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Scalable Controller Design for Consensus with Performance Considerations: A Phase-Theoretical Approach

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Abstract— **In this paper, we study the problem of designing a uniform controller for heterogeneous multi-agent systems to achieve consensus, taking into account transient performance requirement characterized by convergence rate and damping. Two main issues are addressed: 1) Under what conditions the problem is solvable? 2) When the problem is solvable, how to design such a uniform controller? To answer these questions, we define a measure of diversity of the agents through simultaneous phase alignment of a set of matrices, and define a measure of interaction quality using the essential phase of the Laplacian matrix of a graph. The main finding of the paper is a critical trade-off among the diversity of the agents, the interaction quality among them, and the desired damping performance that constitutes the solvability condition. We also propose a method to design the controller when the condition is satisfied. The analysis of departure angles of multivariable root loci plays a useful role in our study.**

I. INTRODUCTION

Consensus plays a fundamental role in control of multi-agent systems, with broad application scenarios such as distributed optimization, robotic formation control, and power grid stability. One is referred to textbooks and monographs [1]–[5] for a comprehensive review. Consensus algorithms foster collective behavior through local interactions, ensuring all agents asymptotically converge to a common state through a consensus protocol over a network topology.

Early studies of consensus started with analysis problem, i.e., finding conditions under which a given set of agents, homogeneous or heterogeneous, achieve consensus with given controllers and given network topology [3], [5]–[7]. Later, efforts have been devoted to designing controllers driving the agents toward consensus, for both homogeneous agents [8]–[12] and heterogeneous agents [13]–[16]. An important issue in controller design is scalability, particularly in large-scale networks of highly diverse agents. Scalability can be achieved by designing a small number of distinct controllers, significantly fewer than the number of agents, or in the extreme, a uniform controller shared by all agents. In our previous work [16], both analysis and controller synthesis for consensus of heterogeneous multi-agent systems have been studied. By exploiting a recently developed phase theory of multivariable systems, we established a condition under which a uniform controller can be constructed so that agents converge to consensus asymptotically.

On top of the convergence, it is often desirable that the transient performance of consensus meets some requirement. Two important performance metrics are convergence rate and damping. The convergence rate, often captured by the largest real part of the closed-loop poles, is relatively easy to deal with and has attracted cosniderable attention in the literature (e.g. [1], [17], [18]). By contrast, damping, which captures oscillations in the consensus process and has a crucial effect on overshoot, has been much less considered. Incorporating damping into consensus design is both practically relevant and conceptually novel.

In this paper, we study the consensus design with performance considerations using a phase-theoretic approach. We consider a heterogeneous multi-agent system consisting of single integrators with different gains and design a uniform controller that guarantees consensus with prescribed convergence rate and damping. We show that the phase approach provides a natural fit for considering damping requirement. A solvability condition for the design problem is established, formulated through a trade-off among the diversity of the agents, the interaction quality among them, and the desired damping. In addition, a uniform controller is constructed by solving a simultaneous phase alignment problem.

The rest of this paper is organized as follows. The consensus design problem is formulated in Section II. Section III introduces measures of matrix diversity and interaction quality. Section IV presents our main results. Simulations are given in Section V. Section VI concludes this paper.

Graph preliminaries: A graph, denoted by $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, consists of a set of nodes $\mathcal{V} = \{1, \ldots, N\}$ and a set of edges $\mathcal{E}$. We use (i, j) to represent the edge directed from node i to node j. A path from node i_1 to node i_k is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ with $(i_j, i_{j+1}) \in \mathcal{E}$ for $j \in \{1, \ldots, k-1\}$. A graph $\mathbb{G}$ is said to be strongly connected if every node has paths to all the other nodes in the graph.

In a weighted graph, each edge is associated with a weight. Denote by $a_{ji} > 0$ the weight of edge (i, j), where a_{ji} is understood to be zero when there is no edge from node i to j. The indegree and outdegree of node i are given by

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$d_{\text{in}}(i) = \sum_{j=1}^{n} a_{ij}$ and $d_{\text{out}}(i) = \sum_{j=1}^{n} a_{ji}$ respectively. A graph is said to be weight-balanced if $d_{\text{in}}(i) = d_{\text{out}}(i)$ for all $i \in \mathcal{V}$. The Laplacian matrix $L = [l_{ij}]$ of a graph is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j=1, j\neq i}^{n} a_{ij}, & i = j. \end{cases}$$

The Laplacian matrix L has all its eigenvalues in the closed right half plane. Also, it has a zero eigenvalue with a corresponding eigenvector being $\mathbf{1}$, which denotes a vector will all entries being 1. Furthermore, if $\mathbb{G}$ is strongly connected, then 0 is a simple eigenvalue.

