$\begin{aligned}
&\mathsf{p}_{\epsilon^n}\left(\sigma_k = 0 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0\right)\\
&= \mathsf{p}_{\epsilon^n}\left(u_k = 1 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0\right)\\
&\quad \times \mathsf{p}_{\epsilon^n}\left(u_k\gamma_k = 0 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0, u_k = 1\right)\\
&\quad + \mathsf{p}_{\epsilon^n}\left(u_k = 0 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0\right)\\
&\quad \times \mathsf{p}_{\epsilon^n}\left(u_k\gamma_k = 0 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0, u_k = 0\right)\\
&= \lambda'_k\, \mathsf{p}_{\epsilon^n}\left(u_k = 1 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0\right)\\
&\quad + \mathsf{p}_{\epsilon^n}\left(u_k = 0 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0\right)\\
&= \lambda'_k + \lambda_k\, \mathsf{p}_{\epsilon^n}\left(u_k = 0 \middle| \varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0\right) \quad (44)
\end{aligned}$$

where in the first equality we used the law of total probability and in the second equality the fact that γ_k is independent of ϖ_k, $\boldsymbol{\sigma}_{0:k-1}$, and u_k. Define $l'_{\epsilon^n}(\varpi_k) := \mathsf{p}_{\epsilon^n}(\sigma_k = 0|\varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0)\, \mathsf{p}_{\epsilon^n}(\varpi_k|\boldsymbol{\sigma}_{0:k-1} = 0)$ and $l'_{\epsilon^s}(\varpi_k) := \mathsf{p}_{\epsilon^s}(\sigma_k = 0|\varpi_k, \boldsymbol{\sigma}_{0:k-1} = 0)\, \mathsf{p}_{\epsilon^s}(\varpi_k|\boldsymbol{\sigma}_{0:k-1} = 0)$. Hence, using the identity in (44), we can rewrite (43) as

$$\int_{\mathcal{B}(r)} l'_{\epsilon^s}(\varpi_k) d\varpi_k = \int_{\mathcal{B}(r)} l'^*_{\epsilon^n}(\varpi_k) d\varpi_k \quad (45)$$

for all $r \geq 0$ with $l'_{\epsilon^s}(\varpi_k)$ as a radially symmetric function of ϖ_k. By Lemma 8, to prove that $\Phi(\epsilon^s, \delta^o) \leq \Phi(\epsilon^n, \delta^o)$, it suffices to prove that $\Omega^M_{\epsilon^s}(\tilde{e}_{d-1}) \leq \Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ for any $M \in \mathbb{N}_{[d-1,N]}$ and for any $\tilde{e}_{d-1}$. Note that $\tilde{e}_{d-1}$ is the same under both ϵ^n and ϵ^s, and that u_0 has no effects on state estimation when the time horizon is $d-1$. Hence, the claim holds for the time horizon $d-1$. We assume that the claim also holds for all time horizons from d to $M-1$, and will show in the rest of this step, i.e., in four substeps, that the terms in $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ are not less that those in $\Omega^M_{\epsilon^s}(\tilde{e}_{d-1})$.

Step (II.A): In this substep, we show that the probability coefficients in $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ are equal to those in $\Omega^M_{\epsilon^s}(\tilde{e}_{d-1})$. More specifically, we have

$$\begin{aligned}
&\mathsf{p}_{\epsilon^n}(\sigma_{k-d} = 0|\boldsymbol{\sigma}_{0:k-d-1} = 0)\\
&= \int_{\mathbb{R}^m} \mathsf{p}_{\epsilon^n}(\sigma_{k-d} = 0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1} = 0)\\
&\quad \times \mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d-1} = 0) d\varpi_{k-d}\\
&= \int_{\mathbb{R}^m} \mathsf{p}_{\epsilon^s}(\sigma_{k-d} = 0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1} = 0)\\
&\quad \times \mathsf{p}_{\epsilon^s}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d-1} = 0) d\varpi_{k-d}\\
&= \mathsf{p}_{\epsilon^s}(\sigma_{k-d} = 0|\boldsymbol{\sigma}_{0:k-d-1} = 0)
\end{aligned}$$

where the second equality comes from (45) when $r \to \infty$, following the fact that $\int_{\mathbb{R}^m} l'^*_{\epsilon^n}(\varpi_k) d\varpi_k = \int_{\mathbb{R}^m} l'_{\epsilon^n}(\varpi_k) d\varpi_k$. This also implies that $\mathsf{p}_{\epsilon^n}(\boldsymbol{\sigma}_{0:k-d} = 0) = \mathsf{p}_{\epsilon^s}(\boldsymbol{\sigma}_{0:k-d} = 0)$ and $\mathsf{p}_{\epsilon^n}(\boldsymbol{\sigma}_{0:k-d} = 0, \sigma_{k-d+1} = 1) = \mathsf{p}_{\epsilon^s}(\boldsymbol{\sigma}_{0:k-d} = 0, \sigma_{k-d+1} = 1)$.

