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Phase of Multivariable Systems: A Revisit via $\mathcal{H}_2^T$-Dissipativity

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***Abstract*— A new notion of phase of multi-input multi-output (MIMO) systems was recently defined and studied, leading to new understandings in various fronts including a formulation of small phase theorem, a performance criterion named $\mathcal{H}_\infty$ phase sector, and a sectored real lemma, etc. In this paper, we define a new notion of $\mathcal{H}_2^T$-dissipativity and show the connection between the phase of a multivariable linear time-invariant (LTI) system and the $\mathcal{H}_2^T$-dissipativity. The $\mathcal{H}_2^T$-dissipativity, roughly speaking, is dissipativity restricted to the time-domain $\mathcal{H}_2$ space which consists of $\mathcal{L}_2$ signals with only positive frequency components. In addition, by exploiting the newly defined $\mathcal{H}_2^T$-dissipativity, we also study the phase of a feedback system and provide a physical interpretation of the sectored real lemma.**

I. Introduction

The concepts of magnitude and phase play equally important roles in classical frequency-domain single-input single-output (SISO) feedback control system theory. The magnitude response and phase response combined together are essential in understanding how the system responds to different types of input signals. Moreover, the notions of gain margin and phase margin are widely used in characterizing closed-loop stability and performance.

As for MIMO LTI systems, the magnitude-based control theory has been well-developed. Particularly, the $\mathcal{H}_\infty$ norm and the well-known small gain theorem lays the foundation of the $\mathcal{H}_\infty$ optimal control theory. However, the development of a phasic counterpart has fallen behind. How to define the phase of a MIMO LTI system has been an unsettled issue for a long time. Several notable attempts along this direction have been made, such as introducing the notion of principal phase that resulted in an early version of small phase theorem [1], extending Bode gain-phase relation to MIMO systems [2]–[4], and developing phase-related notions based on the numerical range [5]–[7].

Recently, a new definition of phase response of frequency-wise semi-sectorial MIMO LTI systems was proposed [8]–[10]. The $\mathcal{H}_\infty$ phase sector, as a counterpart to the $\mathcal{H}_\infty$ norm, was also defined, which measures the range of phase shift between the output signal and input signal over all directions and all nonnegative frequencies. A small phase theorem was developed as a counterpart of the small gain theorem. Also, the phase concept extends the well-known notion of passivity.

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It is well known that both the gain and passivity is closely connected to the dissipativity theory. Dissipativity, a concept emerged in many subjects including thermodynamics, circuit, and mechanical theories, has undergone extensive developments in the control field and evolved into a comprehensive dissipativity theory over the past half century [11]–[13]. It is naturally expected that the phase of a MIMO LTI system, as a sister notion of gain, is also connected to some kind of dissipativity. It is the purpose of this paper to explore such connection and its implications.

In this paper, we define a constrained dissipativity called $\mathcal{H}_2^T$-dissipativity, i.e., dissipativity restricted to time domain $\mathcal{H}_2$ space that consists of $\mathcal{L}_2$ signals containing only positive frequency components. We show that phase bounded systems can be described by such $\mathcal{H}_2^T$-dissipativity. This is done by first showing that the $\mathcal{H}_\infty$ phase sector of a MIMO LTI system coincides with the phase of its associated time domain operator restricted to the invariant subspace $\mathcal{H}_2^T$. Moreover, by using the newly defined $\mathcal{H}_2^T$-dissipativity, the phase of a feedback system is studied. A physical interpretation of the sectored real lemma established in [9] is also provided.

The rest of this paper is organized as follows. Section 2 introduces preliminaries on phases of bounded linear operators and phases of MIMO LTI systems. Time domain interpretation of the $\mathcal{H}_\infty$ phase sector and its connection to $\mathcal{H}_2^T$-dissipativity are established in Section 3. Section 4 discusses the phase of a feedback system and presents an interpretation of the sectored real lemma. Section 5 concludes this paper. The notation used in this paper is more or less standard and will be made clear as we move forward.

II. Preliminaries

In this section, we briefly review some basics of the phases of bounded linear operators and phase response of MIMO LTI systems. One can refer to [8]–[10], [14] for more details.

A. *Phases of bounded linear operators*

Let $\boldsymbol{T}$ be a bounded linear operator on a complex Hilbert space $\mathcal{X}$. The numerical range of $\boldsymbol{T}$ is given by [15]

$$W(\boldsymbol{T}) = \{\langle x, \boldsymbol{T}x\rangle : x \in \mathcal{X}, \|x\| = 1\}. \tag{1}$$

In the finite-dimensional vector space $\mathbb{C}^n$, it reduces to the numerical range of a complex matrix $C \in \mathbb{C}^{n\times n}$ [16]

$$W(C) = \{x^*Cx : x\in\mathbb{C}^n, \|x\| = 1\}.$$

According to the Toeplitz-Hausdorff theorem, $W(\boldsymbol{T})$ is always a convex set [15]. In general $W(\boldsymbol{T})$ may not be closed. An important property of $\boldsymbol{T}$ is that its spectrum is included

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in the closure of its numerical range, i.e., $\sigma(\boldsymbol{T}) \subset \mathrm{cl}\{W(\boldsymbol{T})\}$, where cl denotes closure [15].