II. PROBLEM FORMULATION

Consider a multi-agent system with N agents. The dynamics of agent i is given by

$$\dot{x}_i(t) = M_i u_i(t), \quad x_i(0) = x_{i0}, \tag{1}$$

where $x_i \in \mathbb{R}^m$ is state. Assume $M_i \in \mathbb{R}^{m\times m}$ is nonsingular. The transfer function of the agent i is $P_i(s) = \dfrac{M_i}{s}$.

The agents communicate over the following protocol

$$u_i(t) = \sum_{j=1}^{n} a_{ij} F(x_j(t) - x_i(t)), \tag{2}$$

where $a_{ij}, i, j = 1, \ldots, N$, are nonnegative edge weights of the underlying graph of the network, capturing the topology and strength of the interactions, and $F \in \mathbb{R}^{m\times m}$ is a uniform state feedback gain applied to the agent. The edge weights a_{ij} are given *a priori*. The uniform feedback gain matrix F needs to be designed. Assume the underlying network is strongly connected. Let

$$\begin{aligned} \boldsymbol{P}(s) &= \operatorname{diag}\{P_1(s), \ldots, P_N(s)\}, \\ u(t) &= \begin{bmatrix} u_1(t)' & \cdots & u_N(t)' \end{bmatrix}', \\ x(t) &= \begin{bmatrix} x_1(t)' & \cdots & x_N(t)' \end{bmatrix}'. \end{aligned}$$

Then, the transfer function from w to x is given by

$$G(s) = (I + \boldsymbol{P}(s)(L \otimes F))^{-1} \boldsymbol{P}(s), \tag{3}$$

as depicted in Fig. 1, where $\otimes$ represents the Kronecker product, w is a non-persistent signal due to initial conditions of the agents and L is the Laplacian matrix of the network.

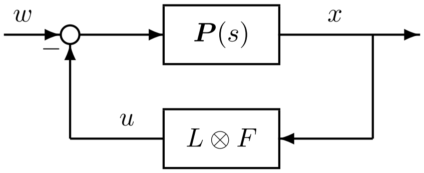


Fig. 1. Block diagram of consensus problem.

In this paper, we will study the problem of designing state-feedback controller F so that the agents reach consensus with prescribed transient performance. Two key performance criteria are considered, i.e., convergence rate and damping. The convergence rate characterizes how fast trajectories approach the steady state and depends on the real parts of the closed-loop poles. The damping measures how oscillatory the trajectories are and provides a characterization of the convergence quality. It depends on the phases of the closed-loop poles.

Let $\alpha > 0$ and $\zeta \in (0, 1]$. As shown in Fig. 2, if the poles of the closed system (excluding the N poles at the origin that correspond to the consensus modes) are contained in the intersection of a sector pointing to the left with opening angle $2\cos^{-1}\zeta$ and a vertical half plane with distance α to the left of the imaginary axis, then the agents will reach consensus with convergence rate α and damping ζ, i.e., $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ and $x_i(t) - x_j(t)$ decays exponentially to zero with rate no less than α and damping at least ζ for all $i, j = 1, 2, \ldots, N$.

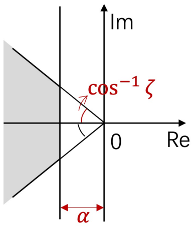


Fig. 2. Pole location corresponding to consensus with desired convergence rate and damping.

The problem to be considered in this paper is formulated as follows.

Problem: Let $\alpha > 0$ and $\zeta \in (0, 1]$. Design feedback gain F, if exists, so that the agents of the form (1) achieve consensus under protocol (2) with convergence rate α and damping ζ.

We will study this consensus design problem by exploiting the recently developed phase theory. We will see that the phase notion provides a natural and precise fit for analyzing damping characteristics of dynamic systems. Solvability conditions of the problem will be provided. When the problem is solvable, controller design method will be given.

