ON SIMULTANEOUS H_2-OPTIMIZATION OF SEVERAL PERFORMANCE BOUNDS

B. Bernhardsson and K.H. Johansson

Department of Automatic Control, Lund Institute of Technology, Box 118, S-221 00 Lund, Sweden

Abstract. Simultaneous optimization of several criteria is an important and interesting subject. Even the standard H_2-case is non-trivial. The problem is here treated for the general MIMO case for two conflicting objectives (the results directly generalize to several performance measures). A semi-norm D_2 is introduced to describe the situation. The state-space solution technique in [Khargonekar and Rotea, 1991] is simplified and generalized slightly. It is shown that the method simplifies when the controller is SISO, SIMO or MISO. A polynomial method is also given, together with an example that illustrates the results.

1. INTRODUCTION

The case in controller design is often to optimize conflicting criteria. One possible form of a problem is

$$\min_K \|T_0(K)\|_A; \qquad \text{with } \|T_1(K)\|_B \leq \gamma \quad (1)$$

where $\|\cdot\|_X$ could be the H_2-norm, see e. g. [Rotea and Khargonekar, 1990]. We will show via Q-parametrization a way to treat some design problems like (1). We recall a standard result in convexity.

LEMMA 1

$$\min_K \|T_0(K)\|_A; \qquad \text{with } \|T_1(K)\|_B \leq \gamma \quad (2)$$

for varying $\gamma > 0$ is equivalent to solving

$$\min_K \|T_0(K)\|_A + \rho^2 \|T_1(K)\|_B \quad (3)$$

for varying $\rho \in (0, \infty)$. □

Proof: See e. g. [Luenberger, 1969].

This theorem can easily be generalized to n constraints, $\gamma_1, \ldots, \gamma_n$, in (2), and thus $n+1$ terms in (3) with $\rho_1^2, \ldots, \rho_n^2$.

In the main part of this paper, we will treat problems of the form (3). However, Lemma 1 makes our results more general.

For a stable system the H_2-norm is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{tr } G(j\omega)^* G(j\omega)\, d\omega\right)^{1/2}$$

2. A VIEW OF THE PROBLEM

Suppose that we want to minimize the two conflicting criteria given by $\|T_{00}\|_2$ and $\|T_{11}\|_2$, where T_{ii} denotes the transfer function from w_i to z_i, see Fig. 2. This can be done by e. g. solving a problem on the form (1) or minimizing

$$\|T_{00}\|_2^2 + \|T_{11}\|_2^2 \quad (4)$$

It is trivial to write (4) as a standard H_2-problem when $z_0 = z_1$. All we have to do is to introduce

$$z = \begin{pmatrix} T_{00} & T_{11} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

The case when $w_0 = w_1$ is solved in the dual way.

EXAMPLE 1

Consider Fig. 1. Assume we want to minimize the 2-norm of

$$\begin{pmatrix} KS \\ S \end{pmatrix}$$

where $S = (I - PK)^{-1}$ is the sensitivity function. This is equal to minimize

$$\|T_{uv}\|_2^2 + \|T_{zv}\|_2^2 \quad (5)$$

which is the dual case mentioned above, and thus can be treated as a standard H_2-problem. □

Many standard mixed problems can be dealt with in the way shown in Example 1. This is the case for criteria given by

$$\begin{pmatrix} S \\ T \end{pmatrix}; \begin{pmatrix} S \\ KS \end{pmatrix}; \begin{pmatrix} S \\ SP \end{pmatrix}; \begin{pmatrix} KS \\ T \end{pmatrix}; \begin{pmatrix} SP \\ T \end{pmatrix} \quad (6)$$

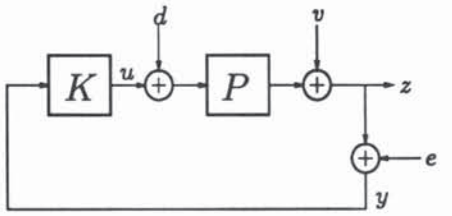

Fig. 1. Fundamental block diagram for Example 1 and the polynomial solution in Section 5.

where $T = I - S$ is the complementary sensitivity function. The above problems can therefore all be solved by your favorite LQG-method for "the standard problem", see e. g. [Doyle *et al.*, 1989], [Hunt *et al.*, 1991] or [Anderson and Moore, 1990].