If $0 \notin \mathrm{cl}\{W(\boldsymbol{T})\}$, then $\mathrm{cl}\{W(\boldsymbol{T})\}$ is contained in an open half complex plane due to its convexity. In this case, $\boldsymbol{T}$ is said to be a sectorial operator [14]. Given a sectorial $\boldsymbol{T}$, there exist two unique supporting rays of $\mathrm{cl}\{W(\boldsymbol{T})\}$ that subtend an angle less than π at the origin, called the field angle of $\boldsymbol{T}$ and denoted by $\delta(\boldsymbol{T})$. An illustration is in Fig. 1. The angles from the positive real axis to the two supporting rays correspond to the supremum and infimum phases of $\boldsymbol{T}$, denoted by $\overline{\phi}(\boldsymbol{T})$ and $\underline{\phi}(\boldsymbol{T})$ respectively. Mathematically, $\overline{\phi}(\boldsymbol{T})$ and $\underline{\phi}(\boldsymbol{T})$ are defined as

$$\overline{\phi}(\boldsymbol{T}) = \sup_{x\in\mathcal{X}, \|x\|=1} \angle\langle x, \boldsymbol{T}x\rangle, \ \underline{\phi}(\boldsymbol{T}) = \inf_{x\in\mathcal{X}, \|x\|=1} \angle\langle x, \boldsymbol{T}x\rangle$$

such that $\overline{\phi}(\boldsymbol{T}) - \underline{\phi}(\boldsymbol{T}) < \pi$. Note that $\overline{\phi}(\boldsymbol{T})$ and $\underline{\phi}(\boldsymbol{T})$ are not uniquely determined, but are rather determined modulo 2π. After one selects a value of $\gamma(\boldsymbol{T}) = [\underline{\phi}(\boldsymbol{T}) + \overline{\phi}(\boldsymbol{T})]/2$, called the phase center of $\boldsymbol{T}$, in $\mathbb{R}$, then one can uniquely determine the values of $\overline{\phi}(\boldsymbol{T})$ and $\underline{\phi}(\boldsymbol{T})$, respectively. The phases $\overline{\phi}(\boldsymbol{T})$ and $\underline{\phi}(\boldsymbol{T})$ are said to take the principal values if $\gamma(\boldsymbol{T})$ takes the principal value in $[-\pi, \pi)$.

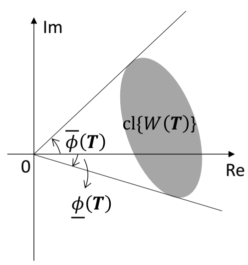


Fig. 1. The phases of a sectorial operator.

Now we extend the phase definition to a broader class of operators. A bounded linear operator $\boldsymbol{T}$ is said to be semi-sectorial if $\mathrm{cl}\{W(\boldsymbol{T})\}$ is contained in a closed half complex plane. Fig. 2 shows the numerical range of a typical semi-sectorial operator. One can see that the origin is on the boundary of $\mathrm{cl}\{W(\boldsymbol{T})\}$. Clearly, positive semidefinite linear operators are semi-sectorial. We define the supremum and infimum phases of a semi-sectorial $\boldsymbol{T}$ as

$$\begin{aligned}\overline{\phi}(\boldsymbol{T}) &= \sup_{x\in\mathcal{X}, \|x\|=1, \langle x, \boldsymbol{T}x\rangle \neq 0} \angle\langle x, \boldsymbol{T}x\rangle \\ \underline{\phi}(\boldsymbol{T}) &= \inf_{x\in\mathcal{X}, \|x\|=1, \langle x, \boldsymbol{T}x\rangle \neq 0} \angle\langle x, \boldsymbol{T}x\rangle.\end{aligned}$$

We view the set of sectorial operators as a subset of that of semi-sectorial operators.

For operators on $\mathbb{C}^n$, the supremum and infimum phases defined above coincide with those of (semi-)sectorial matrices [9], [17]–[19].

B. Phases of MIMO LTI systems

Here we review the phase response of MIMO LTI systems recently developed in [8]–[10]. Let G be an $m \times m$ real rational proper stable transfer matrix, i.e., $G \in \mathcal{RH}_\infty^{m\times m}$. Then, G is said to be frequency-wise semi-sectorial if

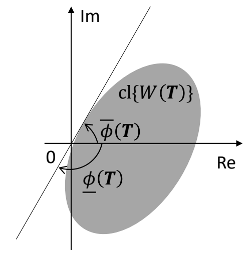


Fig. 2. The phases of a semi-sectorial operator.

1) $G(j\omega)$ is semi-sectorial for all $\omega \in [-\infty, \infty]$;
2) there exists an $\epsilon^* > 0$ such that for all $\epsilon \leq \epsilon^*$, $G(s)$ has a constant rank and is semi-sectorial along the indented imaginary axis shown in Fig. 3, where the half-circle detours with radius ϵ are taken at the finite zeros of $G(s)$ at the frequency axis and a half-circle detour with radius $1/\epsilon$ is taken if infinity is a zero of $G(s)$.

Furthermore, $G \in \mathcal{RH}_\infty^{m\times m}$ is said to be frequency-wise sectorial if $G(j\omega)$ is sectorial for all $\omega \in [-\infty, \infty]$. Clearly, a frequency-wise sectorial system does not have transmission zeros on the imaginary axis.

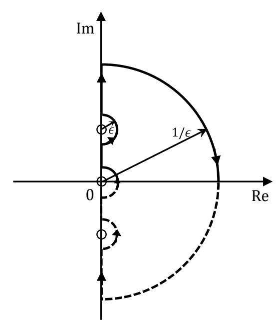


Fig. 3. Indented $j\omega$-axis: "o" denote the $j\omega$-axis zeros.

Let G be frequency-wise semi-sectorial. If 0 is not a zero of G, then $G(0)$ can be accretive, i.e., $G(0) + G^*(0) \geq 0$, anti-accretive, or indefinite Hermitian. For simplicity, we assume in this paper that $G(0)$ is accretive such that the DC phase center $\gamma(G(0)) = 0$. We let $\gamma(G(s))$ be defined continuously along the indented imaginary axis. If 0 is a zero of G, we look at $G(\epsilon)$ instead. Again, we assume that $G(\epsilon)$ is accretive and define $\gamma(G(s))$ continuously along the indented imaginary axis. With this construction, $\phi(G(s))$ is odd with respect to the indented imaginary axis. Hence we can focus on the part of the phase response corresponding to the upper half of the indented imaginary axis.