Step (II.B): In this substep, we show that the terms including the expected value of the scheduling variable in $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ are equal to those in $\Omega^M_{\epsilon^s}(\tilde{e}_{d-1})$. More specifically, we have

$$\begin{aligned}
&\mathsf{E}_{\epsilon^n}\left[u_{k-d+1} \middle| \boldsymbol{\sigma}_{0:k-d} = 0\right]\\
&= 1 - \mathsf{p}_{\epsilon^n}(u_{k-d+1} = 0|\boldsymbol{\sigma}_{0:k-d} = 0)\\
&= \frac{1}{\lambda_{k-d+1}} - \frac{1}{\lambda_{k-d+1}} \mathsf{p}_{\epsilon^n}(\sigma_{k-d+1} = 0|\boldsymbol{\sigma}_{0:k-d} = 0)\\
&= \frac{1}{\lambda_{k-d+1}} - \frac{1}{\lambda_{k-d+1}} \mathsf{p}_{\epsilon^s}(\sigma_{k-d+1} = 0|\boldsymbol{\sigma}_{0:k-d} = 0)\\
&= 1 - \mathsf{p}_{\epsilon^s}(u_{k-d+1} = 0|\boldsymbol{\sigma}_{0:k-d} = 0)\\
&= \mathsf{E}_{\epsilon^s}\left[u_{k-d+1} \middle| \boldsymbol{\sigma}_{0:k-d} = 0\right]
\end{aligned}$$

where in the second and the fourth equalities we used an identity similar to what we derived in (44).

Step (II.C): In this substep, we show that the terms including the expected value of the quadratic estimation mismatch in $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ are greater than or equal to those in $\Omega^M_{\epsilon^s}(\tilde{e}_{d-1})$. Note that, by Lemma 4, when the channel is with packet-loss detection and $\boldsymbol{\sigma}_{0:k-d} = 0$, $\tilde{e}_t$ satisfies the recursive equation

$$\tilde{e}_t = A_{t-1}\tilde{e}_{t-1} + K_t\nu_t - (1 - u_{t-d})\imath_{t-1} - u_{t-d}\jmath_{t-1}$$

for $t \in \mathbb{N}_{[d,k]}$. However, when the channel is without packet-loss detection and $\boldsymbol{\sigma}_{0:k-d} = 0$, $\tilde{e}_t$ satisfies the recursive equation

$$\tilde{e}_t = A_{t-1}\tilde{e}_{t-1} + K_t\nu_t - \varsigma_{t-1}$$

for $t \in \mathbb{N}_{[d,k]}$. Therefore, in either case, we can find proper matrixes E_k and F_k and a proper vector c_k, all independent of $\boldsymbol{\nu}_{0:k}$, such that $\tilde{e}_k = E_k\boldsymbol{\nu}_{0:k-d} + F_k\boldsymbol{\nu}_{k-d+1:k} + c_k$ under ϵ^n. We know that ϵ^s is symmetric with respect to $\boldsymbol{\nu}_{0:k}$. Therefore, by Lemma 5, we deduce that $\tilde{e}_k = E_k\boldsymbol{\nu}_{0:k-d} + F_k\boldsymbol{\nu}_{k-d+1:k}$ under ϵ^s. Then, we can write

$$\begin{aligned}
&\mathsf{E}_{\epsilon^n}\left[\tilde{e}_k^T\tilde{e}_k \middle| \boldsymbol{\sigma}_{0:k-d} = 0\right]\\
&= \mathsf{E}_{\epsilon^n}\Big[(E_k\boldsymbol{\nu}_{0:k-d} + F_k\boldsymbol{\nu}_{k-d+1:k} + c_k)^T\\
&\quad \times (E_k\boldsymbol{\nu}_{0:k-d} + F_k\boldsymbol{\nu}_{k-d+1:k} + c_k)\Big|\boldsymbol{\sigma}_{0:k-d} = 0\Big]\\
&= \mathsf{E}_{\epsilon^n}\Big[(E_k\boldsymbol{\nu}_{0:k-d} + c_k)^T(E_k\boldsymbol{\nu}_{0:k-d} + c_k)\\
&\quad + \boldsymbol{\nu}_{k-d+1:k}^T F_k^T F_k \boldsymbol{\nu}_{k-d+1:k}\Big|\boldsymbol{\sigma}_{0:k-d} = 0\Big]
\end{aligned}$$