III. MATRIX DIVERSITY AND NETWORK ASYMMETRY

The diversity of a set of square matrices has been defined in [19] by exploiting the simultaneous phase alignment. We will extend these notions to a set of wide matrices. Moreover, we will review the notion of the essential phase for a Laplacian matrix introduced in [20] and propose to use it as a measure of interaction quality.

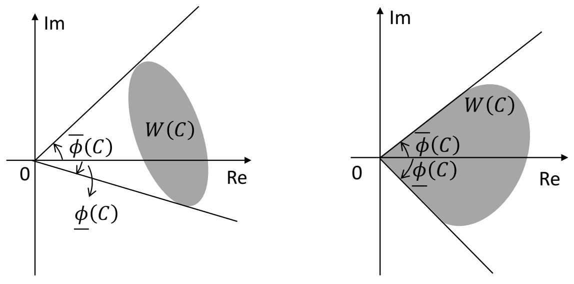
A. *Matrix phases*

The numerical range of a matrix $C \in \mathbb{C}^{m\times m}$ is defined as $W(C) = \{x^* C x : x \in \mathbb{C}^m, \|x\| = 1\}$. It is convex and contains the spectrum of C [21]. If $0 \notin W(C)$, due to the convexity, $W(C)$ is contained in an open half complex plane. In this case, C is called a sectorial matrix. Then, the (largest and smallest) phases of C are defined as

$$\overline{\phi}(C) = \max_{z\in W(C)} \angle z, \quad \underline{\phi}(C) = \min_{z\in W(C)} \angle z$$

so that $\overline{\phi}(C) - \underline{\phi}(C) < \pi$ [22]. One can select the value of the phase center of C, defined by $\gamma(C) = [\overline{\phi}(C) + \underline{\phi}(C)]/2$, from $[-\pi, \pi)$, then the values of the phases are determined.

A graphic interpretation of phases of a sectorial matrix is illustrated in Fig. 3(a). The two angles from the positive real axis to each of the two supporting rays of $W(C)$ are $\overline{\phi}(C)$ and $\underline{\phi}(C)$ respectively. The angle subtended by the two supporting rays is defined to be the field angle of C, denoted by $\delta(C)$. Clearly, $\delta(C) < \pi$.



(a) A sectorial matrix. (b) A quasi-sectorial matrix.

Fig. 3. Geometric interpretation of $\overline{\phi}(C)$ and $\underline{\phi}(C)$.

The phase definition can be extended to a larger class of matrices [20]. A matrix C is said to be quasi-sectorial if $\delta(C) < \pi$. The quasi-sectorial C has a decomposition $C = U\mathrm{diag}\{0_{m-r}, C_s\}U^*$, where U is unitary, $C_s \in \mathbb{C}^{r\times r}$ is sectorial, and $r = \mathrm{rank}(C)$ [20]. The phases of C are then defined as the phases of C_s, i.e., $\overline{\phi}(C) = \overline{\phi}(C_s)$, $\underline{\phi}(C) = \underline{\phi}(C_s)$. An example of the numerical range of a quasi-sectorial matrix is shown in Fig. 3(b). The phase interval of a quasi-sectorial C is defined as $\Phi(C) = [\underline{\phi}(C), \overline{\phi}(C)]$.

The phases defined above have many nice properties. One may refer to [20], [22] for more details. Here we list one property that will be useful in later development.

Lemma 1 ([20]): Let $A, B \in \mathbb{C}^{m\times m}$ be quasi-sectorial and sectorial with phase centers $\gamma(A)$ and $\gamma(B)$ respectively. Then AB has $\mathrm{rank}(A)$ nonzero eigenvalues $\lambda_i(AB)$, and $\angle\lambda_i(AB)$ can take values in $(\gamma(A) + \gamma(B) - \pi, \gamma(A) + \gamma(B) + \pi)$. Moreover, $\angle\lambda_i(AB) \in \Phi(A) + \Phi(B)$.

B. Simultaneous alignment and matrix diversity

Let $\mathcal{A} = \{A_i \in \mathbb{C}^{m\times n} : m \leq n, i = 1, \ldots, N\}$. Consider $\mathrm{rank}(A_i)$ as the vitality of A_i.

Definition 1: Let $\alpha \in [0, \frac{\pi}{2})$. The set $\mathcal{A}$ is said to be simultaneously α-alignable if there exists a nonzero matrix $K \in \mathbb{C}^{n\times m}$ such that $\Phi(A_iK) \subset [-\alpha, \alpha]$ and $\mathrm{rank}(A_iK) = \mathrm{rank}(A_i)$ for $i = 1, \ldots N$.