For frequency-wise semi-sectorial G, its maximum and minimum phases are defined to be

$$\overline{\phi}(G) = \sup_{\omega\in[0,\infty]} \overline{\phi}(G(j\omega)), \quad \underline{\phi}(G) = \inf_{\omega\in[0,\infty]} \underline{\phi}(G(j\omega))$$

and its $\mathcal{H}_\infty$ phase sector, also called Φ_∞ sector, is defined to be

$$\Phi_\infty(G) = [\underline{\phi}(G), \overline{\phi}(G)].$$

We say $G \in \mathcal{RH}_\infty$ is semi-sectorial if $\Phi_\infty(G) \subset [\alpha, \alpha+\pi]$ for some α and it is sectorial if $\Phi_\infty(G) \subset (\alpha, \beta)$ with $0 <$

$\beta - \alpha \leq \pi$. Note that the twisted positive-real system with twisting angle ψ defined in [20] is a SISO sectrorial system with Φ_∞ sector $(\psi - \frac{\pi}{2}, \psi + \frac{\pi}{2})$.

III. Phase and $\mathcal{H}_2^T$-Dissipativity

A system $G \in \mathcal{RH}_\infty^{m\times m}$ corresponds in time domain to a bounded linear causal operator $\boldsymbol{G}$ mapping an input signal space to an output signal space. It is well-known that the $\mathcal{H}_\infty$ norm of G coincides with the operator norm of $\boldsymbol{G}$ [21]. In this section, we will show that similar things can be said for Φ_∞ sector.

Denote by $\mathcal{L}_2^T(-\infty,\infty)$ the Hilbert space of complex-valued bilateral time functions. Recall that Fourier transform on $\mathcal{L}_2^T(-\infty,\infty)$ is an isometry onto $\mathcal{L}_2^\Omega(-\infty,\infty)$, i.e., the Hilbert space of complex-valued bilateral frequency functions with the inner product

$$\langle u(j\omega), v(j\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(j\omega)v(j\omega)d\omega$$

for $u(j\omega), v(j\omega) \in \mathcal{L}_2^\Omega(-\infty,\infty)$. Decompose $\mathcal{L}_2^\Omega(-\infty,\infty)$ into

$$\mathcal{L}_2^\Omega(-\infty,\infty) = \mathcal{L}_2^\Omega(0,\infty) \oplus \mathcal{L}_2^\Omega(-\infty,0).$$

This is clearly an orthogonal decomposition, which in turn leads to a natural orthogonal decomposition in $\mathcal{L}_2^T(-\infty,\infty)$

$$\mathcal{L}_2^T(-\infty,\infty) = \boldsymbol{F}^{-1}\mathcal{L}_2^\Omega(0,\infty) \oplus \boldsymbol{F}^{-1}\mathcal{L}_2^\Omega(-\infty,0),$$

where $\boldsymbol{F}$ denotes the Fourier transform and $\boldsymbol{F}^{-1}$ denotes the inverse Fourier transform.

The time functions in the first subspace contain only positive frequency component and can be analytically extended to the upper half complex plane [22], i.e., the first subspace is the $\mathcal{H}_2$ space in time domain. Correspondingly, the time functions in the second subspace contain only negative frequency component and can be analytically extended to the lower half complex plane. We denote the first subspace by $\mathcal{H}_2^T$ and consequently the second subspace by $\mathcal{H}_2^{T^\perp}$. Denote by $\boldsymbol{P}$ the orthogonal projection onto $\mathcal{L}_2^\Omega(0,\infty)$ and $\boldsymbol{Q}$ be the orthogonal projection onto $\mathcal{H}_2^T$. A full picture of the relationships among these signal spaces is summarized in the commutative diagram in Fig. 4.

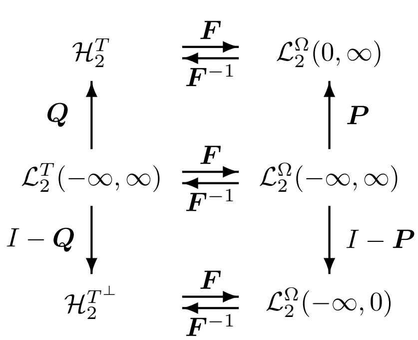


Fig. 4. A commutative diagram.

Now let $\boldsymbol{G}: \mathcal{L}_2^T(-\infty,\infty) \to \mathcal{L}_2^T(-\infty,\infty)$ be the bounded linear operator corresponding to $G(s) \in \mathcal{RH}_\infty$. Since $\mathcal{H}_2^T$ is an invariant subspace of $\boldsymbol{G}$, the restriction of $\boldsymbol{G}$ to $\mathcal{H}_2^T$, denoted by $\boldsymbol{G}|_{\mathcal{H}_2^T}$, is a bounded linear operator on $\mathcal{H}_2^T$.

The following theorem provides a connection between $\Phi_\infty(G)$ defined in frequency domain and the phases of $\boldsymbol{G}|_{\mathcal{H}_2^T}$ defined in time domain.