The problem is however non-trivial in the case both $z_0 \neq z_1$ and $w_0 \neq w_1$! As an example take the very natural problem of optimizing the H_2-norm of

$$\begin{pmatrix} SP \\ KS \end{pmatrix} \tag{7}$$

To the knowledge of the authors, this cannot directly be written as a standard H_2-problem. This can also be seen if we use the linear fractional transformation (LFT) defined by

$$LFT(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

If (7) is written as an LFT of an extended system we get a restriction on the controller structure

$$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

3. THE D_2-NORM

Consider the system in Fig. 2, which in state-space form is

$$\begin{aligned} \dot{x} &= Ax + B_0w_0 + B_1w_1 + B_2u \\ z_0 &= C_0x + D_{00}w_0 + D_{01}w_1 + D_{02}u \\ z_1 &= C_1x + D_{10}w_0 + D_{11}w_1 + D_{12}u \\ y &= C_2x + D_{20}w_0 + D_{21}w_1 + D_{22}u \end{aligned}$$

which includes all weighting functions. We assume $D_{00}, D_{01}, D_{10}, D_{11}$ to be zero since there is otherwise a white noise component in z_i. By setting $y' := y - D_{22}u$ we can often also assume $D_{22} = 0$. After closing the loop we get

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \tag{8}$$

Standard LQG now minimizes the sum of squares of the H_2-norm of *all* matrix elements. But we are here only interested in the diagonal, i.e. $||T_{00}||_2^2 + ||T_{11}||_2^2$.

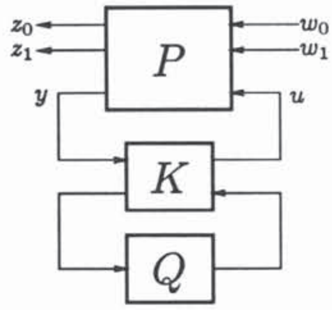

Fig. 2. Block diagram for the system in Section 3.

DEFINITION 1
The D_2 (diagonal-H_2) semi-norm of a stable square transfer matrix G is defined as

$$||G||_{D_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_i ||G_{ii}(j\omega)||_2^2 \, d\omega \right)^{1/2}$$

□

The objective is now to find the controller K such that the closed loop system is internally stable and has minimal D_2-norm. The minimization can be performed using very similar techniques to these for the H_2-norm (completion of squares), as we will see in the next section.

4. Q-PARAMETRIZATION SOLUTION

We will need the famous Q-parametrization.

LEMMA 2
Assume that G is stabilizable. All proper controllers that stabilize the system can be written

$$K = (Y - MQ)(X - NQ)^{-1}$$

where Q ranges over all stable transfer functions of the same size as K. The corresponding closed loop systems are given by $T = T_1 - T_2QT_3$, where T_i are stable. □

Proof: See e. g. [Francis, 1987].

State-space formulas for T_i, M, N, X, Y are given in [Francis, 1987] as well. We will also need a result on inner-outer and co-inner-outer factorization.

LEMMA 3
Every proper, rational, stable matrix has an inner-outer and a co-inner-outer factorization

$$G = G_iG_o = G_{co}G_{ci}$$

where G_i is called inner and fulfills $G_i^*G_i = I$ and G_o outer and has full row rank. G_{co} and G_{ci} are named in a similar way. G_{co}^T is outer and G_{ci}^T is inner. □

Proof: See e. g. [Francis, 1987], where state-space formulas also are given.

Consider the closed loop system (8). It follows from Lemma 2 that we can write $T_{00} = T_1 - T_2 Q T_3$ and $T_{11} = \bar{T}_1 - \bar{T}_2 Q \bar{T}_3$. Hence, the problem is to minimize

$$\left\| \begin{pmatrix} T_1 - T_2 Q T_3 \\ \bar{T}_1 - \bar{T}_2 Q \bar{T}_3 \end{pmatrix} \right\|_2 \qquad (9)$$

Note that (9) can not be considered as two separate optimizations since the Q:s should be the same. It is also impossible to decompose the problem into the form $T_{1e} - T_{2e} Q T_{3e}$ directly.

The problem is quadratic in the elements of Q, so it is no surprise that the problem is solvable using completion of squares. One solution method uses Kroenecker products. If we introduce

$$A = \begin{pmatrix} \text{vec}(T_1) \\ \text{vec}(\bar{T}_1) \end{pmatrix} \quad B = \begin{pmatrix} T_3^T \otimes T_2 \\ \bar{T}_3^T \otimes \bar{T}_2 \end{pmatrix}$$

we get $\|T\|_{D_2} = \|A - B\text{vec}(Q)\|_2$. From this we easily obtain

THEOREM 1
Minimum for (9) is given by

$$\text{vec}(Q) = B_o^+ (B_i^* A)_+$$

where X^+ denotes the pseudo inverse. $(X)_+$ is the stable part of X, including direct terms. The minimization has a unique minima if and only if the matrix B has full column rank. □

This is a slight generalization of the results in [Khargonekar and Rotea, 1991] who only give sufficient conditions for the minima to be unique, and require that either T or $\bar{T}$ satisfies a full rank condition.