where in the second equality we used the fact that $\boldsymbol{\nu}_{k-d+1:k}$ has zero mean and is independent of $\boldsymbol{\nu}_{0:k-d}$ and $\boldsymbol{\sigma}_{0:k-d}$. Choose $T_{k-d} = E_k$, and define $f_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k}) := (\varpi_{k-d} + c_k)^T(\varpi_{k-d} + c_k) + \boldsymbol{\nu}_{k-d+1:k}^T F_k^T F_k \boldsymbol{\nu}_{k-d+1:k}$, $f_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k}) := \varpi_{k-d}^T\varpi_{k-d} + \boldsymbol{\nu}_{k-d+1:k}^T F_k^T F_k \boldsymbol{\nu}_{k-d+1:k}$, $g_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k}) := z - \min_z\{z, f_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\}$, and $g_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k}) := z - \min_z\{z, f_{\epsilon^s}$

$(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\}$. Note that $g_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})$ and $g_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})$ both vanish at infinity for any fixed z. It follows that

$$\begin{aligned}
&\mathsf{E}_{\epsilon^n}\left[\tilde{e}_k^T\tilde{e}_k \middle| \boldsymbol{\sigma}_{0:k-d}=0\right]\\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{m\times d}} f_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\\
&\quad \times \mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d}=0)\,\mathsf{p}(\boldsymbol{\nu}_{k-d+1:k})d\varpi_{k-d}d\boldsymbol{\nu}_{k-d+1:k}.
\end{aligned}$$

In addition, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^m} g_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\,\mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d-1}=0)\\
&\quad\times \mathsf{p}_{\epsilon^n}(\sigma_{k-d}=0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1}=0)d\varpi_{k-d}\\
&\le \int_{\mathbb{R}^m} g^*_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\,(\mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d-1}=0)\\
&\quad\times \mathsf{p}_{\epsilon^n}(\sigma_{k-d}=0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1}=0))^*\,d\varpi_{k-d}\\
&= \int_{\mathbb{R}^m} g_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\,(\mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d-1}=0)\\
&\quad\times \mathsf{p}_{\epsilon^n}(\sigma_{k-d}=0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1}=0))^*\,d\varpi_{k-d}\\
&\le \int_{\mathbb{R}^m} g_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\,\mathsf{p}_{\epsilon^s}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d-1}=0)\\
&\quad\times \mathsf{p}_{\epsilon^s}(\sigma_{k-d}=0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1}=0)d\varpi_{k-d}
\end{aligned}$$

where in the first inequality we used the Hardy–Littlewood inequality with respect to ϖ_{k-d}, in the equality the fact that $g^*_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k}) = g_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})$, and in the second inequality (45) and Lemma 7. This implies that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \min_z\{z, f_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\}\\
&\quad\times \mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d}=0)d\varpi_{k-d}\\
&\ge \int_{\mathbb{R}^m} \min_z\{z, f_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\}\\
&\quad\times \mathsf{p}_{\epsilon^s}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d}=0)d\varpi_{k-d} \qquad (46)
\end{aligned}$$

where we used the facts that

$$\begin{aligned}
&\mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d}=0)\\
&= \frac{\mathsf{q}_{k-d}(\varpi_{k-d})\,\mathsf{p}_{\epsilon^n}(\sigma_{k-d}=0|\varpi_{k-d}, \boldsymbol{\sigma}_{0:k-d-1}=0)}{\mathsf{p}_{\epsilon^n}(\sigma_{k-d}=0|\boldsymbol{\sigma}_{0:k-d-1}=0)}
\end{aligned}$$

and that $\mathsf{p}_{\epsilon^n}(\sigma_{k-d}=0|\boldsymbol{\sigma}_{0:k-d-1}=0) = \mathsf{p}_{\epsilon^s}(\sigma_{k-d}=0|\boldsymbol{\sigma}_{0:k-d-1}=0)$. Thus, we deduce that