Theorem 1: For a semi-sectorial G, it holds that

$$\Phi_\infty(G) = \left[\underline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^T}), \overline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^T})\right].$$

Proof: The result follows from the fact that

$$\mathrm{cl}\,\mathrm{conv}\,\{W(G(j\omega)) : \omega \in [0,\infty]\} = \mathrm{cl}\left\{W(\boldsymbol{G}|_{\mathcal{H}_2^T})\right\},$$

which was shown in [9]. ■

We wish to emphasize the advantage of considering $\boldsymbol{G}|_{\mathcal{H}_2^T}$ instead of $\boldsymbol{G}$. For a real system, $W(\boldsymbol{G})$ is symmetric over the real axis. This means that if $\boldsymbol{G}$ is required to be semi-sectorial, then $\mathrm{cl}\{W(\boldsymbol{G})\}$ is contained in either the right half or the left half complex plane, and $[\underline{\phi}(\boldsymbol{G}), \overline{\phi}(\boldsymbol{G})]$ is a symmetric interval and could only be a subset of $[-\pi/2, \pi/2]$ or $[-3\pi/2, -\pi/2]$. This limits the applicability of phase theory. Nevertheless, considering $\boldsymbol{G}|_{\mathcal{H}_2^T}$ can diminish this issue, as $\mathcal{H}_2^T$ contains signals with only positive frequency components and $W(\boldsymbol{G}|_{\mathcal{H}_2^T})$ is not necessarily symmetric with respect to the real axis. When $\boldsymbol{G}|_{\mathcal{H}_2^T}$ is semi-sectorial, $[\underline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^T}), \overline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^T})]$ is possibly asymmetric and could be any subset of $[-3\pi/2, 3\pi/2]$ with length no larger than π and containing 0 or $-\pi$.

Note that such understanding for phase delivered by Theorem 1 is in parallel with the understanding for gain. To be specific, it is known that for system $G \in \mathcal{RH}_\infty$, it holds that $\|G\|_\infty = \|\boldsymbol{G}\|$. In fact, there holds further that

$$\|G\|_\infty = \|\boldsymbol{G}\| = \|\boldsymbol{G}|_{\mathcal{H}_2^T}\|,$$

where the second equality is due to the conjugate symmetry of the LTI system, i.e., $G(-j\omega) = \overline{G(j\omega)}$.

With Theorem 1, we are able to establish the connection between the Φ_∞ sector of LTI systems and the concept of dissipativity. A stable system with associated bounded operator $\boldsymbol{G}$ is said to be ultimately dissipative with respect to the (Q,S,R)-supply rate, where Q and R are Hermitian matrices, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} dt \geq 0 \qquad (2)$$

for all $u \in \mathcal{L}_2^T(-\infty,\infty)$ [11].

In the classical definition of dissipativity, it is required that the inequality (2) holds for all $u \in \mathcal{L}_2^T(-\infty,\infty)$. Now we introduce a new notion called $\mathcal{H}_2^T$-dissipativity. A stable system is said to be ultimately $\mathcal{H}_2^T$-dissipative with respect to the (Q,S,R)-supply rate if the dissipation inequality (2) holds for all $u \in \mathcal{H}_2^T$. The $\mathcal{H}_2^T$-dissipativity can be viewed as a constrained dissipativity where u is restricted to be in the subspace $\mathcal{H}_2^T$ in the dissipation inequality.

The next theorem builds the connection between $\Phi_\infty(G)$ and the ultimate $\mathcal{H}_2^T$-dissipativity for sectorial system G.

Theorem 2: Let G be sectorial. Then $\Phi_\infty(G) \subset (\alpha,\beta)$ for $0 < \beta - \alpha \leq \pi$ if and only if $\boldsymbol{G}$ is ultimately $\mathcal{H}_2^T$-dissipative with respect to the $(-\epsilon I, e^{j(\alpha+\frac{\pi}{2})}I, -\delta I)$-supply rate and $(-\epsilon I, e^{j(\beta-\frac{\pi}{2})}I, -\delta I)$-supply rate for some $\epsilon, \delta > 0$.

Proof: According to Theorem 1, $\Phi_\infty(G) \subset (\alpha, \beta)$ is equivalent to

$$\left[\underline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^T}), \overline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^T})\right] \subset (\alpha, \beta). \quad (3)$$

From the definition of phases of $\boldsymbol{G}|_{\mathcal{H}_2^T}$, we know that (3) holds if and only if for all $u \in \mathcal{H}_2^T, u \neq 0$, there holds

$$\alpha < \angle \frac{\langle u, \boldsymbol{G}|_{\mathcal{H}_2^T} u\rangle}{\|u\|_2^2} = \angle \frac{\langle u, \boldsymbol{G}u\rangle}{\|u\|_2^2} < \beta,$$

which is equivalent to that there exists $\delta > 0$ such that

$$\operatorname{Re}\left\{e^{-j(\frac{\pi}{2}+\alpha)} \frac{\langle u, \boldsymbol{G}u\rangle}{\|u\|_2^2}\right\} \geq \delta, \quad (4)$$

$$\operatorname{Re}\left\{e^{-j(\beta-\frac{\pi}{2})} \frac{\langle u, \boldsymbol{G}u\rangle}{\|u\|_2^2}\right\} \geq \delta. \quad (5)$$

Let $\sigma = \|G\|_\infty^2$. Then $\|\boldsymbol{G}u\|_2^2 \leq \sigma \|u\|_2^2$. Let $\epsilon = \frac{\delta}{\sigma}$. Then

$$\delta\|u\|_2^2 = \frac{\delta}{2\sigma}\sigma\|u\|_2^2 + \frac{\delta}{2}\|u\|_2^2 \geq \frac{\epsilon}{2}\|\boldsymbol{G}u\|_2^2 + \frac{\delta}{2}\|u\|_2^2.$$

Combined with inequalities (4) and (5), we have

$$\operatorname{Re}\left\{e^{-j(\frac{\pi}{2}+\alpha)}\langle u, \boldsymbol{G}u\rangle\right\} \geq \frac{\epsilon}{2}\|\boldsymbol{G}u\|_2^2 + \frac{\delta}{2}\|u\|_2^2,$$
$$\operatorname{Re}\left\{e^{-j(\beta-\frac{\pi}{2})}\langle u, \boldsymbol{G}u\rangle\right\} \geq \frac{\epsilon}{2}\|\boldsymbol{G}u\|_2^2 + \frac{\delta}{2}\|u\|_2^2.$$