The drawback of the method is that the solution, although analytical, is rather implicit in problem data. It also requires the solution of a large dimensional H_2-problem. The goal is to obtain formulas that are as explicit in problem data as possible and also computationally efficient.

4.1 SISO, SIMO and MISO Controllers

When the controller is SISO, SIMO or MISO, the problem is less complex to solve. As usual, by multiplying from left and right with inner and co-inner matrices, we can reduce the problem to minimization of the 2-norm of

$$\begin{pmatrix} R_{11} - T_{2o} Q T_{3co} & R_{12} \\ R_{21} & R_{22} \\ \bar{R}_{11} - \bar{T}_{2o} Q \bar{T}_{3co} & \bar{R}_{12} \\ \bar{R}_{21} & \bar{R}_{22} \end{pmatrix}$$

which reduces to minimization of

$$\left\| \begin{pmatrix} R_{11} - T_{2o} Q T_{3co} \\ \bar{R}_{11} - \bar{T}_{2o} Q \bar{T}_{3co} \end{pmatrix} \right\|_2$$

The matrices around Q are stable and have stable inverses in H_2 for full rank T_2, $\bar{T}_2$ and T_3, $\bar{T}_3$.

Assume the controller is SIMO. Then we get scalar T_{3co} and $\bar{T}_{3co}$, and the problem can therefore be rewritten as minimization of the 2-norm of

$$\begin{pmatrix} R_{11} \\ \bar{R}_{11} \end{pmatrix} - \begin{pmatrix} T_{2o} T_{3co} \\ \bar{T}_{2o} \bar{T}_{3co} \end{pmatrix} Q =: R_e - T_{2e} Q$$

which can be solved using standard H_2-methods. The dual problem with a MISO controller gives scalar T_{2o} and $\bar{T}_{2o}$, and can be solved in the same way.

5. POLYNOMIAL SOLUTIONS OF A MIXED PROBLEM

In this section we will develop a polynomial solution method for the minimization of (4). Consider the SISO system in Fig. 1 again. Assume that we want to minimize the influence of the disturbance d on the output z and that we simultaneously want to minimize the influence of the measurement noise e on the control signal u. We are however *not* (for some reason) particularly interested in the transfer functions from d to u or from e to y. The problem is therefore to find K that stabilizes the system and minimizes the H_2-norm of

$$\begin{pmatrix} T_{zd} \\ \rho T_{ue} \end{pmatrix} = \begin{pmatrix} SP \\ \rho KS \end{pmatrix} \qquad (10)$$

Assume the controller $K = -S/R$ to be designed for the plant $P = B/A$. We use the observer polynomial A_o and the model polynomial A_m, where $\deg A_o = \deg R$ and $\deg A_m = \deg A$. Lemma 2 gives that all proper controllers that stabilize the plant are given by

$$S = \frac{S_0}{A_o} + Q\frac{A}{A_m}, \quad R = \frac{R_0}{A_o} - Q\frac{B}{A_m}$$

where S_0/R_0 is a nominal stabilizing controller. If we optimize the H_2-norm of (10) we get

$$Q = -\frac{1}{M}\left(\frac{N}{M^*}\right)_+$$

where

$$MM^* = A_o A_o^* (\rho^2 A^2 (A^*)^2 + B^2 (B^*)^2)$$
$$N = A_m A_o^* (\rho^2 A (A^*)^2 S_0 - B (B^*)^2 R_0)$$

and where M is stable.

EXAMPLE 2

Consider the simple plant $G = 1/(s+1)$. Choose $A_m = s+1$, $A_o = 1$, and $\rho = 0.001$. The optimal D_2-regulator and H_2-regulator are

$$\frac{956.3(s+1)}{s^2 + 44.7s + 43.7} \quad \text{and} \quad \frac{413.8}{s + 1000.4}$$

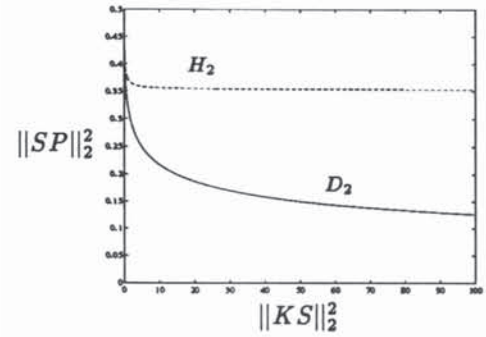

Fig. 3. The D_2-regulator (solid) compared to the H_2-regulator (dashed) in Example 2.