Obviously, if $\mathcal{A}$ is α-alignable for some $\alpha \in [0, \frac{\pi}{2})$, then it is β-alignable for all $\beta \in (\alpha, \frac{\pi}{2})$. The feasibility of the simultaneous alignment problem can be verified and the aligning K can be found by solving a group of linear matrix inequalities (LMIs). Let $\mathrm{Re}\, C = \frac{C+C^*}{2}$ and $\mathrm{Im}\, C = \frac{C-C^*}{2j}$ be the Hermitian part and skew Hermitian part of C respectively. The following lemma can be proved similarly to the case where all matrices in $\mathcal{A}$ are square [19].

Lemma 2: The set $\mathcal{A}$ is simultaneously α-alignable if and only if the following LMIs are feasible

$$\mathrm{Re}(A_iK) \geq A_iA_i^*,$$
$$-\tan\alpha\, \mathrm{Re}(A_iK) \leq \mathrm{Im}(A_iK) \leq \tan\alpha\, \mathrm{Re}(A_iK).$$

The aligning matrix K is not unique, for example, if K is a solution, then kK with $k > 0$ is also a solution. By using the simultaneous alignment, we have the following definition.

Definition 2: Given a matrix set $\mathcal{A}$, let

$$\alpha(\mathcal{A}) = \left\{\alpha \in [0, \frac{\pi}{2}) : \mathcal{A} \text{ is simultaneously } \alpha\text{-alignable}\right\}.$$

The diversity of $\mathcal{A}$ is defined as

$$\mathrm{div}(\mathcal{A}) = \begin{cases} \inf \alpha(\mathcal{A}) & \text{if } \alpha(\mathcal{A}) \text{ is nonempty,} \\ \frac{\pi}{2} & \text{otherwise.} \end{cases}$$

An intriguing aspect of this definition is that diversity is not directly defined by the differences among individuals. Instead, it is defined as how similar they can be made through a common uniform transformation. In this case, the transformation is the right multiplication by K. It is easy to see that if $\tilde{\mathcal{A}}$ is a subset of $\mathcal{A}$, then $\mathrm{div}(\tilde{\mathcal{A}}) \leq \mathrm{div}(\mathcal{A})$.

Example 1:

1) If $\mathcal{A} = \{A\}$, then $\mathrm{div}(\mathcal{A}) = 0$ with an aligning $K = A^\dagger$, where $A^\dagger$ denotes the Moore–Penrose generalized inverse of A.
2) If $A_i = A$ for all i, then $\mathrm{div}(\mathcal{A}) = 0$ with an aligning $K = A^\dagger$.
3) If $A_i = \mu_i A$ with $\mu_i > 0, i = 1, \ldots, N$, then $\mathrm{div}(\mathcal{A}) = 0$ with an aligning $K = A^\dagger$.
4) If A_i are all positive semi-definite, then $\mathrm{div}(\mathcal{A}) = 0$ with an aligning $K = I$.
5) For nonzero matrix A, $\mathrm{div}\{A, -A\} = \pi/2$ with an aligning $K = jA^\dagger$.

One can easily approximate $\mathrm{div}(\mathcal{A})$ with arbitrary precision by a bisection iterative of checking the feasibility of LMIs in Lemma 2.

C. Network symmetry and interaction quality

Given $A \in \mathbb{C}^{m\times m}$, its largest and smallest essential phases are given by

$$\overline{\phi}_{\mathrm{ess}}(A) = \inf_{D\in\mathcal{D}} \overline{\phi}(D^{-1}AD), \quad \underline{\phi}_{\mathrm{ess}}(A) = \sup_{D\in\mathcal{D}} \underline{\phi}(D^{-1}AD),$$

where $\mathcal{D}$ is the set of positive definite diagonal matrices such that $D^{-1}AD$ is quasi-sectorial [20]. The Laplacian of a strongly connected graph is quasi-sectorial if and only if the graph is weight-balanced [20]. For an unbalanced graph, an analytic expression for the essential phase of the Laplacian can be obtained. To be specific, let v be a positive left eigenvector of L corresponding to zero eigenvalue. Let $V = \mathrm{diag}\{v\}$ and $D_o = V^{-1/2}$.