These two inequalities can be rewritten as

$$\int_{-\infty}^{\infty} \begin{bmatrix}\boldsymbol{G}u \\ u\end{bmatrix}^* \begin{bmatrix} -\epsilon I & e^{j(\alpha+\frac{\pi}{2})I} \\ e^{-j(\alpha+\frac{\pi}{2})}I & -\delta I\end{bmatrix} \begin{bmatrix}\boldsymbol{G}u \\ u\end{bmatrix} dt \geq 0, \quad (6)$$

$$\int_{-\infty}^{\infty} \begin{bmatrix}\boldsymbol{G}u \\ u\end{bmatrix}^* \begin{bmatrix} -\epsilon I & e^{j(\beta-\frac{\pi}{2})I} \\ e^{-j(\beta-\frac{\pi}{2})}I & -\delta I\end{bmatrix} \begin{bmatrix}\boldsymbol{G}u \\ u\end{bmatrix} dt \geq 0, \quad (7)$$

which completes the proof. ■

Such connection between Φ_∞ sector and $\mathcal{H}_2^T$-dissipativity can be extended to semi-sectorial systems. See the theorem below. The proof is similar to Theorem 2 and is omitted for brevity.

Theorem 3: Let G be semi-sectorial. Then $\Phi_\infty(G) \subset [\alpha, \alpha+\pi]$ if and only if $\boldsymbol{G}$ is ultimately $\mathcal{H}_2^T$-dissipative with respect to the $(0, e^{j(\alpha+\frac{\pi}{2})}I, 0)$-supply rate.

Theorems 2 and 3 convey a message that in general the phase of an LTI system is connected to $\mathcal{H}_2^T$-dissipativity rather than the classical dissipativity. This is in fact very intuitive as $\Phi_\infty(G)$ concerns only the phase over the positive frequency, thereby only signals in the subspace $\mathcal{H}_2^T$ need to be considered in the dissipation inequality.

Nevertheless, we wish to mention that in the special case where $\alpha = -\beta$ in Theorem 2, a sectorial G with $\Phi_\infty(G) \subset (-\beta, \beta)$ is in fact connected to the classical dissipativity. See the following theorem.

Theorem 4: Let G be sectorial. Then $\Phi_\infty(G) \subset (-\beta, \beta)$, $\beta \in (0, \frac{\pi}{2}]$ if and only if $\boldsymbol{G}$ is ultimately dissipative with respect to the $(-\epsilon I, e^{j(\beta-\frac{\pi}{2})}I, -\delta I)$-supply rate.

Proof: We first show the sufficiency. If $\boldsymbol{G}$ is ultimately dissipative with the $(-\epsilon I, e^{j(\beta-\frac{\pi}{2})}I, -\delta I)$-supply rate, then it is ultimately $\mathcal{H}_2^T$-dissipative with the same supply rate. This means that (7) holds for all $u \in \mathcal{H}_2^T, u \neq 0$. Thus,

$$\operatorname{Re}\left\{e^{-j(\beta-\frac{\pi}{2})}\langle u, \boldsymbol{G}|_{\mathcal{H}_2^T} u\rangle\right\} = \operatorname{Re}\left\{e^{-j(\beta-\frac{\pi}{2})}\langle u, \boldsymbol{G}u\rangle\right\}$$
$$\geq \frac{\epsilon}{2}\|\boldsymbol{G}u\|_2^2 + \frac{\delta}{2}\|u\|_2^2 \geq \frac{\delta}{2}\|u\|_2^2.$$

Then we have $-\beta < \angle \frac{\langle u, \boldsymbol{G}|_{\mathcal{H}_2^T} u\rangle}{\|u\|_2^2} < \beta$. In view of Theorem 1, G is sectorial and $\Phi_\infty(G) \subset (-\beta, \beta)$.

Next we show the necessity. Since $\Phi_\infty(G) \subset (-\beta, \beta)$, from Theorem 2, $\boldsymbol{G}$ is ultimately $\mathcal{H}_2^T$-dissipative with the $(-\epsilon I, e^{j(\beta-\frac{\pi}{2})}I, -\delta I)$-supply rate. Then, it suffices to show $\boldsymbol{G}$ is ultimately $\mathcal{H}_2^{T^\perp}$-dissipative with the same supply rate, i.e., (7) holds for all $u \in \mathcal{H}_2^{T^\perp}, u \neq 0$. In view of the conjugate symmetry of G, we know that $W(G(j\omega))$ and $W(G(-j\omega))$ are symmetric over the real axis. Thus,

$$\left[\inf_{\omega\in[-\infty,0]} \underline{\phi}(G(j\omega)), \sup_{\omega\in[-\infty,0]} \overline{\phi}(G(j\omega))\right] \subset (-\beta, \beta). \quad (8)$$

On the other hand, by applying the same techniques used in [9, Proposition 5.1], it can be shown that

$$\operatorname{cl}\operatorname{conv}\{W(G(j\omega)) : \omega \in [-\infty, 0]\} = \operatorname{cl}\left\{W(\boldsymbol{G}|_{\mathcal{H}_2^{T^\perp}})\right\}.$$

Together with (8), we have $\left[\underline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^{T^\perp}}), \overline{\phi}(\boldsymbol{G}|_{\mathcal{H}_2^{T^\perp}})\right] \subset (-\beta, \beta)$. Hence, similar to Theorem 2, it can be shown that (7) holds for all $u \in \mathcal{H}_2^{T^\perp}, u \neq 0$. The proof is complete. ■

When $\beta = \frac{\pi}{2}$, Theorem 4 reduces to the known fact that a stable G is very strictly passive if and only if it is ultimately dissipative with respect to the $(-\epsilon I, I, -\delta I)$-supply rate [23].

IV. Implications and Discussions

A. Phases of feedback systems

By exploiting the $\mathcal{H}_2^T$-dissipativity introduced in the last section, we study the phase of feedback systems.

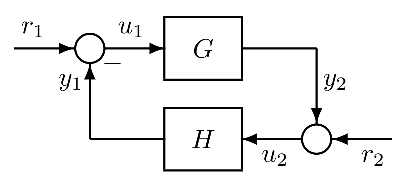


Fig. 5. A standard feedback system.

