Distributed Online Convex Optimization With Time-Varying Constraints: Tighter Cumulative Constraint Violation Bounds Under Slater's Condition

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***Abstract*—This article considers distributed online convex optimization with time-varying constraints. In this setting, a network of agents makes decisions at each round, and then, only a portion of the loss function and a coordinate block of the constraint function are privately revealed to each agent. The loss and constraint functions are convex and can vary arbitrarily across rounds. The agents collaborate to minimize static network regret and network cumulative constraint violation. A novel distributed online algorithm with a vanishing stepsize is proposed and it achieves an $\mathcal{O}(T^{\max\{c,1-c\}})$ static network regret bound and an $\mathcal{O}(T^{1-c/2})$ network cumulative constraint violation bound, where T is the number of rounds and $c \in (0,1)$ is a user-defined tradeoff parameter. When Slater's condition holds (i.e., there is a point that strictly satisfies the inequality constraints), the network cumulative constraint violation bound is reduced to $\mathcal{O}(T^{1-c})$. Moreover, if the loss functions are strongly convex, then static network regret bound is reduced to $\mathcal{O}(\log(T))$, and the network cumulative constraint violation bound is reduced to $\mathcal{O}(\sqrt{\log(T)T})$ and $\mathcal{O}(\log(T))$ without and with Slater's condition, respectively. To the best of the authors' knowledge, this article is the first to achieve tighter (network) cumulative constraint violation bounds for (distributed) online convex optimization with time-varying constraints under Slater's condition. Finally, the theoretical results are verified through numerical simulations.**

***Index Terms*—Cumulative constraint violation, distributed optimization, online convex optimization, Slater's condition, time-varying constraints.**

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I. Introduction

Online convex optimization is a sequential decision-making problem with a sequence of arbitrarily varying convex loss functions. Specifically, at each round t, a decision maker selects a decision $x_t \in \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}^p$ is a known closed convex set with p being a positive integer. After the selection, a convex loss function $l_t : \mathbb{R}^p \to \mathbb{R}$ is revealed. The goal of the decision maker is to minimize the cumulative loss across T rounds. The standard performance measure is static regret

$$\sum_{t=1}^{T} l_t(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} l_t(x)$$

which is the difference between the accumulative loss and the loss obtained by the best fixed decision in hindsight. Due to its wide applications, such as online display advertising [1], online linear regression [2], and reactive power management [3], online convex optimization has been extensively studied over the past decades; see, e.g., [4], [5], [6], [7], [8], and [9]. For more applications and background, refer to the monographs [10] and [11].

A. Online Convex Optimization With Time-Varying Constraints

It is well-known that the projection-based online gradient descent algorithm

$$x_{t+1} = \mathcal{P}_{\mathcal{X}}(x_t - \alpha \nabla l_t(x_t)) \tag{1}$$

where $\mathcal{P}_{\mathcal{X}}(\cdot)$ is the projection onto set $\mathcal{X}$ and $\alpha > 0$ is the stepsize, achieves an $\mathcal{O}(\sqrt{T})$ static regret bound for convex loss functions with bounded subgradients [4], which is a tight bound up to constant factors [5]. Static regret bound can be reduced under more stringent strong convexity conditions on the loss functions [5], [10], [11]. Although the algorithm (1) seems simple, the projection operator can yield heavy computation and/or storage burden when the constraint set is complicated. For example, in practice, the constraint set $\mathcal{X}$ is often characterized by inequality constraints, i.e.,

$$\mathcal{X} = \{x : g(x) \le \mathbf{0}_m, \, x \in \mathbb{X}\} \tag{2}$$

where m is the positive integer, $\mathbb{X} \subseteq \mathbb{R}^p$ is the closed convex set that normally is a simple set, e.g., a box or a ball, and $g(x) : \mathbb{R}^p \to \mathbb{R}^m$ is the constraint function that is convex. To tackle this challenge, online convex optimization with long-term constraints has been considered, see, e.g., [12], [13], and [14]. In this new problem, the decisions are selected from the simple set $\mathbb{X}$ and the inequality constraints should be satisfied in the long term on average. This is measured by constraint violation

$$\left\| \left[\sum_{t=1}^{T} g(x_t) \right]_+ \right\| \tag{3}$$

which is the violation of the accumulative constraint function. Here, $[\,\cdot\,]_+$ is the projection onto the nonnegative space. In this case, the goal of the decision maker is to minimize both static regret and constraint violation.

The problem above has been further extended to the time-varying constraints setting, i.e., the constraint function can be arbitrarily and adversarially designed, and is revealed to the decision maker after selecting its decision at each round. In this case, let $g_t : \mathbb{R}^p \to \mathbb{R}^m$ denote the constraint function of the tth round and then the constraint violation metric is $\|[\sum_{t=1}^{T} g_t(x_t)]_+\|$. Online convex optimization with time-varying constraints has also been extensively studied, e.g., [15], [16], [17], [18], [19], [20], [21], and [22]. These works usually proposed online primal–dual algorithms and achieved sublinear static regret and constraint violation bounds. For example, Sun et al. [16] achieved an $\mathcal{O}(\sqrt{T})$ static regret bound and an $\mathcal{O}(T^{3/4})$ constraint violation bound. In [17] and [18], the bound for constraint violation was reduced to $\mathcal{O}(\sqrt{T})$ under Slater's condition (i.e., there exists a point $x_s \in \mathbb{X}$ and a constant $\epsilon_s > 0$, such that $g_t(x_s) \le -\epsilon_s \mathbf{1}_m$ for all t). Slater's condition is an example of a constraint qualification that guarantees strong duality in convex optimization problems with inequality constraints [23]. This condition guarantees that the optimal solution of the primal problem can be achieved without violating any of the constraints. In the context of online convex optimization, under Slater's condition, algorithms can adaptively adjust the decision at each round to ensure that the constraints are not violated significantly, even if the optimization problem is evolving dynamically.

Note that the constraint violation metric defined in (3) takes the summation across rounds before the projection operation $[\cdot]_+$. As a result, it allows strict feasible decisions that have large margins compensate constraint violations at many rounds. In this way, even if the constraint violation grows sublinearly, the constraints could be violated at many rounds. To avoid this potential drawback, stricter forms of constraint violation metric have been proposed in [24] and [25]. For instance, Yuan and Lamperski [24] proposed cumulative constraint violation

$$\left\| \sum_{t=1}^{T} [g(x_t)]_+ \right\| \tag{4}$$

and cumulative squared constraint violation

$$\sum_{t=1}^{T} \|[g(x_t)]_+\|^2. \tag{5}$$

Both forms of metrics (4) and (5) take into account all constraints that are not satisfied, and the metric (4) is stricter than the constraint violation metric defined in (3). In [24], an $\mathcal{O}(T^{\max\{c,1-c\}})$ static regret bound and an $\mathcal{O}(T^{1-c/2})$ cumulative constraint violation bound were achieved, where $c \in (0,1)$ is a user-defined tradeoff parameter enabling the tradeoff between these two bounds. Moreover, when the loss functions are strongly convex, static regret and cumulative constraint violation bounds were reduced to $\mathcal{O}(\log(T))$ and $\mathcal{O}(\sqrt{\log(T)T})$, respectively. The key idea to achieve these results is to use the clipped constraint function $[g]_+$ to replace the original constraint function g. As pointed out in [24], with this idea, the bounds for constraint violation achieved in some existing works, e.g., [12] and [13], still hold when using the stricter metric (4) to replace the standard metric (3). However, this idea becomes ineffective when extending the constraint violation bounds achieved in [17] and [18], as the use of the clipping operation renders Slater's condition ineffective. More specifically, the clipped function is nonnegative, and thus, it is impossible to find a point where the value of the clipped function is strictly less than zero. It remains an open problem how to achieve tighter cumulative constraint violation bounds for online convex optimization with time-varying constraints under Slater's condition.

B. Distributed Online Convex Optimization With Time-Varying Constraints

Noting that distributed paradigm can address critical issues in centralized processing, such as data privacy, data security, and single point failures, distributed online convex optimization has also been extensively studied, e.g., [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], and [36]. In distributed online convex optimization, there is a group of agents (decision makers). At each round t, each agent i selects a decision $x_{i,t} \in \mathcal{X}$, and after the selection, a portion of the global loss function l_t is revealed to agent i only. The goal of the agents is to minimize the network-wide accumulated loss, and the corresponding performance measure is the static network regret $\frac{1}{n}\sum_{i=1}^{n}(\sum_{t=1}^{T} l_t(x_{i,t}) - \min_{x\in\mathcal{X}} \sum_{t=1}^{T} l_t(x))$.

Similarly, in order to avoid the potential computation and/or storage challenge caused by the projection operator when using projection-based algorithms, distributed online convex optimization with long-term constraints has also been considered, e.g., [37], [38], and [39]. In this problem, the constraint set $\mathcal{X}$ is characterized by inequality constraints as described in (2), and each agent knows the simple set $\mathbb{X}$ and the constraint function g in advance. Similar to the centralized case, the decisions are selected from $\mathbb{X}$ instead of $\mathcal{X}$ and the inequality constraints should be satisfied in the long-term on average, which is measured by network constraint violation $\frac{1}{n}\sum_{i=1}^{n}\|[\sum_{t=1}^{T}g(x_{i,t})]_+\|$ or network cumulative constraint violation $\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g(x_{i,t})]_+\|$. The static network regret and constraint violation bounds achieved in [37], [38], and [39] are similar to their centralized counterparts. Specifically, Yuan et al. [37] achieved an $\mathcal{O}(T^{0.5+\beta})$ static network regret bound and an $\mathcal{O}(T^{1-\beta/2})$ network constraint violation bound, where $\beta \in (0, 0.5)$ is a user-defined parameter. When the loss functions are quadratic and the constraint function is linear, Yuan et al. [38] achieved an $\mathcal{O}(T^{\max\{c,1-c\}})$ static network regret bound and an $\mathcal{O}(T^{1-c/2})$ network cumulative constraint violation bound. The quadratic loss and linear constraint functions were relaxed by convex functions in [39] and the same static network regret and network cumulative constraint violation bounds were still achieved.