Lemma 3 ([20]): The essential phases of a Laplacian L of a strongly connected graph are given by

$$[\underline{\phi}_{\mathrm{ess}}(L), \overline{\phi}_{\mathrm{ess}}(L)] = [\underline{\phi}(D_o^{-1}LD_o), \overline{\phi}(D_o^{-1}LD_o)] \subset (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Since L is a real matrix, $\underline{\phi}_{\mathrm{ess}}(L) = -\overline{\phi}_{\mathrm{ess}}(L)$. From now on, we denote $\overline{\phi}_{\mathrm{ess}}(L)$ by $\phi_{\mathrm{ess}}(L)$ for notational simplicity. The essential phase provides a measure of the network asymmetry. It holds that $\phi_{\mathrm{ess}}(L) = 0$ if and only if the graph is essentially undirected, i.e., VL is symmetric. We will explain later that in the consensus problem, the more symmetric a network is, the higher the interaction quality is.

IV. MAIN RESULTS

Consensus is essentially a stability problem on a disagreement subspace. We will study the consensus design problem by analyzing the root loci of the poles at the origin.

A. Departure angles of root loci

Let $G(s)$ be an $m \times m$ real rational proper system with state space description:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$. Assume that (A, B) is controllable and (C, A) is observable. Also assume that B and C have full rank.

Consider the feedback system in Fig. 4, where $k > 0$ is the feedback gain. The state matrix of the closed-loop system is given by $A_c(k) = A - kBC$. As k varies from 0 to ∞, the closed-loop poles form n root loci starting from the n open loop poles. It has been shown that for a simple open-loop pole λ, the branch of root loci starting from λ has departure angle equal to $\angle(-v^*BCu)$, where v and u are right and left eigenvectors of A corresponding to λ such that $v^*u = 1$ [23], [24].

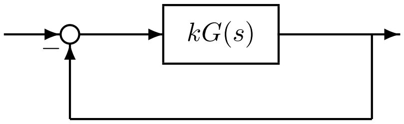


Fig. 4. A feedback system.

Now, let λ be a semi-simple open-loop pole with multiplicity l, i.e., λ is an eigenvalue of A with both algebraic and geometric multiplicities l. Let $U, V \in \mathbb{C}^{n\times l}$ be such that their columns span the right and left eigenspaces of A corresponding to λ and $V^*U = I$.

Lemma 4: If V^*BCU is nonsingular, then the departure angles of the l branches of the root loci starting from λ are $\angle(-\lambda_j(V^*BCU)), j = 1, \dots, l$.

Proof: By matrix perturbation theory [25, Theorem 2.3], for small k, the l eigenvalues of $A_c(k)$ that depart from λ are of the form $\lambda - k\lambda_j(V^*BCU) + o(k^2), j = 1, \dots, l$, from which the desired result follows. ■

It turns out that one can also express the departure angles in terms of the residue matrix. By doing partial fractional expansion, one can write $G(s)$ into the form $G(s) = \frac{R}{s-\lambda} + \tilde{G}(s)$, where R is the residue matrix at λ which has rank l. One can find $X \in \mathbb{C}^{m\times l}$ and $Y \in \mathbb{C}^{l\times m}$ such that $R = XY$. Let $\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]$ be a minimal realization of $\tilde{G}$. Then $G(s)$ has a realization of the form

$$\left[\begin{array}{c|c} \begin{bmatrix} \lambda I_l & \\ & \tilde{A} \end{bmatrix} & \begin{bmatrix} Y \\ \tilde{B} \end{bmatrix} \\ \hline \begin{bmatrix} X & \tilde{C} \end{bmatrix} & 0 \end{array}\right].$$

Suppose 0 is a semi-simple eigenvalue of R. Then

$$\begin{bmatrix} I_l & 0 \end{bmatrix} \begin{bmatrix} Y \\ \tilde{B} \end{bmatrix} \begin{bmatrix} X & \tilde{C} \end{bmatrix} \begin{bmatrix} I_l \\ 0 \end{bmatrix} = YX$$

is nonsingular with eigenvalues being exactly the nonzero eigenvalues of R. Then from Lemma 4, the departure angles of the l root loci starting from λ are given by $\angle(-\lambda_j(R))$, where $\lambda_j(R), j = 1, \dots, l$ are the nonzero eigenvalues of R. This is summarized in the following lemma.