Consider the feedback interconnections of G and H shown in Fig. 5. Denote by $S = (I + GH)^{-1}$ the sensitivity function. Then we have the following result.

Theorem 5: Let G, H be sectorial with $\Phi_\infty(G) \subset (\alpha, \beta)$ and $\Phi_\infty(H) \subset (-\beta, -\alpha)$ with $\beta - \alpha \leq \pi$ respectively. Then $\Phi_\infty(SG) \subset (\alpha, \beta)$ and $\Phi_\infty(HS) \subset (-\beta, -\alpha)$.

Proof: We will show that $\Phi_\infty(SG) \subset (\alpha, \beta)$. The validity of $\Phi_\infty(H) \subset (-\beta, -\alpha)$ can be shown similarly.

Let $r_2 = 0$ in Fig. 5. Then $u_2 = y_2$. Next we show that $\underline{\phi}(SG) > \alpha$. Since $\underline{\phi}(G) > \alpha$ and $\overline{\phi}(H) < -\alpha$, from Theorem 2, we have

$$e^{j(\frac{\pi}{2}+\alpha)}\langle y_2, u_1\rangle + e^{-j(\frac{\pi}{2}+\alpha)}\langle u_1, y_2\rangle \geq \epsilon_1\|y_2\|_2^2 + \delta_1\|u_1\|_2^2,$$
$$e^{j(-\alpha-\frac{\pi}{2})}\langle y_1, y_2\rangle + e^{j(\alpha+\frac{\pi}{2})}\langle y_2, y_1\rangle \geq \epsilon_2\|y_1\|_2^2 + \delta_2\|y_2\|_2^2,$$

for some $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$. Adding both sides of these two inequalities yields

$$e^{j(\frac{\pi}{2}+\alpha)}\langle y_2, u_1 + y_1\rangle + e^{-j(\frac{\pi}{2}+\alpha)}\langle u_1 + y_1, y_2\rangle \geq \epsilon_1\|y_2\|_2^2 + \delta_1\|u_1\|_2^2 + \epsilon_2\|y_1\|_2^2 + \delta_2\|y_2\|_2^2. \quad (9)$$

Let $\delta = \min\{\delta_1, \epsilon_2\}$. Then, due to the parallelogram law of norm, we have

$$\delta_1\|u_1\|_2^2 + \epsilon_2\|y_1\|_2^2 \geq \delta(\|u_1\|_2^2 + \|y_1\|_2^2) = \frac{\delta}{2}(\|u_1+y_1\|_2^2+\|u_1-y_1\|_2^2) \geq \frac{\delta}{2}\|u_1+y_1\|_2^2 = \frac{\delta}{2}\|r_1\|_2^2.$$

Let $\epsilon = \epsilon_1 + \delta_2$. In view of (9), it holds

$$e^{j(\frac{\pi}{2}+\alpha)}\langle y_2, r_1\rangle + e^{-j(\frac{\pi}{2}+\alpha)}\langle r_1, y_2\rangle \geq \epsilon\|y_2\|_2^2 + \frac{\delta}{2}\|r_1\|_2^2.$$

Again, by Theorem 2, it follows that $\underline{\phi}(SG) > \alpha$. Similarly, it can be shown that $\overline{\phi}(SG) < \beta$. The proof is complete. ■

When $\alpha = -\beta$, Theorem 5 becomes that if both $\Phi_\infty(G)$ and $\Phi_\infty(H)$ are subsets of $(-\beta, \beta)$, then $\Phi_\infty(SG)$ and $\Phi_\infty(HS)$ are also subsets of $(-\beta, \beta)$, which means that phase is preserved under the feedback interconnection. Further, when $\beta = \pi/2$, Theorem 5 reduces to a known fact that if both G and H are strongly positive real, then SG and HS are strongly positive real as well [13].

Example 1: Let

$$G = \begin{bmatrix} \frac{9s^3+17s^2+23s+5}{2s^3+15s^2+19s+6} & \frac{4s^2+s+2}{s^2+7s+6} \\ \frac{4s^2+2s+1}{s^2+7s+6} & \frac{4s^2+s+1}{s^2+7s+6} \end{bmatrix},$$

$$H = \begin{bmatrix} \frac{s^3+3s^2+9s+13}{s^3+5s^2+5s+4} & \frac{s^3+3s^2+15s+25}{s^3+5s^2+5s+4} \\ \frac{s^3+4s^2+16s+24}{s^3+5s^2+5s+4} & \frac{s^3+6s^2+30s+49}{s^3+5s^2+5s+4} \end{bmatrix}.$$

We can see from Fig. 6 (a)-(b) that $\Phi_\infty(G) \subset (-20°, 115°)$ and $\Phi_\infty(H) \subset (-115°, 20°)$. By Theorem 5, we should have $\Phi_\infty(SG) \subset (-20°, 115°)$ and $\Phi_\infty(HS) \subset (-115°, 20°)$. This is indeed the case, as is shown in Fig. 6 (c)-(d).

B. Interpretation of sectored real lemma via $\mathcal{H}_2^T$-dissipativity

A sectored real lemma was devised in [9], which provides a state-space characterization of a sectorial system G satisfying $\Phi_\infty(G) \subset (\alpha, \beta)$ with $0 < \beta - \alpha \leq \pi$. Therein the sectored real lemma was obtained by applying the generalized KYP lemma, which connects frequency domain inequalities to time domain linear matrix inequalities. Here we will give an interpretation of the sectored real lemma by exploiting the connections between Φ_∞ sector and $\mathcal{H}_2^T$-dissipativity.