$$\begin{aligned}
&\mathsf{E}_{\epsilon^n}\left[\tilde{e}_k^T\tilde{e}_k \middle| \boldsymbol{\sigma}_{0:k-d}=0\right]\\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{m\times d}} f_{\epsilon^n}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\\
&\quad \times \mathsf{p}_{\epsilon^n}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d}=0)\,\mathsf{p}(\boldsymbol{\nu}_{k-d+1:k})d\varpi_{k-d}d\boldsymbol{\nu}_{k-d+1:k}\\
&\ge \int_{\mathbb{R}^m}\int_{\mathbb{R}^{m\times d}} f_{\epsilon^s}(\varpi_{k-d}, \boldsymbol{\nu}_{k-d+1:k})\\
&\quad \times \mathsf{p}_{\epsilon^s}(\varpi_{k-d}|\boldsymbol{\sigma}_{0:k-d}=0)\,\mathsf{p}(\boldsymbol{\nu}_{k-d+1:k})d\varpi_{k-d}d\boldsymbol{\nu}_{k-d+1:k}\\
&= \mathsf{E}_{\epsilon^s}\left[\tilde{e}_k^T\tilde{e}_k \middle| \boldsymbol{\sigma}_{0:k-d}=0\right]
\end{aligned}$$

where the inequality is obtained from (46) after taking z to infinity.

Step (II.D): In this substep, we show that the terms including the expected value of the cost-to-go in $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ are greater than or equal to those in $\Omega^M_{\epsilon^s}(\tilde{e}_{d-1})$. By Lemma 4, when $\sigma_{k-d+1}=1$, regardless of the existence of packet-loss detection, $\tilde{e}_{k+1}$ satisfies

$$\tilde{e}_{k+1} = \sum_{t=1}^{d}\left(\prod_{t'=1}^{t-1} A_{k-t'+1}\right) K_{k-t+2}\nu_{k-t+2}.$$

Hence, $\tilde{e}_{k+1}$ is the same under both ϵ^n and ϵ^s. Note that

$$\begin{aligned}
&\mathsf{E}_{\epsilon^n}\left[\Omega^{k+1,M}_{\epsilon^n}(\tilde{e}_{k+1}) \middle| \boldsymbol{\sigma}_{0:k-d}=0, \sigma_{k-d+1}=1\right]\\
&= \int_{\mathbb{R}^{m\times d}} \Omega^{k+1,M}_{\epsilon^n}(\tilde{e}_{k+1})\\
&\quad\times \mathsf{p}_{\epsilon^n}(\boldsymbol{\nu}_{k-d+2:k+1}|\boldsymbol{\sigma}_{0:k-d}=0, \sigma_{k-d+1}=1)d\boldsymbol{\nu}_{k-d+2:k+1}.
\end{aligned}$$

Let $\bar{\Omega}^M_{\epsilon^n}(\tilde{e}_{d-1})$ denote a loss function that is structurally similar to $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1})$ but with new value of $\boldsymbol{\theta}_{0:N-d+1}$. Clearly, if $\Omega^M_{\epsilon^n}(\tilde{e}_{d-1}) \ge \Omega^M_{\epsilon^s}(\tilde{e}_{d-1})$ for any $\boldsymbol{\theta}_{0:N-d+1}$, then $\bar{\Omega}^M_{\epsilon^n}(\tilde{e}_{d-1}) \ge \bar{\Omega}^M_{\epsilon^s}(\tilde{e}_{d-1})$. Accordingly, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^{m\times d}} \Omega^{k+1,M}_{\epsilon^n}(\tilde{e}_{k+1})\\
&\quad\times \mathsf{p}_{\epsilon^n}(\boldsymbol{\nu}_{k-d+2:k+1}|\boldsymbol{\sigma}_{0:k-d}=0, \sigma_{k-d+1}=1)d\boldsymbol{\nu}_{k-d+2:k+1}\\
&= \int_{\mathbb{R}^{m\times d}} \bar{\Omega}^{M-k+d-2}_{\epsilon^n}(\boldsymbol{\nu}_{k-d+2:k+1})\\
&\quad\times \mathsf{p}(\boldsymbol{\nu}_{k-d+2:k+1})d\boldsymbol{\nu}_{k-d+2:k+1}\\
&\ge \int_{\mathbb{R}^{m\times d}} \bar{\Omega}^{M-k+d-2}_{\epsilon^s}(\boldsymbol{\nu}_{k-d+2:k+1})\\
&\quad\times \mathsf{p}(\boldsymbol{\nu}_{k-d+2:k+1})d\boldsymbol{\nu}_{k-d+2:k+1}\\
&= \int_{\mathbb{R}^{m\times d}} \Omega^{k+1,M}_{\epsilon^s}(\tilde{e}_{k+1})\\
&\quad\times \mathsf{p}_{\epsilon^s}(\boldsymbol{\nu}_{k-d+2:k+1}|\boldsymbol{\sigma}_{0:k-d}=0, \sigma_{k-d+1}=1)d\boldsymbol{\nu}_{k-d+2:k+1}
\end{aligned}$$