The distributed online convex optimization with long-term constraints setting was extended to a more general scenario in [40], where the constraint function is time-varying, and at each round, only a coordinate block of the global constraint function is privately revealed to each agent after selecting its decision. The same static network regret and cumulative constraint violation bounds as achieved in [38] and [39], were also established. However, similar to the centralized case, it is still unclear *how to reduce network cumulative constraint violation bounds for distributed online convex optimization with time-varying constraints under Slater's condition*, which is the main motivation behind this article.

C. Main Contributions

In this article, similar to [40], we study the general distributed online convex optimization with time-varying constraints and adopt static network regret and cumulative constraint violation as performance measures. However, different from [40], we also consider the scenario where Slater's condition holds. The main challenge is to consider cumulative constraint violation and Slater's condition at the same time. As mentioned before, to consider cumulative constraint violation, existing papers all replace the original constraint function with the clipped constraint function; however, the clipping operation renders Slater's condition ineffective. To tackle this problem, we propose a novel distributed online primal–dual composite mirror descent algorithm, which updates the dual variables by directly maximizing the regularized Lagrangian function and thus can be explicitly calculated using the clipped constraint function. Consequently, for the scenario without Slater's condition, all squared constraint violations can be accumulated, leveraging the fact that $a^\top[a]_+ = \|[a]_+\|^2$ for any vector a, and state-of-the-art (network) cumulative constraint violation bounds can be achieved. Moreover, for the scenario with Slater's condition, all constraint violations can be accumulated directly by summing the dual variables. As a result, we can demonstrate that tighter (network) cumulative constraint violation bounds can be achieved under Slater's condition through a properly designed analysis, which is presented for the first time since no papers have achieved such a result.

The detailed performance guarantees for the proposed algorithm are summarized as follows.

1) We show in Theorem 1 that the proposed algorithm with a vanishing stepsize achieves an $\mathcal{O}(T^{\max\{c,1-c\}})$ static network regret bound and an $\mathcal{O}(T^{1-c/2})$ network cumulative constraint violation bound as achieved in [40], which generalizes the results in [24], [16], [38], and [39] to the more general settings, and thus also improves the results in [37].
2) When Slater's condition holds, we show in Theorem 2 that network cumulative constraint violation bound is reduced to $\mathcal{O}(T^{1-c})$, which generalizes the results in [17] and [18], and thus solves the open problem left in [24]. To the best of the authors' knowledge, this article is the first to achieve a tighter (network) cumulative constraint violation bound for (distributed) online convex optimization with time-varying constraints under Slater's condition.
3) When the loss functions are strongly convex, we show in Theorem 4 that the proposed algorithm with a vanishing stepsize achieves an $\mathcal{O}(\log(T))$ static network regret bound and an $\mathcal{O}(\sqrt{\log(T)T})$ network cumulative constraint violation bound, which generalizes the results in [24] and [39] and improves the results in [37] and [40]. Moreover, if Slater's condition holds in addition, network cumulative constraint violation bound is reduced to $\mathcal{O}(\log(T))$. Again, to the best of the authors' knowledge, it is the first time to achieve such a result.

The detailed comparison of this article to related works is summarized in Table I.

Outline: The rest of this article is organized as follows. Section II formulates the considered problem. Section III proposes the novel distributed online primal–dual composite mirror descent algorithm to solve the problem. Section IV analyzes static network regret and cumulative constraint violation bounds for the proposed algorithm. Section V gives numerical simulations. Finally, Section VI concludes this article, and and proofs are given in Appendix.

Notations: All inequalities and equalities throughout this article are understood componentwise. $\mathbb{R}^n$ and $\mathbb{R}^n_+$ stand for the set of n-dimensional vectors and nonnegative vectors, respectively. $\mathbb{N}_+$ denotes the set of all positive integers. $[n]$ represents the set $\{1,\dots,n\}$ for any $n \in \mathbb{N}_+$. $\|\cdot\|$ ($\|\cdot\|_1$) stands for the Euclidean norm (1-norm) for vectors and the induced 2-norm (1-norm) for matrices. $x^\top$ denotes the transpose of a vector or a matrix. $\langle x, y\rangle$ represents the standard inner product of two vectors x and y. $\mathbf{0}_n$ ($\mathbf{1}_n$) denotes the column zero (one) vector with dimension n. $\mathrm{col}(z_1,\dots,z_k)$ is the concatenated column vector of $z_i \in \mathbb{R}^{n_i}$, $i \in [k]$. For a closed convex set $\mathbb{K} \subseteq \mathbb{R}^p$ and any $x \in \mathbb{R}^p$, $\mathcal{P}_{\mathbb{K}}(x)$ is the projection of x onto $\mathbb{K}$, i.e., $\mathcal{P}_{\mathbb{K}}(x) = \arg\min_{y\in\mathbb{K}}\|x-y\|^2$. For simplicity, $[x]_+$ is used to denote $\mathcal{P}_{\mathbb{R}^p_+}(x)$. For a function f, let $\nabla f(x)$ denote the (sub)gradient of f at x. $\mathcal{U}(a,b)$ is the uniform distribution over the interval $[a,b]$ with $a \le b \in \mathbb{R}$. Given two scalar sequences $\{\alpha_t,\, t \in \mathbb{N}_+\}$ and $\{\beta_t > 0,\, t \in \mathbb{N}_+\}$, $\alpha_t = \mathcal{O}(\beta_t)$ means that there exists a constant $a > 0$ such that $\alpha_t \le a\beta_t$ for all t, while $\alpha_t = \mathbf{o}(t)$ means that there exist two constants $a > 0$ and $\kappa \in (0,1)$ such that $\alpha_t \le at^\kappa$ for all t.

II. Problem Formulation

This article studies distributed online convex optimization with time-varying constraints. Specifically, consider a network of n agents indexed by $i \in [n]$. At each round t, each agent i

TABLE I
COMPARISON OF THIS ARTICLE TO RELATED WORKS ON ONLINE CONVEX OPTIMIZATION WITH LONG-TERM AND TIME-VARYING CONSTRAINTS

<table>
<tr><th>Reference</th><th>Problem type</th><th>Loss functions</th><th>Constraint functions</th><th>Slater's condition</th><th>Regret</th><th>Constraint violation</th><th>Cumulative constraint violation</th></tr>
<tr><td>[16]</td><td>Centralized</td><td>Convex</td><td>Convex</td><td>No</td><td>$\mathcal{O}(\sqrt{T})$</td><td>$\mathcal{O}(T^{3/4})$</td><td>Not given</td></tr>
<tr><td>[17], [18]</td><td>Centralized</td><td>Convex</td><td>Convex</td><td>Yes</td><td>$\mathcal{O}(\sqrt{T})$</td><td>$\mathcal{O}(\sqrt{T})$</td><td>Not given</td></tr>
<tr><td rowspan="2">[24]</td><td rowspan="2">Centralized</td><td>Convex</td><td rowspan="2">Convex, time-invariant</td><td rowspan="2">No</td><td>$\mathcal{O}(T^{\max\{c,1-c\}})$</td><td colspan="2">$\mathcal{O}(T^{1-c/2})$</td></tr>
<tr><td>Strongly convex</td><td>$\mathcal{O}(\log(T))$</td><td colspan="2">$\mathcal{O}(\sqrt{\log(T)T})$</td></tr>
<tr><td rowspan="2">[37]</td><td rowspan="2">Distributed</td><td>Convex</td><td rowspan="2">Convex, time-invariant</td><td rowspan="2">No</td><td>$\mathcal{O}(T^{\max\{0.5+\beta\}})$</td><td>$\mathcal{O}(T^{1-\beta/2})$</td><td rowspan="2">Not given</td></tr>
<tr><td>Strongly convex</td><td>$\mathcal{O}(T^{c})$</td><td>$\mathcal{O}(T^{1-c/2})$</td></tr>
<tr><td>[38]</td><td>Distributed</td><td>Quadratic</td><td>Linear, time-invariant</td><td>No</td><td>$\mathcal{O}(T^{\max\{c,1-c\}})$</td><td colspan="2">$\mathcal{O}(T^{1-c/2})$</td></tr>
<tr><td rowspan="2">[39]</td><td rowspan="2">Distributed</td><td>Convex</td><td rowspan="2">Convex, time-invariant</td><td rowspan="2">No</td><td>$\mathcal{O}(T^{\max\{c,1-c\}})$</td><td colspan="2">$\mathcal{O}(T^{1-c/2})$</td></tr>
<tr><td>Strongly convex</td><td>$\mathcal{O}(\log(T))$</td><td colspan="2">$\mathcal{O}(\sqrt{\log(T)T})$</td></tr>
<tr><td rowspan="2">[40]</td><td rowspan="2">Distributed</td><td>Convex</td><td rowspan="2">Convex</td><td rowspan="2">No</td><td>$\mathcal{O}(T^{\max\{c,1-c\}})$</td><td colspan="2" rowspan="2">$\mathcal{O}(T^{1-c/2})$</td></tr>
<tr><td>Strongly convex</td><td>$\mathcal{O}(T^{c})$</td></tr>
<tr><td rowspan="4">This article</td><td rowspan="4">Distributed</td><td rowspan="2">Convex</td><td rowspan="4">Convex</td><td>No</td><td rowspan="2">$\mathcal{O}(T^{\max\{c,1-c\}})$</td><td colspan="2">$\mathcal{O}(T^{1-c/2})$</td></tr>
<tr><td>Yes</td><td colspan="2">$\mathcal{O}(T^{1-c})$</td></tr>
<tr><td rowspan="2">Strongly convex</td><td>No</td><td rowspan="2">$\mathcal{O}(\log(T))$</td><td colspan="2">$\mathcal{O}(\sqrt{\log(T)T})$</td></tr>
<tr><td>Yes</td><td colspan="2">$\mathcal{O}(\log(T))$</td></tr>
</table>

makes a decision $x_{i,t} \in \mathbb{X}$, where $\mathbb{X} \subseteq \mathbb{R}^p$ is a known set and p is a positive integer. After making the selection, the local loss function $l_{i,t} : \mathbb{R}^p \to \mathbb{R}$ and constraint function $g_{i,t} : \mathbb{R}^p \to \mathbb{R}^{m_i}$ are revealed to agent i only, which, respectively, are a portion of the global loss function $l_t(x) = \frac{1}{n}\sum_{i=1}^n l_{i,t}(x) : \mathbb{R}^p \to \mathbb{R}$ and a coordinate block of the global constraint function $g_t(x) = \mathrm{col}(g_{1,t}(x), \dots, g_{n,t}(x)) : \mathbb{R}^p \to \mathbb{R}^m$. Here, m_i is a positive integer and $m = \sum_{i=1}^n m_i$. The agents' objective is to select the decision sequences $\{x_{i,t}\}$ such that both the static network regret

$$\text{Net-Reg}(T) := \frac{1}{n}\sum_{i=1}^n \sum_{t=1}^T (l_t(x_{i,t}) - l_t(x_T^*)) \tag{6}$$

and the network cumulative constraint violation

$$\frac{1}{n}\sum_{i=1}^n \sum_{t=1}^T \|[g_t(x_{i,t})]_+\| \tag{7}$$

grow sublinearly, where $T \in \mathbb{N}_+$ is the number of rounds and x_T^* is the best fixed decision when all functions $\{l_t\}_{t=1}^T$ and $\{g_t\}_{t=1}^T$ are known a priori. In other words, x_T^* is the solution to the following constrained optimization problem:

$$\min_{x\in\mathbb{X}} \quad \sum_{t=1}^T l_t(x)$$
$$\text{s.t.} \quad g_t(x) \le \mathbf{0}_m \quad \forall t \in [T].$$