Lemma 5: Let λ be a semi-simple pole of $G(s)$ with multiplicity l. Let $R = \lim_{s\to\lambda}(s - \lambda)G(s)$ be the residue matrix at λ. If 0 is an eigenvalue of R, assume it is semi-simple. Then the departure angles of the l root loci starting from λ are given by $\angle(-\lambda_j(R))$, where $\lambda_j(R), j = 1, \dots, l$ are the nonzero eigenvalues of R.

Lemma 5 suggests that the departure angle of root loci starting from a semi-simple pole depends only on the residue matrix at this pole. We will exploit this result to study the consensus design problem later. Note that Lemma 5 itself is a useful result in analyzing feedback stability.

B. Consensus with desired performance

Let $\mathcal{M} = \{M_1, \dots, M_N\}$. The following theorem provides a solution to the consensus design problem.

Theorem 1: The problem of consensus with arbitrary convergence rate and damping ζ can be achieved if

$$\operatorname{div}\{\mathcal{M}\} + \phi_{\text{ess}}(L) + \arcsin\zeta < \frac{\pi}{2}.$$

Moreover, a state feedback controller is $F = kK$, where $k > 0$ is sufficiently large and K is a solution to the simultaneously $\big(\pi/2 - \phi_{\text{ess}}(L) - \arcsin\zeta\big)$-alignment of $\mathcal{M}$.

Proof: We first transform the consensus problem into a stability problem. Define the average state to be

$$x^{\text{ave}}(t) = \frac{1}{N}(\mathbf{1}_N \otimes I_m)'x(t) \tag{4}$$

and the disagreement state to be

$$x^{\text{dis}}(t) = x(t) - \mathbf{1}_N \otimes x^{\text{ave}}(t). \tag{5}$$

Consensus is achieved if and only if $\lim_{t\to 0} x^{\text{dis}}(t) = 0$ and $x^{\text{ave}}(t)$ converges to a constant vector as $t \to \infty$.

Let $Q \in \mathbb{R}^{N\times(N-1)}$ be a matrix whose columns form a basis of $\text{span}\{\mathbf{1}_N\}^\perp$. Then, $Q'Q = I_{N-1}$, $QQ' = I - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N'$. From (4)-(5), one can compute

$$\begin{aligned} x^{\text{dis}}(t) &= x(t) - \frac{1}{N}\mathbf{1}_N \otimes (\mathbf{1}_N' \otimes I_m)x(t) \\ &= (I - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N') \otimes I_m x(t) = (QQ' \otimes I_m)x(t). \end{aligned} \tag{6}$$

Note that

$$\begin{aligned} I + \boldsymbol{P}(s)(L \otimes F) &= \left(\begin{bmatrix} \frac{\mathbf{1}}{\sqrt{N}} & Q \end{bmatrix} \otimes I_m\right) \\ &\cdot \begin{bmatrix} I & (\frac{\mathbf{1}'}{\sqrt{N}} \otimes I_m)\boldsymbol{P}(s)(LQ \otimes F) \\ 0 & I + (Q' \otimes I_m)\boldsymbol{P}(s)(LQ \otimes F) \end{bmatrix} \left(\begin{bmatrix} \frac{\mathbf{1}}{\sqrt{N}} & Q \end{bmatrix} \otimes I_m\right)'. \end{aligned}$$

Thus, we have

$$\begin{aligned} (I + \boldsymbol{P}(s)(L \otimes F))^{-1} &= \left(\begin{bmatrix} \frac{\mathbf{1}}{\sqrt{N}} & Q \end{bmatrix} \otimes I_m\right) \\ &\cdot \begin{bmatrix} I & -\left(\frac{\mathbf{1}'}{\sqrt{N}} \otimes I_m\right)\boldsymbol{P}(s)(LQ \otimes F)\boldsymbol{S}(s) \\ 0 & \boldsymbol{S}(s) \end{bmatrix} \left(\begin{bmatrix} \frac{\mathbf{1}}{\sqrt{N}} & Q \end{bmatrix}' \otimes I_m\right), \end{aligned}$$

where

$$\boldsymbol{S}(s) = (I + (Q' \otimes I_m)\boldsymbol{P}(s)(LQ \otimes F))^{-1}, \qquad (7)$$

which is the sensitivity function of the feedback system shown in Fig. 5. From (3) and (6) we obtain the dynamics: $x^{\mathrm{dis}}(s) = (Q \otimes I_m)\boldsymbol{S}(s)(Q' \otimes I_m)\boldsymbol{P}(s)w(s)$. Thus, $x^{\mathrm{dis}}(t) \to 0$ if the system shown in Fig. 5 is internally stable.