They give the squared D_2-norm 0.0428 and 0.3539, respectively. The matrix

$$\begin{pmatrix} ||T_{zd}||_2^2 & ||T_{ze}||_2^2 \\ \rho^2 ||T_{ud}||_2^2 & \rho^2 ||T_{ue}||_2^2 \end{pmatrix}$$

is for the D_2-regulator and the H_2-regulator

$$\begin{pmatrix} 0.0326 & 10.2185 \\ 0.0000 & 0.0102 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.3538 & 0.0605 \\ 0.0000 & 0.0001 \end{pmatrix}$$

Notice that the D_2-regulator gives much smaller influence of d on z, while the noise in the control signal is only increased marginally. The prize paid (which we defined as less important), is that there is a larger influence of e on z.

In Fig. 3 we have plotted $||SP||_2^2$ versus $||KS||_2^2$ for varying ρ. For all ρ, the D_2-regulator is better than the H_2-regulator in the sense of the D_2-norm. □

Note that the two separate H_2-problems are singular, but the total problem is non-singular. This problem would hence not be covered directly by Theorem 3.4 in [Khargonekar and Rotea, 1991] .

6. CONCLUSIONS AND FUTURE RESEARCH

Simultaneous minimization of several H_2-norms are related to the D_2 semi-norm minimization. We have derived a formula based on Q-parametrization that gives the D_2-optimal controller for two minimization criteria. But the same method works for more than two criteria. It was shown that it is possible to use standard H_2-methods to calculate optimal SIMO and MISO controllers. A polynomial solution method has also been presented.

Multiobjective optimization in general has a long history in control theory. The research in multicriteria optimal control was initiated in [Zadeh, 1963]. Several papers have attempted some sort of LQG-minimization, see e. g. [Makila, 1989] and [Koussoulas and Leondes, 1986]. The recent excellent reference [Khargonekar and Rotea, 1991] tackles a similar problem as ours. They say that the problem formulation is new in its generality. Our results in Section 4, although first obtained unaware of that reference, can be seen as a concretization and simplification of that paper. The example they include has two measurement signals and one control signal and can be solved by the method of SIMO controllers in Section 4.

It would be interesting to have more explicit state-space formulas for the MIMO controller case. The order of the optimal D_2-controller is an open question in the general case. Polynomial methods, [Kučera, 1991] and [Hunt *et al.*, 1991], give insight for singular problems. It would also be interesting to see the solution obtained using other "completion of squares" methods, to see if the solution can be further simplified.

7. REFERENCES

ANDERSON, B. and J. MOORE (1990): *Optimal Control-Linear Quadratic Methods.* Prentice Hall.

DOYLE, J., K. GLOVER, P. KHARGONEKAR, and B. FRANCIS (1989): "State space solutions to standard H_2 and H_∞ problems." *IEEE Trans. Automat. Contr.*, **34**, pp. 831–847.

FRANCIS, B. (1987): *A Course in H_∞ Control Theory.* Springer-Verlag.

HUNT, K., M. ŠEBEK, and V. KUČERA (1991): "Polynomial approach to H_2-optimal control: the multivariable standard problem." In *IEEE Conference on Decision and Control, Brighton,England.*

KHARGONEKAR, P. and M. ROTEA (1991): "Multiple objective optimal control of linear systems: The quadratic norm case." *AC*, **36:1**, pp. 14–24.

KOUSSOULAS, N. and C. LEONDES (1986): "The multiple linear quadratic gaussian problem." *Int. J. Control*, **43:2**, pp. 337–349.

KUČERA, V. (1991): *Analysis and Design of Linear Control Systems.* Academia, Prague and Prentice Hall International, London.

LUENBERGER, D. (1969): *Optimization by Vector Space Methods.* John Wiley & Sons, New York.

MAKILA, P. (1989): "On multiple criteria stationary linear quadratic control." *IEEE Trans. Automat. Contr.*, **34:12**, pp. 1311–1313.

ROTEA, M. and P. KHARGONEKAR (1990): "Simultaneous H_2/H_∞ optimal control with state feedback." In *ACC90*, pp. 2380–2384.

ZADEH, L. (1963): "Optimality and non-scalar-valued performance criteria." *IEEE Trans. Automat. Contr.*, **8**, pp. 59–60.