Let $\mathcal{X}_T$ denote the feasible set, i.e.,

$$\mathcal{X}_T = \{x : x \in \mathbb{X},\, g_t(x) \le \mathbf{0}_m \quad \forall t \in [T]\}.$$

Similar to existing literature considering time-varying constraints, e.g., [15], [16], [17], [18], [19], [20], [21], [22], and [40], it is assumed that the feasible set $\mathcal{X}_T$ is nonempty for every T.

Without loss of generality, we assume that each local loss function $l_{i,t}$ consists of a private part $f_{i,t}$ and a common part r_t, i.e., $l_{i,t}(x) = f_{i,t}(x) + r_t(x)$. Here, r_t represents the common knowledge in the network. For example, r_t could be the regularization used to influence the structure of the decisions. The above distributed problem setting incorporates various problems studied in the literature. For instance, when $g_{i,t} \equiv \mathbf{0}_{m_i}\ \forall i \in [n],\ t \in \mathbb{N}_+$, the above distributed problem becomes the problem studied in [33]; when $g_{i,t} \equiv \mathbf{0}_{m_i}$ and $r_t \equiv 0\ \forall i \in [n],\ t \in \mathbb{N}_+$, the above distributed problem becomes the problem studied in various existing works, e.g., [26], [27], [28], [29], [30], [31], [32], [34], and [38]; when $g_{i,t} \equiv g$ and $r_t \equiv 0\ \forall i \in [n], t \in \mathbb{N}_+$, with g being a known and predefined constraint function, the above distributed problem becomes the problem studied in [37], [38], and [39]; and when $r_t \equiv 0\ \forall t \in \mathbb{N}_+$, the above distributed problem becomes the problem studied in [40].

Some necessary definitions and assumptions are listed in the following.

A. Graph Theory

In this article, the communication topology for the network of agents is modeled by a time-varying directed graph. Specifically, let $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$ denote the directed graph at the tth round, where $\mathcal{V} = [n]$ is the agent set and $\mathcal{E}_t \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. A directed edge $(j, i) \in \mathcal{E}_t$ means that agent i can receive data from agent j at the tth round. Let $\mathcal{N}_i^{\text{in}}(\mathcal{G}_t) = \{j \in [n] \mid (j, i) \in \mathcal{E}_t\}$ and $\mathcal{N}_i^{\text{out}}(\mathcal{G}_t) = \{j \in [n] \mid (i, j) \in \mathcal{E}_t\}$ be the sets of in- and out-neighbors, respectively, of agent i at the tth round. A directed path is a sequence of consecutive directed edges. A directed graph is strongly connected if there is at least one directed path from any agent to any other agent. The associated adjacency (mixing) matrix $W_t \in \mathbb{R}^{n\times n}$ fulfills $[W_t]_{ij} > 0$ if $(j, i) \in \mathcal{E}_t$ or $i = j$, and $[W_t]_{ij} = 0$, otherwise.

B. Bregman Divergence

For the sake of generality, this article uses the Bregman divergence [41] to measure the distance of two points $x,\ y \in \mathbb{X}$, which is defined as

$$\mathcal{D}_\psi(x,y) = \psi(x) - \psi(y) - \langle \nabla\psi(y), x-y\rangle$$

where $\psi : \mathbb{R}^p \to \mathbb{R}$ is a strongly convex function with convexity parameter $\sigma > 0$ on the set $\mathbb{X}$, i.e.,

$$\psi(x) \geq \psi(y) + \langle \nabla\psi(y), x-y\rangle + \frac{\sigma}{2}\|x-y\|^2.$$

Thus

$$\mathcal{D}_\psi(x,y) \geq \frac{\sigma}{2}\|x-y\|^2 \quad \forall x,y \in \mathbb{X}. \tag{8}$$

Moreover, $\mathcal{D}_\psi(\,\cdot\,,y)$ is a strongly convex function with convexity parameter σ for all fixed $y \in \mathbb{X}$.

Two well-known examples of Bregman divergence are the Euclidean distance $\mathcal{D}_\psi(x,y) = \|x-y\|^2$ generated from $\psi(x) = \|x\|^2$ and the Kullback–Leibler (K–L) divergence $\mathcal{D}_\psi(x,y) = -\sum_{i=1}^p x_i \log(y_i/x_i)$ between two p-dimensional standard unit vectors (with $\mathbb{X}$ being the p-dimensional probability simplex) generated from $\psi(x) = \sum_{i=1}^p x_i \log x_i$.

C. Assumptions

We make the following assumptions on the loss and constraint functions.

Assumption 1: The set $\mathbb{X}$ is closed and convex. For all $i \in [n]$ and $t \in \mathbb{N}_+$, the functions r_t, $f_{i,t}$, and $g_{i,t}$ are convex.

Assumption 2: There exists a positive constant F such that for all $i \in [n]$, $t \in \mathbb{N}_+$, and $x, y \in \mathbb{X}$

$$|l_{i,t}(x) - l_{i,t}(y)| \leq F. \tag{9}$$

Assumption 3: There exist two positive constants G_1 and G_2 such that for all $i \in [n]$, $t \in \mathbb{N}_+$, and $x \in \mathbb{X}$

$$\|\nabla l_{i,t}(x)\| \leq G_1,\ \|\nabla g_{i,t}(x)\| \leq G_2. \tag{10}$$

Assumptions 1 and 3 are standard in the literature of online convex optimization, e.g., [27], [28], [29], [30], [32], [33], [37], [38], and [39]. Note that we do not assume that the local constraint functions $\{g_{i,t}\}$ are uniformly bounded, and the assumption that $\nabla l_{i,t}(x)$ is bounded is slightly weaker than the assumption that both $\nabla f_{i,t}(x)$ and $\nabla r_t(x)$ are bounded. From Assumptions 1 and 3 and [10, Lemma 2.6], it follows that for all $i \in [n]$, $t \in \mathbb{N}_+$, $x,\ y \in \mathbb{X}$

$$|l_{i,t}(x) - l_{i,t}(y)| \leq G_1\|x-y\| \tag{11a}$$

$$\|g_{i,t}(x) - g_{i,t}(y)\| \leq G_2\|x-y\|. \tag{11b}$$

Therefore, Assumption 2 holds if the set $\mathbb{X}$ has a bounded diameter. The latter is widely postulated in the online convex optimization literature, e.g., [27], [28], [29], [32], [33], [37], [38], and [39]. The following commonly used assumption is made on the graph.

Assumption 4: For any $t \in \mathbb{N}_+$, the directed graph $\mathcal{G}_t$ satisfies the following conditions.

a) There exists a constant $w \in (0,1)$, such that $[W_t]_{ij} \geq w$ if $[W_t]_{ij} > 0$.

b) The mixing matrix W_t is doubly stochastic, i.e., $\sum_{i=1}^n [W_t]_{ij} = \sum_{j=1}^n [W_t]_{ij} = 1\ \forall i,j \in [n]$.

c) There exists an integer $B > 0$ such that the directed graph $(\mathcal{V}, \cup_{l=0}^{B-1}\mathcal{E}_{t+l})$ is strongly connected.

Some assumptions on the Bergman divergence are stated as follows.

Assumption 5: For any $x \in \mathbb{X}$, $\mathcal{D}_\psi(x,\,\cdot\,) : \mathbb{X} \to \mathbb{R}$ is convex.

Assumption 6: There exists a positive constant K such that

$$\mathcal{D}_\psi(x,y) \leq K \quad \forall x,y \in \mathbb{X}. \tag{12}$$

Assumption 5 is satisfied for commonly used Bregman divergences, such as the Euclidean distance and the K–L divergence. Assumption 6 is essentially employed to ensure that the set $\mathbb{X}$ has a bounded diameter. However, in certain scenarios, as demonstrated later, this assumption is eliminated.

The objective of this article is to design an algorithm that can solve distributed online convex optimization with time-varying constraints, ensuring that both the static network regret and the network cumulative constraint violation bounds grow sublinearly under the aforementioned assumptions. More importantly, this article aims to show that tighter (network) cumulative constraint violation bounds can be achieved under Slater's condition, thereby addressing the open problem in the literature. We formally introduce Slater's condition as follows.

Assumption 7 (Slater's condition): There exists a point $x_s \in \mathbb{X}$ and a constant $\epsilon_s > 0$ such that

$$g_t(x_s) \leq -\epsilon_s \mathbf{1}_m \quad \forall t \in \mathbb{N}_+. \tag{13}$$

Slater's condition is a sufficient condition for strong duality to hold in convex optimization problems [23], which has also been utilized in online convex optimization problems to show that tighter constraint violation bounds can be achieved, e.g., [14], [17], [18], and [42]. However, to the best of the authors' knowledge, existing literature has not achieved tighter (network) cumulative constraint violation bounds under Slater's condition. The key reason for this is that existing literature studying (network) cumulative constraint violation, e.g., [24], [38], [39], and [40], often replaces the original constraint function with the clipped constraint function, which makes Slater's condition ineffective. Therefore, the main challenge lies in addressing the contradiction between the need for the clipping operation when using the stricter cumulative constraint violation metrics, as defined in (4) and (7), and the fact that it renders Slater's condition ineffective.