Lemma 1 ([9]): Let G be sectorial with a minimal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then $\Phi_\infty(G) \subset (\alpha, \beta)$, $0 < \beta - \alpha \leq \pi$, if and only if there exist Hermitian matrices X_i, Y_i, $i = 1, 2$, such that $Y_i \geq 0$ and

$$\begin{bmatrix} A^T(X_i+jY_i)+(X_i-jY_i)A & (X_i-jY_i)B \\ B^T(X_i+jY_i) & 0 \end{bmatrix} - M_i < 0, \quad (10)$$

where

$$M_1 = \begin{bmatrix} 0 & e^{j(\alpha+\frac{\pi}{2})}C^T \\ e^{-j(\alpha+\frac{\pi}{2})}C & e^{-j(\alpha+\frac{\pi}{2})}D + e^{j(\alpha+\frac{\pi}{2})}D^T \end{bmatrix}, \quad (11)$$

$$M_2 = \begin{bmatrix} 0 & e^{j(\beta-\frac{\pi}{2})}C^T \\ e^{-j(\beta-\frac{\pi}{2})}C & e^{-j(\beta-\frac{\pi}{2})}D + e^{j(\beta-\frac{\pi}{2})}D^T \end{bmatrix}. \quad (12)$$

A physical interpretation of the sectored real lemma is given below. Assume $u(t)$ is an arbitrary signal in $\mathcal{H}_2^T$. Since $\mathcal{H}_2^T$ is an invariant subspace of $\boldsymbol{G}$, it follows that both $x(t)$ and $y(t)$ are in $\mathcal{H}_2^T$ as well. In view of (10), there exists $\epsilon > 0$ such that

$$\begin{bmatrix} A^T(X_i+jY_i)+(X_i-jY_i)A & (X_i-jY_i)B \\ B^T(X_i+jY_i) & 0 \end{bmatrix} - M_i \leq -\epsilon I.$$

Left-multiplying and right-multiplying both sides of the above inequality by $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^*$ and $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ and integrating both sides yields

$$\int_{-\infty}^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* M_i \begin{bmatrix} x \\ u \end{bmatrix} dt \geq \epsilon(\|u\|_2^2 + \|x\|_2^2) + \int_{-\infty}^{\infty} (x^* X_i \dot{x} + \dot{x}^* X_i x + j(\dot{x}^* Y_i x - x^* Y_i \dot{x}))\, dt.$$

Since G is stable, $x(-\infty) = x(\infty) = 0$, and thus

$$\int_{-\infty}^{\infty} (x^* X_i \dot{x} + \dot{x}^* X_i x) dt = x^* X_i x|_{-\infty}^{\infty} = 0.$$

Therefore, we have

$$\int_{-\infty}^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* M_i \begin{bmatrix} x \\ u \end{bmatrix} dt \geq \epsilon(\|u\|_2^2 + \|x\|_2^2) + \int_{-\infty}^{\infty} j(\dot{x}^* Y_i x - x^* Y_i \dot{x}) dt = \epsilon(\|u\|_2^2 + \|x\|_2^2) + \int_{-\infty}^{\infty} 2wX^*(j\omega) Y_i X(j\omega) d\omega,$$

where $X(j\omega)$ is the Fourier transform of $x(t)$. Since $x(t) \in \mathcal{H}_2^T$, it follows that $X(j\omega) = 0$ for all $\omega < 0$. Therefore,

$$\int_{-\infty}^{\infty} 2wX^*(j\omega) Y_i X(j\omega) d\omega = \int_{0}^{\infty} 2wX^*(j\omega) Y_i X(j\omega) d\omega \geq 0,$$

and then we have

$$\int_{-\infty}^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* M_i \begin{bmatrix} x \\ u \end{bmatrix} dt \geq \epsilon(\|u\|_2^2 + \|x\|_2^2).$$

Since G is stable, there exists $\delta > 0$ such that

$$\int_{-\infty}^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* M_i \begin{bmatrix} x \\ u \end{bmatrix} dt \geq 2\delta(\|u\|_2^2 + \|y\|_2^2).$$

Substituting M_i with (11) and (12), we obtain the inequalities (6) and (7) for $\delta > 0$. In view of Theorem 2, we know G is sectorial with $\Phi_\infty(G) \subset (\alpha, \beta)$.

V. Conclusion

In this paper, we defined the notion of $\mathcal{H}_2^T$-dissipativity and built its connection to Φ_∞ sector of a stable phase-bounded system, which was shown coinciding with the phases of the operator associated with the system being restricted to $\mathcal{H}_2^T$. By exploiting the $\mathcal{H}_2^T$-dissipativity, we studied the phase of a feedback system and provided a physical interpretation of the sectored real lemma.

In the future, we will extend the $\mathcal{H}_2^T$-dissipativity to unstable systems, especially semi-stable systems, and establish its connections to phase-bounded systems. We will also explore connections between our work and other phase-related works, e.g., scaled relative graph [24], incremental dissipativity [25], etc.

REFERENCES

[1] I. Postlethwaite, J. Edmunds, and A. G. MacFarlane, “Principal gains and principal phases in the analysis of linear multivariable feedback systems,” *IEEE Trans. Autom. Control*, vol. 26, no. 1, pp. 32–46, 1981.

[2] J. S. Freudenberg and D. P. Looze, *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*. Lecture Notes in Control and Information Sciences, vol. 104, Berlin, Germany: Springer-Verlag, 1988.

[3] B. D. Anderson and M. Green, “Hilbert transform and gain/phase error bounds for rational functions,” *IEEE Trans. Circuits Syst.*, vol. 35, no. 5, pp. 528–535, 1988.

[4] J. Chen, “Multivariable gain-phase and sensitivity integral relations and design tradeoffs,” *IEEE Trans. Autom. Control*, vol. 43, no. 3, pp. 373–385, 1998.

[5] D. H. Owens, “The numerical range: A tool for robust stability studies?,” *Syst. Control Lett.*, vol. 5, no. 3, pp. 153–158, 1984.