where in the equalities we used the facts that $\Omega^{k+1,M}_{\epsilon^n}(\tilde{e}) = \bar{\Omega}^{M-k+d-2}_{\epsilon^n}(\tilde{e})$ for any Gaussian variable $\tilde{e}$ and an appropriate selection of $\boldsymbol{\theta}_{0:N-d+1}$, and that $\boldsymbol{\nu}_{k-d+2:k+1}$ is independent of $\boldsymbol{\sigma}_{0:k-d}=0$; and in the inequality we used the hypothesis $\Omega^{M-k+d-2}_{\epsilon^n}(\tilde{e}) \ge \Omega^{M-k+d-2}_{\epsilon^s}(\tilde{e})$ for any Gaussian variable $\tilde{e}$.

Therefore,

$$\mathsf{E}_{\epsilon^n}\left[\Omega_{\epsilon^n}^{k+1,M}(\tilde{e}_{k+1})\middle|\boldsymbol{\sigma}_{0:k-d}=0,\sigma_{k-d+1}=1\right]$$
$$\geq \mathsf{E}_{\epsilon^s}\left[\Omega_{\epsilon^s}^{k+1,M}(\tilde{e}_{k+1})\middle|\boldsymbol{\sigma}_{0:k-d}=0,\sigma_{k-d+1}=1\right].$$

This establishes $\Omega_{\epsilon^s}^{M}(\tilde{e}_{d-1}) \leq \Omega_{\epsilon^n}^{M}(\tilde{e}_{d-1})$, and that $\Phi(\epsilon^s,\delta^o) \leq \Phi(\epsilon^n,\delta^o)$.

Step (III): Finally, in the third step, we show that for the policy profile in the claim we have $\Phi(\epsilon^\star,\delta^\star) \leq \Phi(\epsilon^s,\delta^o)$. Note that by Lemmas 1, 3, and 5, when ϵ^s is used, δ^o must satisfy

$$\hat{x}_k = u_{k-d}\gamma_{k-d}\left(\prod_{t=1}^{d}A_{k-t}\right)\check{x}_{k-d} + (1-u_{k-d}\gamma_{k-d})A_{k-1}\hat{x}_{k-1} \quad (47)$$

for $k\in\mathbb{N}_{[d,N]}$ with initial conditions $\hat{x}_\tau = (\prod_{t=1}^{\tau}A_{\tau-t})m_0 = 0$ for $\tau\in\mathbb{N}_{[0,d-1]}$. Moreover, from the definition of the value function $V_k(\mathcal{I}_k^e)$ in (14), we can write

$$V_k(\mathcal{I}_k^e) = \min_{u_k}\Big\{\theta_k u_k + \mathsf{E}[\tilde{e}_{k+d}^T\tilde{e}_{k+d}|\mathcal{I}_k^e] + \mathrm{tr}Q_{k+d} + \mathsf{E}[V_{k+1}(\mathcal{I}_{k+1}^e)|\mathcal{I}_k^e]\Big\}$$

for $k\in\mathbb{N}_{[0,N-d]}$ with initial condition $V_{N-d+1}(\mathcal{I}_{N-d+1}^e)=0$, where we used the additivity property of $V_k^e(\mathcal{I}_k^e)$ and the fact that $\mathsf{E}[\hat{e}_{k+d}^T\hat{e}_{k+d}|\mathcal{I}_k^e] = \mathsf{E}[\tilde{e}_{k+d}^T\tilde{e}_{k+d}|\mathcal{I}_k^e] + \mathrm{tr}Q_{k+d}$. The minimizing scheduling policy is then obtained as