III. Algorithm Description

In this section, we propose a novel algorithm for the distributed online convex optimization problem with time-varying constraints, as introduced in the previous section, and analyze its performance in the next section.

Recall that at the tth round, the global loss and constraint functions are l_t and g_t, respectively. The associated regularized Lagrangian function is

$$\mathcal{L}_t(x_t, q_t) := \frac{1}{n}\sum_{i=1}^n f_{i,t}(x_t) + r_t(x_t) + q_t^\top g_t(x_t) - \frac{1}{2\gamma_t}\|q_t\|^2$$

where $x_t \in \mathbb{R}^p$ and $q_t \in \mathbb{R}_+^m$ represent the primal and dual variables, respectively, and γ_t is the regularization parameter. We first update the dual variable by directly maximizing $\mathcal{L}_t(x_t, q)$ over all $q \in \mathbb{R}_+^m$, i.e.,

$$q_{t+1} = \mathrm{argmax}_{q\in\mathbb{R}_+^m}\, \mathcal{L}_t(x_t, q) = \gamma_t[g_t(x_t)]_+ \tag{14}$$

which follows the idea of updating dual variables in [24]. The same idea has also been adopted in [39]. Actually, the updating rule (14) can also be derived through the projected gradient ascent with γ_t being the stepsize, i.e.,

$$q_{t+1} = \left[q_t + \gamma_t \frac{\partial \mathcal{L}_t(x_t,q)}{\partial q}\Big|_{q=q_t}\right]_+.$$

Algorithm 1: Distributed Online Primal–Dual Composite Mirror Descent.

Input: nonincreasing sequence $\{\alpha_t > 0\}$ and nondecreasing sequence $\{\gamma_t > 0\}$; differentiable and strongly convex function ψ.
Initialize: $z_{i,1} \in \mathbb{X}$ for all $i \in [n]$.
for $t = 1, \dots$ **do**
  for $i = 1, \dots, n$ in parallel **do**
    Broadcast $z_{i,t}$ to $\mathcal{N}_i^{\text{out}}(\mathcal{G}_t)$ and receive $z_{j,t}$ from $j \in \mathcal{N}_i^{\text{in}}(\mathcal{G}_t)$.
    Select

$$x_{i,t} = \sum_{j=1}^{n} [W_t]_{ij} z_{j,t}. \tag{16}$$

    Observe $\nabla f_{i,t}(x_{i,t})$, $\nabla g_{i,t}(x_{i,t})$, $g_{i,t}(x_{i,t})$, and $r_t(\cdot)$.
    Update

$$q_{i,t+1} = \gamma_t [g_{i,t}(x_{i,t})]_+ \tag{17a}$$

$$\omega_{i,t+1} = \nabla f_{i,t}(x_{i,t}) + (\nabla g_{i,t}(x_{i,t}))^\top q_{i,t+1} \tag{17b}$$

$$z_{i,t+1} = \arg\min_{x \in \mathbb{X}} \{\alpha_t \langle x, \omega_{i,t+1} \rangle + \alpha_t r_t(x) + \mathcal{D}_\psi(x, x_{i,t})\}. \tag{17c}$$

  end for
end for
Output: $\{x_{i,t}\}$.

Then, instead of using the projected gradient descent

$$x_{t+1} = \mathcal{P}_{\mathbb{X}} \left(x_t - \alpha_t \frac{\partial \mathcal{L}_t(x, q_{t+1})}{\partial x} \Big|_{x = x_t} \right)$$

to update the primal variable, where $\alpha_t > 0$ is the stepsize, we update it as follows:

$$x_{t+1} = \arg\min_{x \in \mathbb{X}} \{\alpha_t \langle x, \omega_{t+1} \rangle + \alpha_t r_t(x) + \mathcal{D}_\psi(x, x_t)\} \tag{15}$$

where

$$\omega_{t+1} = \frac{1}{n} \sum_{i=1}^{n} \nabla f_{i,t}(x_t) + (\nabla g_t(x_t))^\top q_{t+1}.$$

The updating rule (15) is inspired by the composite objective mirror descent in [43], which has also been adopted in [33] and [42]. In the following, we introduce how to implement the updating rules (14) and (15) in a distributed manner.

We use $x_{i,t}$ to denote the local copy of the primal variable x_t. If we rewrite the dual variable in an agentwise manner, i.e., $q_t = \text{col}(q_{1,t}, \dots, q_{n,t})$ with each $q_{i,t} \in \mathbb{R}_+^{m_i}$, then the updating rule (14) can be executed in an agentwise manner as (17a). Note that $\omega_{i,t+1}$ defined in (17b) can be understood as a portion of ω_{t+1} that is available to agent i. Then, each $z_{i,t+1}$ updated by (17c) can be understood as a local estimate of x_{t+1} updated by (15). In this case, for each agent i, $x_{i,t+1}$ computed by the consensus protocol (16) is used to track the average $\frac{1}{n}\sum_{i=1}^{n} z_{i,t+1}$, and thus estimates x_{t+1} more accurately. As a result, the updating rules (14) and (15) can be executed in a distributed manner, which is summarized in pseudocode as Algorithm 1. This algorithm is called the distributed online primal–dual composite mirror descent algorithm.

The minimization problem (17c) is strongly convex, so it can be solved with a linear convergence rate and closed-form solutions are available in special cases. For example, if r_t is a linear mapping and Euclidean distance is used as Bregman distance, i.e., $\mathcal{D}_\psi(x, y) = \|x - y\|^2$, then, as shown by [18, Lemma 1], the convex minimization problem (17c) can be solved by the projection

$$z_{i,t+1} = \mathcal{P}_{\mathbb{X}} \left(x_{i,t} - \frac{\alpha_t}{2} (\omega_{i,t+1} + \nabla r_t) \right).$$

To end this section, we would like to emphasize the key novelty of the proposed algorithm. Different from previous approaches in [24], [38], [39], and [40], our algorithm does not simply replace the original constraint function with the clipped constraint function. Instead, it utilizes the clipped constraint function solely for updating the dual variables, derived from directly maximizing the regularized Lagrangian function, see (14) and (17a). This guarantees the achievement of state-of-the-art (network) cumulative constraint violation bounds without Slater's condition. More importantly, it serves as the key to solving the open problem of achieving tighter (network) cumulative constraint violation bounds under Slater's condition through properly designed analysis, since it enables the use of the stricter cumulative constraint violation metric while preserving the effectiveness of Slater's condition. The detailed explanations are elaborated in the next section.

IV. Performance Analysis

This section analyzes the static network regret and cumulative constraint violation bounds for Algorithm 1 under different scenarios.

A. Preliminary Results

We first bound local regret and (squared) cumulative constraint violation, the accumulated (squared) consensus error, and the changes caused by composite mirror descent in the following.

Lemma 1: Suppose Assumptions 1–5 hold. For all $i \in [n]$, let $\{x_{i,t}\}$ be the sequences generated by Algorithm 1 with $\gamma_t = \gamma_0/\alpha_t$, where $\gamma_0 \in (0, \sigma/(4G_2^2)]$ is a constant. Then, for any $T \in \mathbb{N}_+$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(l_{i,t}(x_{i,t}) - l_{i,t}(y) + \frac{\sigma\|\epsilon_{i,t}^z\|^2}{4\alpha_t}\right) \leq \sum_{t=1}^{T}\frac{2G_1^2\alpha_t}{\sigma} + \frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\Delta_{i,t}(y) \quad \forall y \in \mathcal{X}_T \tag{18a}$$

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{1}{2}\left(\frac{q_{i,t+1}^\top g_{i,t}(x_{i,t})}{\gamma_t} + \frac{\sigma\|\epsilon_{i,t}^z\|^2}{2\gamma_0}\right) \leq h_T(y) + \tilde{h}_T(y) \quad \forall y \in \mathcal{X}_T \tag{18b}$$

$$\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_{i,t} - x_{j,t}\| \leq n\varepsilon_1 + \tilde{\varepsilon}_2\sum_{t=1}^{T}\sum_{i=1}^{n}\|\epsilon_{i,t}^z\| \tag{18c}$$

$$\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_{i,t} - x_{j,t}\|^2 \leq \tilde{\varepsilon}_3 + \tilde{\varepsilon}_4\sum_{t=1}^{T}\sum_{i=1}^{n}\|\epsilon_{i,t}^z\|^2 \tag{18d}$$

$$\|\epsilon_{i,t}^z\| \leq \frac{1}{\sigma}(G_2\gamma_0\|[g_{i,t}(x_{i,t})]_+\| + G_1\alpha_t) \tag{18e}$$

where

$$\Delta_{i,t}(y) = \frac{1}{\alpha_t}(\mathcal{D}_\psi(y, x_{i,t}) - \mathcal{D}_\psi(y, x_{i,t+1}))$$

$$h_T(y) = \sum_{i=1}^n \sum_{t=1}^T \frac{q_{i,t+1}^\top g_{i,t}(y)}{\gamma_t}, \ \epsilon_{i,t}^z = z_{i,t+1} - x_{i,t}$$

$$\tilde{h}_T(y) = \sum_{t=1}^T \frac{nF}{\gamma_t} + \sum_{t=1}^T \frac{2n\gamma_0 G_1^2}{\sigma\gamma_t^2} + \sum_{i=1}^n \frac{\mathcal{D}_\psi(y, x_{i,1})}{\gamma_0}$$

$$\varepsilon_1 = \frac{2\tau}{\lambda(1-\lambda)} \sum_{i=1}^n \|z_{i,1}\|, \ \tilde{\varepsilon}_2 = \frac{2(n\tau + 2 - 2\lambda)}{1-\lambda}$$

$$\tilde{\varepsilon}_3 = \frac{16n\tau^2}{\lambda^2(1-\lambda^2)} \left(\sum_{i=1}^n \|z_{i,1}\|\right)^2, \ \tilde{\varepsilon}_4 = \frac{16n^2\tau^2}{(1-\lambda)^2} + 32$$

$$\tau = (1 - w/4n^2)^{-2} > 1, \ \lambda = (1 - w/4n^2)^{1/B} \in (0, 1).$$

Proof: See Appendix B. □

As mentioned in the previous section, the key innovation of Algorithm 1 lies in the utilization of the clipped constraint function solely for updating the dual variables, rather than directly replacing the original constraint function. With Lemma 1, in the subsequent discussion, we will demonstrate how to obtain static network regret and cumulative constraint violation bounds for Algorithm 1. We will specifically highlight the significance of the aforementioned novelty in achieving (network) cumulative constraint violation bounds in both scenarios: without and with Slater's condition. In addition, we will provide an elucidation of why (network) cumulative constraint violation bounds can be reduced under Slater's condition.