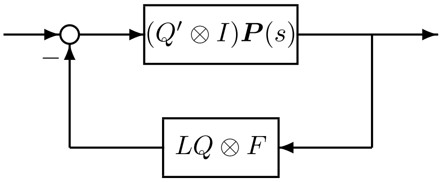


Fig. 5. Stability problem corresponding to consensus problem.

Next we prove the internal stability by analyzing the root locus. Let $\boldsymbol{M} = \mathrm{diag}\{M_1, \dots, M_N\}$. It can be computed from (7) that $\boldsymbol{S}(s) = \left(I + (Q' \otimes I_m)\frac{\boldsymbol{M}}{s}(LQ \otimes kK)\right)^{-1} = s(sI + k(Q' \otimes I_m)\boldsymbol{M}(LQ \otimes K))^{-1}$. Then, the closed-loop poles are the eigenvalues of the matrix $-k(Q' \otimes I_m)\boldsymbol{M}(LQ \otimes K)$, which is exactly the residue matrix of the loop transfer function at pole 0 under the controller. Thus, as k increases from 0 to ∞, all branches of the root loci starting from 0 asymptotically approach infinity along straight rays. When k is sufficiently large, the closed-loop poles are sufficiently far from the imaginary axis, which guarantees the convergence rate. Therefore, the problem reduces to computing the departure angles of these loci at the origin.

Let $R_c = k(Q' \otimes I_m)\boldsymbol{M}(LQ \otimes K)$. From the condition of the theorem,

$$\Phi(M_i K) \subset \Big(-\frac{\pi}{2} + \phi_{\mathrm{ess}}(L) + \arcsin\zeta, \\ \frac{\pi}{2} - \phi_{\mathrm{ess}}(L) - \arcsin\zeta\Big), i = 1, \dots, N.$$

Let v be a positive left eigenvector of L corresponding to 0 eigenvalue and $V = \mathrm{diag}\{v\}$. From Lemma 1, the nonzero eigenvalues of $\mathrm{diag}\{M_1K, \dots, M_NK\}(L \otimes I_m)$ satisfy

$$\begin{aligned} &\angle\lambda(\mathrm{diag}\{M_1K, \dots, M_NK\}(L \otimes I_m)) \\ &= \angle\lambda(\mathrm{diag}\{M_1K, \dots, M_NK\}(V^{-1} \otimes I_m)(VL \otimes I_m)) \\ &\le \overline{\phi}(\mathrm{diag}\{M_1K, \dots, M_NK\}(V^{-1} \otimes I_m)) + \phi_{\mathrm{ess}}(L) \\ &< \pi/2 - \arcsin\zeta = \arccos\zeta. \end{aligned}$$

Similarly, $\angle\lambda(\mathrm{diag}\{M_1K, \dots, M_NK\}(L \otimes I_m)) > -\arccos\zeta$. Note that 0 is a simple eigenvalue of L and the kernel of L is $\mathrm{span}\{\mathbf{1}\}$, which is orthogonal to the range of Q. Thus, R_c is nonsingular and its eigenvalues are the nonzero eigenvalues of $k\mathrm{diag}\{M_1K, \dots, M_NK\}(V^{-1} \otimes I_m)$. Therefore,

$$\angle\lambda_i(R_c) \subset (-\arccos\zeta, \arccos\zeta),$$

for $i = 1, \dots, (N-1)m$. According to Theorem 5, the departure angles of the $(N-1)m$ branches of root loci starting from 0 are equal to $\angle\lambda_i(-R_c)$. The poles of the closed-loop system in Fig. 5 are contained in the region depicted in Fig. 2. The proof is completed. ■

The solvability condition in Theorem 1 characterizes an interplay between agents' diversity, network symmetry, and damping requirement, as illustrated in Fig. 6. A reduction in any one of them allows for greater flexibility in the other two. For example, if the network is essentially undirected, then $\phi_{\mathrm{ess}}(L) = 0$ and the condition reduces to be a tradeoff between agents' diversity and performance requirements.