[6] A. L. Tits, V. Balakrishnan, and L. Lee, “Robustness under bounded uncertainty with phase information,” *IEEE Trans. Autom. Control*, vol. 44, no. 1, pp. 50–65, 1999.

[7] K. Laib, A. Korniienko, M. Dinh, G. Scorletti, and F. Morel, “Hierarchical robust performance analysis of uncertain large scale systems,” *IEEE Trans. Autom. Control*, vol. 63, no. 7, pp. 2075–2090, 2018.

[8] W. Chen, D. Wang, S. Z. Khong, and L. Qiu, “Phase analysis of MIMO LTI systems,” in *Proc. 58th IEEE Conf. Decis. Control*, pp. 6062–6067, 2019.

[9] W. Chen, D. Wang, S. Z. Khong, and L. Qiu, “A phase theory of MIMO LTI systems,” *arXiv preprint arXiv:2105.03630v2*, 2021.

[10] X. Mao, W. Chen, and L. Qiu, “Phases of discrete-time LTI multivariable systems,” *Automatica*, vol. 142, p. 110311, 2022.

[11] D. J. Hill and P. J. Moylan, “Dissipative dynamical systems: Basic input-output and state properties,” *J. Franklin Inst.*, vol. 309, no. 5, pp. 327–357, 1980.

[12] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*. Springer Nature Switzerland AG, 3rd ed., 2020.

[13] J. Bao and P. L. Lee, *Process Control: The Passive Systems Approach*. Springer, 2007.

[14] W. Chen, “Phase of linear time-periodic systems,” *Automatica*, vol. 151, p. 110925, 2023.

[15] K. E. Gustafson and D. K. M. Rao, *Numerical Range*. Springer-Verlag New York, 1997.

[16] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[17] D. Wang, W. Chen, S. Z. Khong, and L. Qiu, “On the phases of a complex matrix,” *Linear Algebra Appl.*, vol. 593, pp. 152–179, 2020.

[18] S. Furtado and C. R. Johnson, “Spectral variation under congruence,” *Linear Multilinear Algebra*, vol. 49, pp. 243–259, 2001.

[19] S. Furtado and C. R. Johnson, “Spectral variation under congruence for a nonsingular matrix with 0 on the boundary of its field of values,” *Linear Algebra Appl.*, vol. 359, pp. 67–78, 2003.

[20] V. Bhus, J. Lin, and G. Weiss, “The modular active capacitor for high power ripple attenuation,” *CPSS Transactions on Power Electronics and Applications*, vol. 6, no. 3, pp. 251–262, 2021.

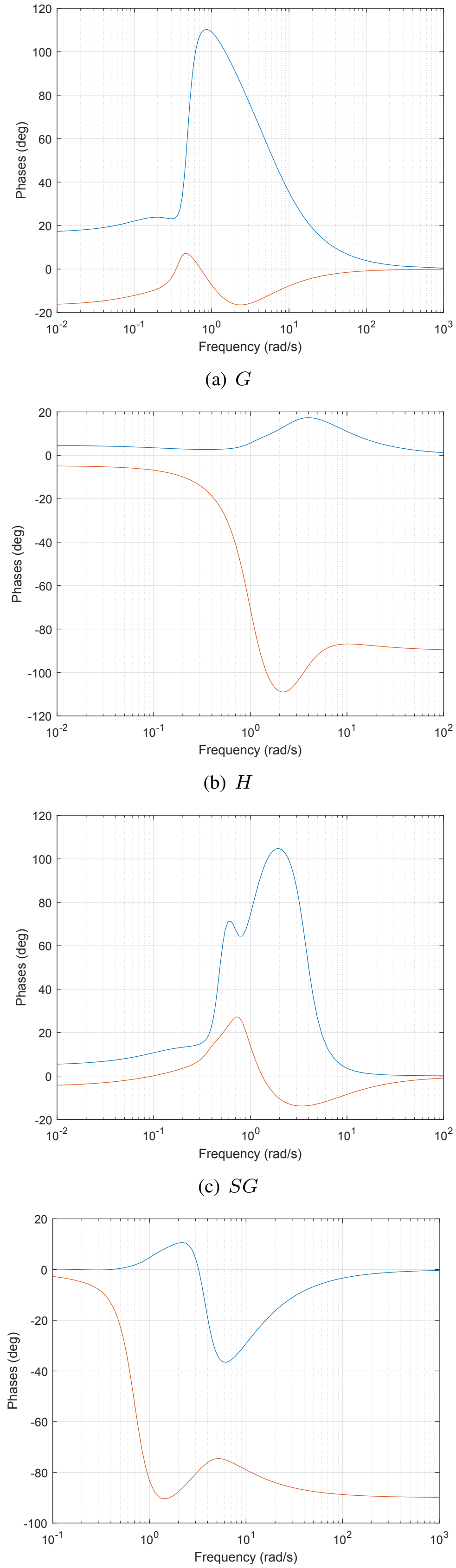
[21] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. New Jersey: Prentice Hall, 1996.

[22] F. W. King, *Hilbert Transforms*. Cambridge University Press, 2009.

[23] N. Kottenstette, M. J. McCourt, M. Xia, V. Gupta, and P. J. Antsaklis, “On relationships among passivity, positive realness, and dissipativity in linear systems,” *Automatica*, vol. 50, no. 4, pp. 1003–1016, 2014.

[24] T. Chaffey, F. Forni, and R. Sepulchre, “Scaled relative graphs for system analysis,” in *Proc. 60th IEEE Conf. Decis. Control*, pp. 3166–3172, 2021.

[25] R. Sepulchre, T. Chaffey, and F. Forni, “On the incremental form of dissipativity,” in *25th IFAC Symposium on Mathematical Theory of Networks and Systems*, pp. 290–294, 2022.



(a) G

(b) H

(c) SG

(d) HS

Fig. 6. Phase of feedback systems.