First, noting that (18a) and (18c), respectively, provide an upper bound for local regret and the accumulated consensus error, an upper bound for the static network regret can be derived by combining (11a), (18a), and (18c).

Second, noting that

$$\frac{q_{i,t+1}^\top g_{i,t}(x_{i,t})}{\gamma_t} = [g_{i,t}(x_{i,t})]_+^\top g_{i,t}(x_{i,t}) = \|g_{i,t}(x_{i,t})\|^2 \quad (19)$$

due to (17a) and the fact that $a^\top[a]_+ = \|[a]_+\|^2$ for any vector a, we know that (18b) provides an upper bound for local squared cumulative constraint violation. Moreover, note that (18d) provides an upper bound for the accumulated squared consensus error, which is $\mathcal{O}(\tilde{h}_T(y))$ since $h_T(y) \le 0 \,\forall y \in \mathcal{X}_T$. Then, we can get an upper bound for network squared cumulative constraint violation by combining (11b) (18b), and (18d). As a result, an $\mathcal{O}((T\tilde{h}_T(y))^{1/2})$ bound for network cumulative constraint violation can be derived since

$$\left(\frac{1}{n}\sum_{i=1}^n \sum_{t=1}^T \|[g_t(x_{i,t})]_+\|\right)^2 \le \frac{T}{n}\sum_{i=1}^n \sum_{t=1}^T \|[g_t(x_{i,t})]_+\|^2 \quad (20)$$

which holds due to the Hölder's inequality.

Third, when Slater's condition holds, noting that

$$h_T(x_s) = \sum_{i=1}^n \sum_{t=1}^T \frac{q_{i,t+1}^\top g_{i,t}(x_s)}{\gamma_t}$$

$$= \sum_{i=1}^n \sum_{t=1}^T [g_{i,t}(x_{i,t})]_+^\top g_{i,t}(x_s) \le -\sum_{i=1}^n \sum_{t=1}^T \epsilon_s [g_{i,t}(x_{i,t})]_+^\top \mathbf{1}_{m_i}$$

$$= -\epsilon_s \sum_{i=1}^n \sum_{t=1}^T \|[g_{i,t}(x_{i,t})]_+\|_1 \le -\epsilon_s \sum_{i=1}^n \sum_{t=1}^T \|[g_{i,t}(x_{i,t})]_+\| \quad (21)$$

where the second equality holds due to (17a) and the first inequality holds due to (13), we know that (18b) provides an upper bound for local cumulative constraint violation, which is $\mathcal{O}(\tilde{h}_T(x_s))$. On the other hand, from (11b), (18c), and (18e), we know that network cumulative constraint violation can be bounded by local cumulative constraint violation. As a result, we can get an $\mathcal{O}(\tilde{h}_T(x_s))$ bound for network cumulative constraint violation.

Note that $\mathcal{O}(\tilde{h}_T(x_s))$, the upper bound for network cumulative constraint violation under Slater's condition, has the same order with respect to T as $\mathcal{O}(\tilde{h}_T(y))$, and thus, it has a smaller order with respect to T than $\mathcal{O}((T\tilde{h}_T(y))^{1/2})$ the upper bound for (network) cumulative constraint violation without Slater's condition, since it can be guaranteed that $\mathcal{O}(\tilde{h}_T(y)) = \mathbf{o}(T)$ by appropriately selecting the stepsize sequence $\{\alpha_t\}$. Therefore, tighter (network) cumulative constraint violation bounds can be achieved under Slater's condition.

With the explanations above, static network regret and network cumulative constraint violation bounds for the general cases are provided in the following lemma.

Lemma 2: Under the same condition as stated in Lemma 1, for any $T \in \mathbb{N}_+$, it holds that

$$\frac{1}{n}\sum_{i=1}^n \sum_{t=1}^T l_t(x_{i,t}) - \sum_{t=1}^T l_t(y)$$

$$\le \varepsilon_1 G_1 + \sum_{t=1}^T \varepsilon_2 \alpha_t + \frac{1}{n}\sum_{t=1}^T \sum_{i=1}^n \Delta_{i,t}(y) \quad \forall y \in \mathcal{X}_T \quad (22a)$$

$$\frac{1}{n}\sum_{i=1}^n \sum_{t=1}^T \|[g_t(x_{i,t})]_+\| \le \sqrt{\varepsilon_3 T + \varepsilon_4 T \tilde{h}_T(y)} \quad \forall y \in \mathcal{X}_T \quad (22b)$$

$$\frac{1}{n}\sum_{i=1}^n \sum_{t=1}^T \|[g_t(x_{i,t})]_+\|$$

$$\le n\varepsilon_1 G_2 + \varepsilon_5 \sum_{t=1}^T \alpha_t + \varepsilon_6 \sum_{t=1}^T \sum_{i=1}^n \|[g_{i,t}(x_{i,t})]_+\| \quad (22c)$$

where

$$\varepsilon_2 = \frac{(\tilde{\varepsilon}_2^2 + 2)G_1^2}{\sigma}, \ \varepsilon_3 = 2G_2^2\tilde{\varepsilon}_3, \ \varepsilon_4 = \frac{4\max\{1, G_2^2\tilde{\varepsilon}_4\}}{\min\{1, \frac{\sigma}{2\gamma_0}\}},$$

$$\varepsilon_5 = \frac{n\tilde{\varepsilon}_2 G_1 G_2}{\sigma}, \ \varepsilon_6 = \frac{\tilde{\varepsilon}_2 G_2^2 \gamma_0 + \sigma}{\sigma}.$$

Proof: See Appendix C. □

B. Convexity Scenario

In this section, we show that the static network regret and network cumulative constraint violation bounds grow sublinearly if the natural vanishing stepsize is used.

Theorem 1: Suppose Assumptions 1–6 hold. For all $i \in [n]$, let $\{x_{i,t}\}$ be the sequences generated by Algorithm 1 with

$$\alpha_t = \frac{1}{t^c}, \ \gamma_t = \frac{\gamma_0}{\alpha_t} \quad \forall t \in \mathbb{N}_+ \quad (23)$$

where $c \in (0,1)$ and $\gamma_0 \in (0, \sigma/(4G_2^2)]$ are constants. Then, for any $T \in \mathbb{N}_+$

$$\text{Net-Reg}(T) = \mathcal{O}(T^{\max\{c,1-c\}}) \tag{24a}$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\| = \mathcal{O}(T^{1-c/2}). \tag{24b}$$

Proof: The explicit expressions of the right-hand sides of (24) and the proof are given in Appendix D. □

Due to space limitations, this article only considers static regret for ease of presentation. With some modifications of the proof, we can show that Algorithm 1 achieves an $\mathcal{O}(\alpha_0 T^{1-c} + \frac{T^c(1+P_T)}{\alpha_0})$ dynamic regret bound, where $\alpha_0 > 0$ can be arbitrarily chosen to design the stepsize $\alpha_t = \alpha_0/t^c$ and $P_T = \sum_{t=1}^{T-1}\|x_{t+1}^* - x_t^*\|$ is the path-length of the optimal decision sequence $\{x_t^*\}$. This bound is standard and has also been achieved in the literature, e.g., [4].

Theorem 1 shows that Algorithm 1 generalizes the results in [16], [24], [38], and [39]. Specifically, by setting $c = 0.5$ in Theorem 1, the result in [16] is recovered, although the algorithm proposed in [16] is centralized and the standard constraint violation metric rather than the stricter metric is used. The bounds achieved in (24) are consistent with the results in [24], [38], and [39], although in [24], [38], and [39], the constraint functions are time-invariant and known in advance. Moreover, in [24], the proposed algorithm is centralized, and in [38], the loss functions are quadratic and the constraint functions are linear. Theorem 1 also shows that Algorithm 1 achieves improved performance compared with the distributed online algorithm proposed in [37], although the global constraint functions in [37] are time-invariant and known in advance by each agent. Specifically, $\text{Net-Reg}(T) = \mathcal{O}(T^{0.5+\beta})$ and $\frac{1}{n}\sum_{i=1}^{n}\|[\sum_{t=1}^{T} g(x_{i,t})]_+\| = \mathcal{O}(T^{1-\beta/2})$ were achieved in [37], where $\beta \in (0, 0.5)$. The same bounds as shown in (24) have also been achieved by the distributed online algorithm proposed in [40]. Compared to the algorithm in [40], the potential drawback of Algorithm 1 is that it uses G_2, the uniform bound for the gradients of the local constraint functions, to design the algorithm parameter γ_0. Note that this bound is available if the global constraint function is time-invariant and known in advance by each agent as assumed in [37], [38], and [39]. The advantage of Algorithm 1 is that it can achieve tighter network cumulative constraint violation bounds when Slater's condition holds, as shown in the following, while the work in [16], [24], [37], [38], [39], and [40] do not have such a result.

To close the present section, similar to the discussions in [39], we could discuss how the different components of the problem, such as the properties of cost/constraint functions as well as the network properties, impact the scaling constants in hidden by the $\mathcal{O}(\cdot)$ notation from its explicit expression given in the proof. For example, it is clear to see that the explicit expression of static regret bound presented in (64) in the proof is monotonically increasing with respect to F, G_1, and G_2, which also matches the intuition. Due to space limitations, we omit the discussions.

C. Slater's Condition Scenario

In this section, we show that Algorithm 1 achieves a tighter network cumulative constraint violation bound under Slater's condition.