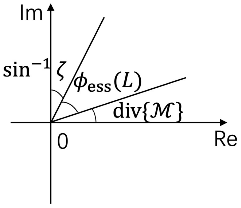


Fig. 6. Illustration of condition in Theorem 1.

In order to check the condition in Theorem 1, one does not need to compute $\mathrm{div}\{\mathcal{M}\}$. Since $\phi_{\mathrm{ess}}(L)$ is given by Lemma 3, it suffices to solve the simultaneous $(\pi/2 - \arcsin\zeta - \phi_{\mathrm{ess}}(L))$-alignment of $\mathcal{M}$. This means one can check the solvability condition and find a uniform state feedback controller simultaneously.

V. SIMULATION

In this section, we use an example to illustrate the main result. Consider a network consisting of four agents. The residue matrices are given by

$$M_1 = \begin{bmatrix} 16 & 0 & 14 \\ 2 & 10 & 18 \\ 6 & 0 & 12 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 12 & 0 & 21 \\ 3 & 15 & 0 \\ 9 & 0 & 18 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 17 & 35 & 14 \\ -1 & 53 & 17 \\ 2 & 23 & 8 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 3 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 10 \end{bmatrix},$$

where M_1 and M_2 are not sectorial, and M_3 and M_4 are sectorial with different phase intervals.

The network is depicted in Fig. 7, of which the corresponding Laplacian has essential phase $\phi_{\mathrm{ess}}(L) = 0.3886$. Let $\zeta = 0.5$. By solving LMIs in Lemma 2, we obtain

$$K = \begin{bmatrix} 2.79 & 0.40 & 0.48 \\ 2.89 & 5.49 & 1.89 \\ 0.88 & 1.58 & 1.49 \end{bmatrix}.$$

The nonzero dominant poles of the closed-loop system are plotted in Fig. 8. A real pole at -2507.8 is not contained in the figure. The trajectories of the agents are shown in Fig. 9.

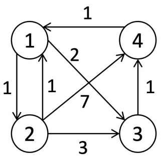


Fig. 7. An example network.

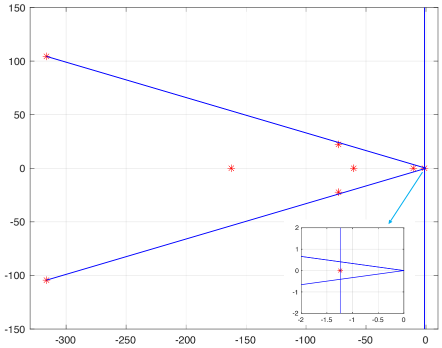


Fig. 8. Nonzero dominant poles of the closed-loop system.

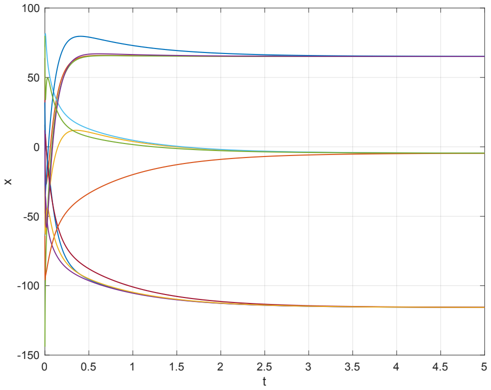


Fig. 9. Trajectories of the agents.

VI. CONCLUSIONS

We have studied the problem of designing a uniform controller for heterogeneous multi-agent systems to achieve consensus with prescribed convergence rate and damping. By exploiting phase theory, we have defined measures of matrix diversity and interaction quality. Based on this, we have derived a solvability condition for the consensus design problem. When this condition holds, we provide a controller which solves the problem.

If the problem is unsolvable, i.e., no uniform controller exists, we propose adopting cluster-based controllers. Specifically, the agents are partitioned into clusters such that agents within the same cluster have similar phases. A common controller is then designed for each cluster, with different controllers applied across clusters. This approach achieves a less conservative design while preserving scalability as much as possible. In the consensus setting, we deal with a single set of residue matrices and thus the agents clustering problem reduces to the matrix clustering problem studied in [26].

The results in this paper can be extended to the synchronization with performance problem, where the agents can have more imaginary-axis poles, and a dynamic controller rather than a static one is needed. Future work includes extending the results to high-order, nonlinear or hybrid agents and addressing more complicated cooperative control problems such as string stability.

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