$$u_k = \mathbb{1}_{\chi_k-\theta_k\geq 0}$$

where $\chi_k = \mathsf{E}[\tilde{e}_{k+d}^T\tilde{e}_{k+d} + V_{k+1}(\mathcal{I}_{k+1}^e)|\mathcal{I}_k^e,\ u_k=0] - \mathsf{E}[\tilde{e}_{k+d}^T\tilde{e}_{k+d} + V_{k+1}(\mathcal{I}_{k+1}^e)|\mathcal{I}_k^e, u_k=1]$. We prove by induction that $V_k^e(\mathcal{I}_k^e)$ can be written in terms of $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k}$, $\boldsymbol{\lambda}_{k-d+1:k}$, and $\boldsymbol{u}_{k-d+1:k-1}$. The claim holds for time $N-d+1$. We assume that it holds for time $k+1$, and will show that it also holds for time k. By the hypothesis, $V_{k+1}^e(\mathcal{I}_{k+1}^e)$ is the function of $\tilde{e}_{k+1}$, $\boldsymbol{\nu}_{k-d+3:k+1}$, $\boldsymbol{\lambda}_{k-d+2:k+1}$, and $\boldsymbol{u}_{k-d+2:k}$. From (21), we can write $\tilde{e}_{k+1}$ in terms of $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k+1}$, u_{k-d+1}, and γ_{k-d+1}. Hence, there exists a function $g(.)$ such that $V_{k+1}^e(\mathcal{I}_{k+1}^e) = g(\tilde{e}_k,\boldsymbol{\nu}_{k-d+2:k+1},\boldsymbol{\lambda}_{k-d+2:k+1},\boldsymbol{u}_{k-d+1:k},\gamma_{k-d+1})$. This implies that $\mathsf{E}[V_{k+1}^e(\mathcal{I}_{k+1}^e)|\mathcal{I}_k^e,u_k]$ is a function of $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k}$, $\boldsymbol{\lambda}_{k-d+1:k}$, and $\boldsymbol{u}_{k-d+1:k-1}$ as ν_{k+1}, γ_{k-d+1}, and λ_{k+1} are averaged out and λ_{k+1} only depends on λ_k. In addition, from (21), we can write $\tilde{e}_{k+d}$ is in terms of $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k+d}$, $\boldsymbol{u}_{k-d+1:k}$, and $\boldsymbol{\gamma}_{k-d+1:k}$. Hence, there exists a function $h(.)$ such that $\tilde{e}_{k+d}^T\tilde{e}_{k+d} = h(\tilde{e}_k,\boldsymbol{\nu}_{k-d+2:k+d},\boldsymbol{u}_{k-d+1:k},\boldsymbol{\gamma}_{k-d+1:k})$. This implies that $\mathsf{E}[\tilde{e}_{k+d}^T\tilde{e}_{k+d}|\mathcal{I}_k^e,u_k]$ is a function of $\tilde{e}_k$, $\boldsymbol{\nu}_{k-d+2:k}$, $\boldsymbol{\lambda}_{k-d+1:k}$, and $\boldsymbol{u}_{k-d+1:k-1}$ as $\boldsymbol{\nu}_{k+1:k+d}$ and $\boldsymbol{\gamma}_{k-d+1:k}$ are averaged out. Therefore, the claim holds. This verifies that $\Phi(\epsilon^\star,\delta^\star) \leq \Phi(\epsilon^s,\delta^o)$, and completes the proof. ∎

H. Proof of Corollary 1

Proof: By Theorem 1 and the definition of ν_0, it is not difficult to see that the optimal estimation policy is given by $\hat{x}_1 = A_0 m_0 + u_0\gamma_0 A_0 K_0\nu_0$. Hence, we only need to obtain the optimal scheduling policy. We can write $\tilde{e}_1$ as

$$\tilde{e}_1 = (1-u_0\gamma_0)A_0K_0\nu_0 + K_1\nu_1. \quad (48)$$

Using (48), we find

$$\begin{aligned}\mathsf{E}\left[\tilde{e}_1^T\tilde{e}_1\middle|\mathcal{I}_0^e,u_0\right] &= \mathsf{E}\left[(1-u_0\gamma_0)^2\nu_0^TK_0^TA_0^TA_0K_0\nu_0\middle|\mathcal{I}_0^e,u_0\right]\\ &+ \mathsf{E}\left[\nu_1^TK_1^TK_1\nu_1\middle|\mathcal{I}_0^e,u_0\right]\\ &+ \mathsf{E}\left[2(1-u_0\gamma_0)\nu_0^TK_0^TA_0^TK_1\nu_1\middle|\mathcal{I}_0^e,u_0\right]\\ &= (1-u_0\lambda_0)\nu_0^TK_0^TA_0^TA_0K_0\nu_0 + \mathrm{tr}(K_1^TK_1N_1).\end{aligned}$$