Theorem 2: Suppose Assumptions 1–7 hold. For all $i \in [n]$, let $\{x_{i,t}\}$ be the sequences generated by Algorithm 1 with (23). Then, for any $T \in \mathbb{N}_+$

$$\text{Net-Reg}(T) = \mathcal{O}(T^{\max\{c,1-c\}}) \tag{25a}$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\| = \mathcal{O}(T^{1-c}). \tag{25b}$$

Proof: The explicit expressions of the right-hand sides of (25) and the proof are given in Appendix E. □

Under Slater's condition, the centralized algorithms proposed in [17] and [18] achieved $\mathcal{O}(\sqrt{T})$ static regret and constraint violation bounds. In [24], it was emphasized that extending the results in [17] and [18] to incorporate the stricter cumulative constraint violation metric remains unclear. This is due to the fact that the clipped constraint functions $\{[g_t]_+\}$ fail to satisfy Slater's condition, even if the original constraint functions $\{g_t\}$ do satisfy it. It is an open problem how to achieve tighter cumulative constraint violation bounds for online convex optimization with time-varying constraints under Slater's condition. Theorem 2 shows that Algorithm 1 extends the results presented in [17] and [18], thereby solving the aforementioned open problem. Specifically, the results in [17] and [18] are recovered when setting $c = 0.5$ in Theorem 2.

D. Strong Convexity Scenario

The static network regret bound provided in Theorems 1 and 2 is at least $\mathcal{O}(\sqrt{T})$ and it can be reduced to strictly less than $\mathcal{O}(\sqrt{T})$ if the local loss functions are strongly convex. Without loss of generality, we assume that the private parts $\{f_{i,t}(x)\}$ are strongly convex.

Assumption 8: For any $i \in [n]$ and $t \in \mathbb{N}_+$, $\{f_{i,t}(x)\}$ are strongly convex over $\mathbb{X}$ with respect to ψ with $\mu > 0$, i.e., for all $x, y \in \mathbb{X}$

$$f_{i,t}(x) \geq f_{i,t}(y) + \langle x - y, \nabla f_{i,t}(y)\rangle + \mu\mathcal{D}_\psi(x,y). \tag{26}$$

Theorem 3: Suppose Assumptions 1–6 and 8 hold. For all $i \in [n]$, let $\{x_{i,t}\}$ be the sequences generated by Algorithm 1 with (23). Then, for any $T \in \mathbb{N}_+$

$$\text{Net-Reg}(T) = \mathcal{O}(T^{1-c}) \tag{27a}$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\| = \mathcal{O}(T^{1-c/2}). \tag{27b}$$

Moreover, if Assumption 7 also holds, then

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\| = \mathcal{O}(T^{1-c}). \tag{28}$$

Proof: The explicit expressions of the right-hand sides of (27) and (28) and the proof are given in Appendix F. □

The same bounds as shown in (27) have also been achieved by the distributed online algorithm proposed in [40]. It should be pointed out that Algorithm 1 achieves a tighter network cumulative constraint violation bound under Slater's condition, as shown in (28), while the algorithm in [40] does not have such a property. Both the static network regret and the cumulative constraint violation bounds can be further reduced if the convex parameter μ is known in advance.

Theorem 4: Suppose Assumptions 1–5 and 8 hold. For all $i \in [n]$, let $\{x_{i,t}\}$ be the sequences generated by Algorithm 1 with

$$\alpha_t = \frac{1}{\mu t},\ \gamma_t = \frac{\gamma_0}{\alpha_t} \quad \forall t \in \mathbb{N}_+ \tag{29}$$

where $\gamma_0 \in (0, \sigma/(4G_2^2)]$ is a constant. Then, for any $T \in \mathbb{N}_+$

$$\text{Net-Reg}(T) = \mathcal{O}(\log(T)) \tag{30a}$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\| = \mathcal{O}(\sqrt{\log(T)T}). \tag{30b}$$

Moreover, if Assumption 7 also holds, then

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\| = \mathcal{O}(\log(T)). \tag{31}$$

Proof: The explicit expressions of the right-hand sides of (30) and (31) and the proof are given in Appendix G. □

The bounds achieved in (30) are consistent with the results in [24] and [39] for strongly convex loss functions, although the proposed algorithm in [24] is centralized, and the constraint functions in [24] and [39] are time-invariant and known in advance. Moreover, when comparing (30) with the results that $\text{Net-Reg}(T) = \mathcal{O}(T^c)$ and $\frac{1}{n}\sum_{i=1}^{n}\|[\sum_{t=1}^{T} g(x_{i,t})]_+\| = \mathcal{O}(T^{1-c/2})$ as achieved in [37] for strongly convex loss functions, we know that our Algorithm 1 achieves improved performance, although the global constraint function in [37] is time-invariant and known in advance by each agent. It should be highlighted that Algorithm 1 achieves an $\mathcal{O}(\log(T))$ network cumulative constraint violation bound under Slater's condition, as shown in (31), which, to the best of the authors' knowledge, is achieved for the first time in the literature.

Note that Assumption 6 is not needed in Theorem 4. This assumption is not needed in Theorems 1 and 2 either when knowing the total number of rounds T in advance and choosing $\alpha_t = 1/T^c$ in (23). In this case, with slight modifications of the proof, we can show that the results stated in Theorems 1 and 2 still hold.

V. Simulations

In this section, we verify the theoretical results through numerical simulations.

Note that, as pointed out in the previous section, without Slater's condition, the proposed algorithm achieves the same static network regret and network cumulative constraint violation bounds as those in [40]. It is not this article's goal to show that the proposed algorithm has better performance without Slater's condition. The key contribution of this article is demonstrating that tighter (network) cumulative constraint violation bounds can be achieved when Slater's condition holds, which is a significant result not found in existing literature. Therefore, we only consider examples where the constraint function satisfies Slater's condition. Specifically, similar to [40], we consider distributed online linear regression problem with time-varying nonlinear inequality constraints formulated as follows:[1]

$$\begin{aligned}\min_{x} \quad & \sum_{t=1}^{T}\sum_{i=1}^{n}\frac{1}{2}\|H_{i,t}x - h_{i,t}\|^2 \\ \text{s.t.} \quad & x \in \mathbb{X},\ b_{i,t} - \log(1+\|x\|^2) \le 0 \quad \forall i \in [n]\ \forall t \in [T]\end{aligned}$$

[1]For a fair comparison, we do not consider regularization in the loss, i.e., set $r_t(\cdot) \equiv 0$.

TABLE II
Input of Each Algorithm

Algorithms	Inputs
Algorithm 1	$\alpha_t = 1/t$, $\gamma_t = 0.15/\alpha_t$, and $\psi(x) = \|x\|^2$
Algorithm 1 in [17]	$\alpha = T$, and $V = \sqrt{\alpha}$
Algorithm 1 in [40]	$\alpha_t = 0.7/t$, $\beta_t = 1/\sqrt{t}$, and $\gamma_t = 1/\sqrt{t}$

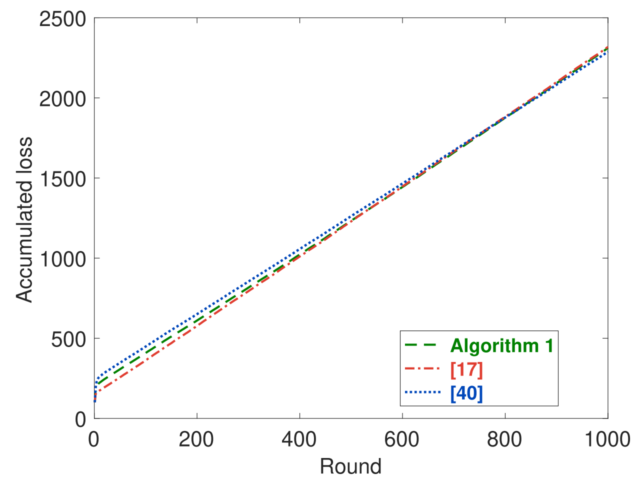

Fig. 1. Trajectories of the accumulated loss.

where $H_{i,t} \in \mathbb{R}^{d_i \times p}$, $h_{i,t} \in \mathbb{R}^{d_i}$, and $b_{i,t} \in \mathbb{R}$ with $d_i \in \mathbb{N}_+$. Each component of $H_{i,t}$ is generated from $\mathcal{U}(-1,1)$ and $h_{i,t} = H_{i,t}\mathbf{1}_p + \varepsilon_{i,t}$, where $\varepsilon_{i,t}$ is a standard normal random vector. Moreover, $b_{i,t}$ is generated from $\mathcal{U}(2,3)$ and thus Slater's condition holds. At each time t, an undirected random graph is used as the communication graph. Specifically, connections between agents are random and the probability of two agents being connected is ρ. To guarantee that Assumption 4 holds, edges $(i, i+1)$, $i \in [n-1]$ are also added and $[W_t]_{ij} = 1/n$ if $(j,i) \in \mathcal{E}_t$ and $[W_t]_{ii} = 1 - \sum_{j=1}^{n}[W_t]_{ij}$.

We compare Algorithm 1 with the centralized algorithm in [17] (which considers Slater's condition but uses the standard constraint violation metric) and the distributed algorithm in [40] (which uses the stricter cumulative constraint violation metric but does not consider Slater's condition). We set $n = 100$, $d_i = 4$, $p = 10$, $\mathbb{X} = [-5,5]^p$, and $\rho = 0.1$. The inputs of these algorithms are listed in Table II. For the considered example, Figs. 1 and 2 illustrate the trajectories of the accumulated loss and the cumulative constraint violation, respectively. From Fig. 1, it is evident that these algorithms exhibit nearly identical accumulated losses, consistent with the theoretical findings. Turning to Fig. 2, we observe a substantial reduction in cumulative constraint violation for the algorithm in [40] compared to [17]. This is expected since Yu et al. [17] employed the standard constraint violation metric rather than the stricter one. More notably, Fig. 2 demonstrates that our proposed algorithm achieves a significantly lower cumulative constraint violation than the one in [40], aligning with the theoretical predictions. As elucidated at the end of Section III, this outcome can be attributed to the fact that Yi et al. [40] directly replaced the original constraint function with the clipped constraint function, rendering Slater's condition ineffective. In contrast, our approach

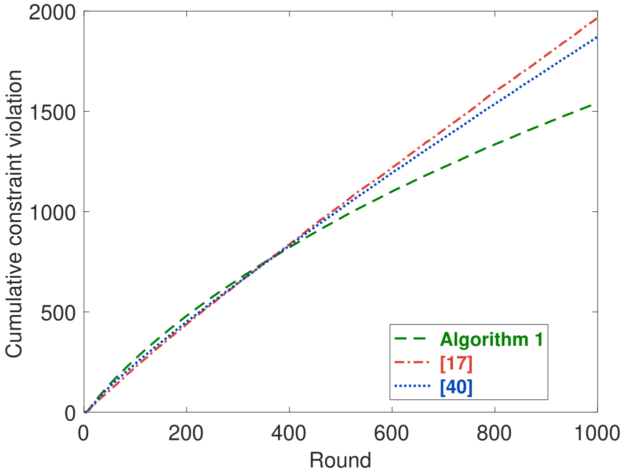

Fig. 2. Trajectories of the cumulative constraint violation.

utilizes the clipped constraint function exclusively for updating dual variables, allowing us to employ the stricter cumulative constraint violation metric while maintaining the effectiveness of Slater's condition.

VI. Conclusion

In this article, we addressed the problem of distributed online convex optimization with time-varying constraints. We proposed a novel distributed online algorithm to solve this problem and conducted an analysis of the static network regret and network cumulative constraint violation bounds for the proposed algorithm under different scenarios. Notably, we demonstrated that, for the first time, tighter (network) cumulative constraint violation bounds can be achieved under Slater's condition, both in convex and strongly convex scenarios. In the future, we plan to explore the challenging bandit setting since in various applications only the values of the loss and constraint functions are available. In addition, we aim to tackle the issue of reducing communication complexity, as previous research has highlighted the significant impact of communication on distributed algorithms.

Appendix

A. Useful Lemmas

We first present some results on the regularized Bregman projection.

Lemma 3 (See [42, Lemma 1]): Suppose that $h : \mathbb{X} \to \mathbb{R}$ is a convex function and $\nabla h(x)\,\forall x \in \mathbb{X}$, exists. Then, for any $y, z \in \mathbb{X}$, the regularized Bregman projection

$$\tilde{x} = \arg\min_{x\in\mathbb{X}}\{h(x) + \mathcal{D}_\psi(x, z)\}$$

satisfies the following inequalities:

$$\langle \nabla h(\tilde{x}), \tilde{x} - y\rangle \le \mathcal{D}_\psi(y, z) - \mathcal{D}_\psi(y, \tilde{x}) - \mathcal{D}_\psi(\tilde{x}, z) \tag{32a}$$

$$\|\tilde{x} - z\| \le \frac{\|\nabla h(z)\|}{\sigma}. \tag{32b}$$

We next quantify the disagreement among the local temporary primal variables $\{z_{i,t}\}$.

Lemma 4: If Assumption 4 holds, for all $i \in [n]$ and $t \in \mathbb{N}_+$, then $z_{i,t}$ generated by Algorithm 1 satisfy

$$\|z_{i,t} - \bar{z}_t\| \le \tau\lambda^{t-2}\sum_{j=1}^{n}\|z_{j,1}\| + \tau\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{j=1}^{n}\|\epsilon^z_{j,s}\| + \|\epsilon^z_{i,t-1}\| + \frac{1}{n}\sum_{j=1}^{n}\|\epsilon^z_{j,t-1}\| \tag{33}$$

where $\bar{z}_t = \frac{1}{n}\sum_{i=1}^n z_{i,t}$.

Proof: From (16) and $\epsilon^z_{i,t-1} = z_{i,t} - x_{i,t-1}$, we have

$$z_{i,t} = \sum_{j=1}^{n}[W_{t-1}]_{ij} z_{j,t-1} + \epsilon^z_{i,t-1}.$$

Then, following the proof of [40, Lemma 4], we know that the result holds. □

We finally analyze regret at one round.

Lemma 5: Suppose Assumptions 1 and 3–5 hold. For all $i \in [n]$, let $\{x_{i,t}\}$ be the sequences generated by Algorithm 1 and y be an arbitrary point in $\mathbb{X}$, then

$$\frac{1}{n}\sum_{i=1}^{n} q_{i,t+1}^\top g_{i,t}(x_{i,t}) + \frac{1}{n}\sum_{i=1}^{n}(l_{i,t}(x_{i,t}) - l_{i,t}(y)) \le \frac{1}{n}\sum_{i=1}^{n} q_{i,t+1}^\top g_{i,t}(y) + \frac{\tilde{\Delta}_t}{n} + \frac{1}{n}\sum_{i=1}^{n}\Delta_{i,t}(y) \tag{34}$$

where

$$\tilde{\Delta}_t = \sum_{i=1}^{n}(G_1 + G_2\|q_{i,t+1}\|)\|\epsilon^z_{i,t}\| - \sum_{i=1}^{n}\frac{\sigma\|\epsilon^z_{i,t}\|^2}{2\alpha_t}.$$

Proof: From Assumptions 1 and 3, we have

$$f_{i,t}(y) \ge f_{i,t}(x) + \langle \nabla f_{i,t}(x), y - x\rangle \quad \forall x, y \in \mathbb{X} \tag{35a}$$

$$r_t(y) \ge r_t(x) + \langle \nabla r_t(x), y - x\rangle \quad \forall x, y \in \mathbb{X} \tag{35b}$$

$$g_{i,t}(y) \ge g_{i,t}(x) + \nabla g_{i,t}(x)(y - x) \quad \forall x, y \in \mathbb{X}. \tag{35c}$$

From (10), (35a), and (35b), it holds that

$$\begin{aligned} & l_{i,t}(x_{i,t}) - l_{i,t}(y) \\ &= f_{i,t}(x_{i,t}) - f_{i,t}(y) + r_t(x_{i,t}) - r_t(z_{i,t+1}) \\ &\quad + r_t(z_{i,t+1}) - r_t(y) \\ &\le \langle \nabla f_{i,t}(x_{i,t}), x_{i,t} - y\rangle + \langle \nabla r_t(x_{i,t}), x_{i,t} - z_{i,t+1}\rangle \\ &\quad + \langle \nabla r_t(z_{i,t+1}), z_{i,t+1} - y\rangle \\ &\le G_1\|\epsilon^z_{i,t}\| + \langle \nabla f_{i,t}(x_{i,t}) + \nabla r_t(z_{i,t+1}), z_{i,t+1} - y\rangle. \end{aligned} \tag{36}$$

For the second term on the right-hand side of (36), we have

$$\begin{aligned} &\langle \nabla f_{i,t}(x_{i,t}) + \nabla r_t(z_{i,t+1}), z_{i,t+1} - y\rangle \\ &= \langle (\nabla g_{i,t}(x_{i,t}))^\top q_{i,t+1}, y - x_{i,t}\rangle \\ &\quad + \langle (\nabla g_{i,t}(x_{i,t}))^\top q_{i,t+1}, x_{i,t} - z_{i,t+1}\rangle \\ &\quad + \langle \omega_{i,t+1} + \nabla r_t(z_{i,t+1}), z_{i,t+1} - y\rangle. \end{aligned} \tag{37}$$

We next to find the upper bound of each term on the right-hand side of (37).

From (35c) and $q_{i,t} \ge \mathbf{0}_{m_i}\ \forall i \in [n]\, \forall t \in \mathbb{N}_+$, it holds that

$$\langle (\nabla g_{i,t}(x_{i,t}))^\top q_{i,t+1}, y - x_{i,t}\rangle$$

$$\leq q_{i,t+1}^\top g_{i,t}(y) - q_{i,t+1}^\top g_{i,t}(x_{i,t}). \tag{38}$$

From the Cauchy–Schwarz inequality and (10), we have

$$\langle (\nabla g_{i,t}(x_{i,t}))^\top q_{i,t+1}, x_{i,t} - z_{i,t+1}\rangle \leq G_2\|q_{i,t+1}\|\|\epsilon_{i,t}^z\|. \tag{39}$$

Applying (32a) to the update (17c) implies that

$$\begin{aligned}
&\langle \omega_{i,t+1} + \nabla r_t(z_{i,t+1}), z_{i,t+1} - y\rangle \\
&\leq \frac{1}{\alpha_t}\left(\mathcal{D}_\psi(y, x_{i,t}) - \mathcal{D}_\psi(y, z_{i,t+1}) - \mathcal{D}_\psi(z_{i,t+1}, x_{i,t})\right) \\
&= \Delta_{i,t}(y) + \frac{1}{\alpha_t}\left(\mathcal{D}_\psi\left(y, \sum_{j=1}^n [W_{t+1}]_{ij} z_{j,t+1}\right)\right. \\
&\quad \left. - \mathcal{D}_\psi(y, z_{i,t+1}) - \mathcal{D}_\psi(z_{i,t+1}, x_{i,t})\right) \\
&\leq \Delta_{i,t}(y) + \frac{1}{\alpha_t}\left(\sum_{j=1}^n [W_{t+1}]_{ij}\mathcal{D}_\psi(y, z_{j,t+1})\right. \\
&\quad \left. - \mathcal{D}_\psi(y, z_{i,t+1}) - \frac{\sigma}{2}\|z_{i,t+1} - x_{i,t}\|^2\right)
\end{aligned} \tag{40}$$

where the equality holds due to (16), and the last inequality holds since W_{t+1} is doubly stochastic, Assumption 5, and (8).

Summing (36)–(40) over $i \in [n]$, dividing by n, using $\sum_{i=1}^n [W_t]_{ij} = 1 \, \forall t \in \mathbb{N}_+$, and rearranging terms yields (34). □

B. Proof of Lemma 1

(i) Noting that $g_{i,t}(y) \leq \mathbf{0}_{m_i} \, \forall i \in [n] \, \forall t \in \mathbb{N}_+$ when $y \in \mathcal{X}_T$, summing (34) over $t \in [T]$ gives

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n\sum_{t=1}^T (l_{i,t}(x_{i,t}) - l_{i,t}(y)) \\
&\leq \frac{1}{n}\sum_{i=1}^n\sum_{t=1}^T\left(-q_{i,t+1}^\top g_{i,t}(x_{i,t}) + \frac{1}{n}\tilde{\Delta}_t + \Delta_{i,t}(y)\right).
\end{aligned} \tag{41}$$

The Cauchy–Schwarz inequality implies that

$$\begin{aligned}
&\sum_{t=1}^T\sum_{i=1}^n (G_1 + G_2\|q_{i,t+1}\|)\|\epsilon_{i,t}^z\| \\
&\leq \sum_{t=1}^T\sum_{i=1}^n\left(\frac{2G_1^2\alpha_t}{\sigma} + \frac{2G_2^2\alpha_t\|q_{i,t+1}\|^2}{\sigma} + \frac{\sigma\|\epsilon_{i,t}^z\|^2}{4\alpha_t}\right).
\end{aligned} \tag{42}$$

From (17a), for all $t \in \mathbb{N}_+$, it holds that

$$\|q_{i,t+1}\| = \gamma_t\|[g_{i,t}(x_{i,t})]_+\|. \tag{43}$$

Then, from (19), (43), and $\gamma_t\alpha_t = \gamma_0$, we have

$$\begin{aligned}
&\frac{2G_2^2\alpha_t\|q_{i,t+1}\|^2}{\sigma} - q_{i,t+1}^\top g_{i,t}(x_{i,t}) \\
&= \left(\frac{2G_2^2\gamma_0}{\sigma} - 1\right)\gamma_t\|[g_{i,t}(x_{i,t})]_+\|^2 \leq 0
\end{aligned} \tag{44}$$

where the inequality holds since $\gamma_0 \leq \sigma/(4G_2^2)$.