Accordingly,

$$\mathsf{E}\left[\tilde{e}_1^T\tilde{e}_1\middle|\mathcal{I}_0^e,u_0=0\right] - \mathsf{E}\left[\tilde{e}_1^T\tilde{e}_1\middle|\mathcal{I}_0^e,u_0=1\right] = \lambda_0\nu_0^TK_0^TA_0^TA_0K_0\nu_0.$$

This implies that $u_0 = \mathbb{1}_{\lambda_0\nu_0^TK_0^TA_0^TA_0K_0\nu_0-\theta_0\geq 0}$, and completes the proof. ∎

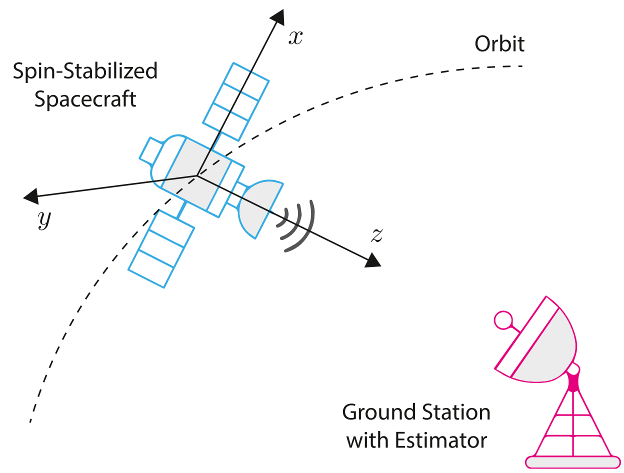

Fig. 2. Angular velocity estimation of a perturbed spin-stabilized spacecraft over a delayed and lossy downlink communication channel at a ground station. The objective is to find an optimal policy profile that minimizes the loss function in the causal frequency-distortion tradeoff.

V. Numerical Example

In this section, we provide a numerical example to demonstrate how the framework developed in the previous sections can be used in a space scenario. Consider a spin-stabilized spacecraft whose body is spinning about the z-axis, i.e., the axis of symmetry, with a constant angular velocity $\omega_z = \omega_0$ (see Fig. 2). For such a vehicle, the Euler equation is written as

$$\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix} = \begin{bmatrix}0 & \frac{I_y-I_z}{I_x}\omega_0 & 0\\ \frac{I_z-I_x}{I_y}\omega_0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} + \begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix}$$

where $(\omega_x,\omega_y,\omega_z)$ is the angular velocity, (I_x,I_y,I_z) is the moment of inertia, and (e_x,e_y,e_z) is a Gaussian disturbance torque acting on the spacecraft. Note that for spin stability,

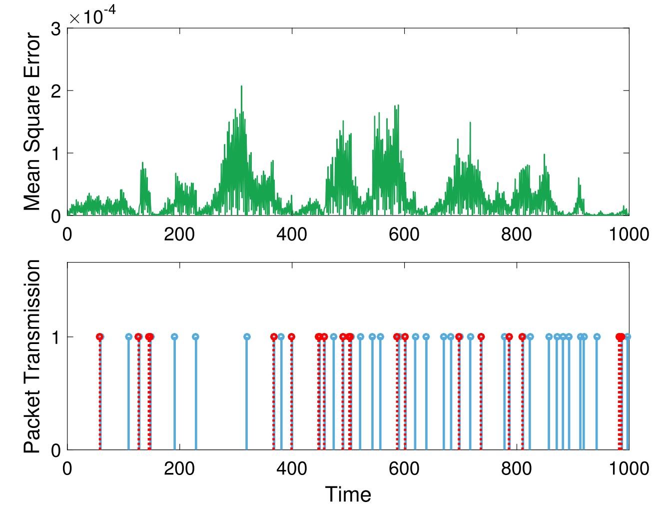

Fig. 3. Mean square error and packet transmission trajectories under the optimal policy profile. The total mean square error is 0.0523, and the total number of packet transmissions is 63, out of which 23 were lost. The solid lines represent successful deliveries, and the dotted lines represent packet losses.

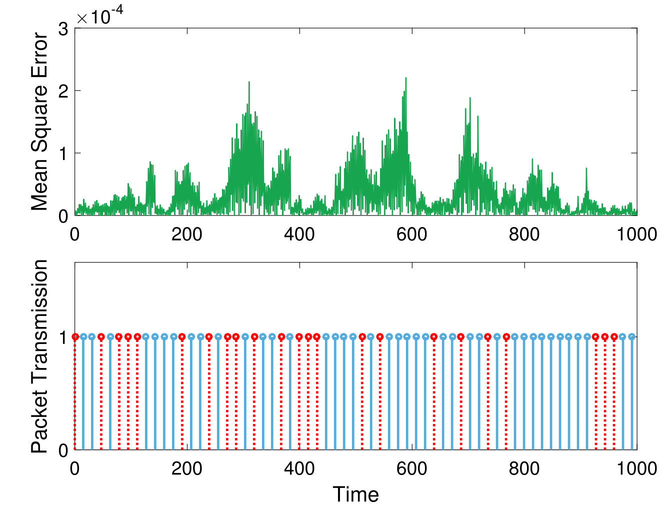

Fig. 4. Mean square error and packet transmission trajectories under a periodic policy profile. The total mean square error is 0.0596, and the total number of packet transmissions is 63, out of which 23 were lost. The solid lines represent successful deliveries, and the dotted lines represent packet losses.

the spacecraft must be spinning either about the major or minor axis of inertia. In our example, the following parameters are used: $I_x = I_y = 20$ kg.m^2, $I_z = 100$ kg.m^2, and $\omega_0 = 2\epsilon$ rad/s; and the Euler equation is discretized over the time horizon $N = 1000$. The state equation of the form (1) is specified by $A_k = [0.4258, 0.4258, 0; 0.4258, 0.4258, 0; 0, 0, 1]$ and $W_k = 10^{-6}\,\mathrm{diag}\{0.2245, 0.2245, 0.0025\}$ for all $k \in \mathbb{N}_{[0,N]}$ with $m_0 = [0; 0; 2\pi]$ and $M_0 = 10\ W_0$. Suppose there is a sensor on the spacecraft that partially observes each component of the angular velocity at each time k. The output equation of the form (2) is specified by $C_k = \mathrm{diag}\{1, 1, 1\}$ and $V_k = 10^{-3}\,\mathrm{diag}\{1, 1, 1\}$ for all $k \in \mathbb{N}_{[0,N]}$. The sensory information should be transmitted over a costly downlink channel to a ground station where the angular velocity is estimated. The downlink channel is subject to packet loss with $\lambda'_k = 0.3$ for all $k \in \mathbb{N}_{[0,N]}$ and time delay with $d = 1$.

For this networked system, we are interested in finding the optimal policy profile $(\epsilon^\star, \delta^\star)$ that minimizes the loss function $\Phi(\epsilon, \delta)$ in the causal frequency-distortion tradeoff with the weighting coefficient $\theta_k = 8 \times 10^{-6}$ for $k \in \mathbb{N}_{[0,N/2]}$ and $\theta_k = 6 \times 10^{-6}$ for $k \in \mathbb{N}_{[N/2+1,N]}$. For a simulated realization of the system, Fig. 3 shows the mean square error and packet transmission trajectories under the optimal policy profile. In this case, the total mean square error is 0.0523, and the total number of packet transmissions is 63, out of which 23 were lost. Besides, for the same realization of the system, Fig. 4 shows the corresponding trajectories under a periodic policy profile with the same total number of packet transmissions. In this case, the total mean square error is 0.0596, and the total number of packet transmissions is again 63, out of which 23 were lost. We observe that the optimal policy profile proved effective in improving the system performance. It is interesting to note that, in comparison with the periodic scheduling policy, the optimal scheduling policy not only transmits sensory information less frequently when the estimation discrepancy is small but also transmits more frequently and more persistently when the estimation discrepancy is large and when there have been some recent packet losses. Moreover, the optimal scheduling policy is adaptive to the weighting coefficient θ_k, and transmits more frequently when the cost of communication decreases.

VI. Conclusion

The aim of this article has been to determine the fundamental performance limit of state estimation of a partially observable Gauss–Markov process over a fixed-delay packet-erasure channel. To that end, we formulated a causal frequency-distortion tradeoff between the packet rate and the mean square error. Associated with this tradeoff, we characterized a globally optimal policy profile, and showed that this policy profile is composed of a symmetric threshold scheduling policy and a non-Gaussian linear estimation policy. We discussed the structural properties of these scheduling and estimation policies, and proved that the globally optimal policy profile remains exactly the same whether the channel is with or without packet-loss detection. For future research, we suggest that one should investigate the effect of random delay on the system performance. In this case, the channel should be modeled as a queue with random service time. An important point regarding this problem is that, depending on the delay distribution, an optimal scheduling policy might avoid transmitting sensory information even if the queue is empty and the communication cost is equal to zero. We believe that tackling such a problem requires development of novel backward-induction techniques.

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