Combining (41), (42), and (44) yields (18a).

(ii) It follows Assumption 2 that

$$l_{i,t}(y) - l_{i,t}(x_{i,t}) \leq F \quad \forall y \in \mathbb{X}. \tag{45}$$

Dividing (34) by γ_t, using (45), and summing over $t \in [T]$ gives

$$\begin{aligned}
&\sum_{t=1}^T\sum_{i=1}^n \frac{q_{i,t+1}^\top g_{i,t}(x_{i,t})}{\gamma_t} \\
&\leq h_T(y) + \sum_{t=1}^T \frac{nF}{\gamma_t} + \sum_{t=1}^T \frac{\tilde{\Delta}_t}{\gamma_t} + \sum_{t=1}^T\sum_{i=1}^n \frac{\Delta_{i,t}(y)}{\gamma_t}.
\end{aligned} \tag{46}$$

The Cauchy–Schwarz inequality implies that

$$\begin{aligned}
&\sum_{t=1}^T\sum_{i=1}^n \frac{(G_1 + G_2\|q_{i,t+1}\|)\|\epsilon_{i,t}^z\|}{\gamma_t} \\
&\leq \sum_{t=1}^T\sum_{i=1}^n\left(\frac{2\gamma_0(G_1^2 + G_2^2\|q_{i,t+1}\|^2)}{\sigma\gamma_t^2} + \frac{\sigma\|\epsilon_{i,t}^z\|^2}{4\gamma_0}\right).
\end{aligned} \tag{47}$$

It follows $\gamma_t\alpha_t = \gamma_0$ that

$$\begin{aligned}
\sum_{t=1}^T \frac{\Delta_{i,t}(y)}{\gamma_t} &= \sum_{t=1}^T \frac{1}{\gamma_0}\left(\mathcal{D}_\psi(y, x_{i,t}) - \mathcal{D}_\psi(y, x_{i,t+1})\right) \\
&\leq \frac{\mathcal{D}_\psi(y, x_{i,1})}{\gamma_0}.
\end{aligned} \tag{48}$$

Combining (19), (43), and (46)–(48) and using $1/2 \geq 2\gamma_0 G_2^2/\sigma$ yield (18b).

(iii) From (16) and $\sum_{i=1}^n [W_t]_{ij} = \sum_{j=1}^n [W_t]_{ij} = 1$, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \|x_{i,t} - x_{j,t}\| \\
&\leq 2\sum_{i=1}^n\left\|\sum_{j=1}^n [W_t]_{ij} z_{j,t} - \bar{z}_t\right\| = 2\sum_{i=1}^n\left\|\sum_{j=1}^n [W_t]_{ij}(z_{j,t} - \bar{z}_t)\right\| \\
&\leq 2\sum_{i=1}^n\sum_{j=1}^n [W_t]_{ij}\|z_{j,t} - \bar{z}_t\| = 2\sum_{i=1}^n \|z_{i,t} - \bar{z}_t\|.
\end{aligned} \tag{49}$$

Clearly

$$\begin{aligned}
\sum_{t=3}^T\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{j=1}^n \|\epsilon_{j,s}^z\| &= \sum_{t=1}^{T-2}\sum_{j=1}^n \|\epsilon_{j,t}^z\| \sum_{s=0}^{T-t-2}\lambda^s \\
&\leq \frac{1}{1-\lambda}\sum_{t=1}^{T-2}\sum_{j=1}^n \|\epsilon_{j,t}^z\|.
\end{aligned} \tag{50}$$

From (33), (49), (50), it holds that

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^T\sum_{i=1}^n\sum_{j=1}^n \|x_{i,t} - x_{j,t}\| \leq \sum_{t=1}^T\sum_{i=1}^n 2\|z_{i,t} - \bar{z}_t\| \\
&\leq \sum_{t=1}^T\sum_{i=1}^n 2\tau\lambda^{t-2}\sum_{j=1}^n \|z_{j,1}\| \\
&\quad + \sum_{t=2}^T\sum_{i=1}^n 2\left(\frac{1}{n}\sum_{j=1}^n \|\epsilon_{j,t-1}^z\| + \|\epsilon_{i,t-1}^z\|\right) \\
&\quad + \sum_{t=3}^T\sum_{i=1}^n 2\tau\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{j=1}^n \|\epsilon_{j,s}^z\|
\end{aligned}$$

$$\leq n\varepsilon_1 + \sum_{t=2}^{T}\sum_{i=1}^{n} 4\|\epsilon^z_{i,t-1}\| + \frac{2n\tau}{1-\lambda}\sum_{t=1}^{T-2}\sum_{j=1}^{n}\|\epsilon^z_{j,t}\|$$

which yields (18c).

(iv)Similar to the way to get (49), from (16), $\sum_{i=1}^n [W_t]_{ij} = \sum_{j=1}^n [W_t]_{ij} = 1$ and $\|\cdot\|^2$ is convex, and we have

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_{i,t} - x_{j,t}\|^2 \leq \sum_{i=1}^{n} 4\|z_{i,t} - \bar{z}_t\|^2. \tag{51}$$

It follows (33) that

$$\begin{aligned}
&4\sum_{t=1}^{T}\sum_{i=1}^{n}\|z_{i,t} - \bar{z}_t\|^2\\
&\leq 4\sum_{i=1}^{n}\sum_{t=1}^{T}\Bigg(\tau\lambda^{t-2}\sum_{j=1}^{n}\|z_{j,1}\| + \|\epsilon^z_{i,t-1}\|\\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\|\epsilon^z_{j,t-1}\| + \tau\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{j=1}^{n}\|\epsilon^z_{j,s}\|\Bigg)^2\\
&\leq 16\sum_{i=1}^{n}\sum_{t=1}^{T}\Bigg(\Bigg(\tau\lambda^{t-2}\sum_{j=1}^{n}\|z_{j,1}\|\Bigg)^2 + \|\epsilon^z_{i,t-1}\|^2\\
&\quad + \Bigg(\frac{1}{n}\sum_{j=1}^{n}\|\epsilon^z_{j,t-1}\|\Bigg)^2 + \Bigg(\tau\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{j=1}^{n}\|\epsilon^z_{j,s}\|\Bigg)^2\Bigg)\\
&\leq 16\sum_{i=1}^{n}\sum_{t=1}^{T}\Bigg(\Bigg(\tau\lambda^{t-2}\sum_{j=1}^{n}\|z_{j,1}\|\Bigg)^2 + \|\epsilon^z_{i,t-1}\|^2\\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\|\epsilon^z_{j,t-1}\|^2\\
&\quad + \tau^2\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{s=1}^{t-2}\lambda^{t-s-2}\Bigg(\sum_{j=1}^{n}\|\epsilon^z_{j,s}\|\Bigg)^2\Bigg)\\
&\leq 16\sum_{i=1}^{n}\sum_{t=1}^{T}\Bigg(\Bigg(\tau\lambda^{t-2}\sum_{j=1}^{n}\|z_{j,1}\|\Bigg)^2 + 2\|\epsilon^z_{i,t-1}\|^2\\
&\quad + \frac{n\tau^2}{1-\lambda}\sum_{s=1}^{t-2}\lambda^{t-s-2}\sum_{j=1}^{n}\|\epsilon^z_{j,s}\|^2\Bigg)\\
&\leq \tilde{\varepsilon}_3 + \tilde{\varepsilon}_4\sum_{t=1}^{T}\sum_{i=1}^{n}\|\epsilon^z_{i,t}\|^2
\end{aligned} \tag{52}$$

where the third inequality holds due to the Hölder's inequality.

Thus, from (51) and (52), (18d) holds.

(v) Applying (32b) to the update (17c) gives

$$\begin{aligned}
\|\epsilon^z_{i,t}\| &= \|z_{i,t+1} - x_{i,t}\| \leq \frac{\alpha_t\|\omega_{i,t+1} + \nabla r_t(x_{i,t})\|}{\sigma}\\
&= \frac{\alpha_t\|\nabla f_{i,t}(x_{i,t}) + (\nabla g_{i,t}(x_{i,t}))^\top q_{i,t+1} + \nabla r_t(x_{i,t})\|}{\sigma}\\
&\leq \frac{1}{\sigma}(G_2\gamma_0\|[g_{i,t}(x_{i,t})]_+\| + G_1\alpha_t)
\end{aligned}$$

where the second equality holds due to (17b); the last inequality holds due to (43) and (10); and the last equality holds due to $\alpha_t\gamma_t = \gamma_0$. Thus, (18e) holds.

C. Proof of Lemma 2

(i) From $l_t(x) = \frac{1}{n}\sum_{j=1}^n l_{j,t}(x)$, it holds that

$$\begin{aligned}
\sum_{i=1}^{n} l_t(x_{i,t}) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} l_{j,t}(x_{i,t})\\
&\leq \sum_{i=1}^{n} l_{i,t}(x_{i,t}) + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} G_1\|x_{i,t} - x_{j,t}\|
\end{aligned} \tag{53}$$

where the inequality holds due to (11a).

It follows (18c) that

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n} G_1\|x_{i,t} - x_{j,t}\|\\
&\leq n\varepsilon_1 G_1 + \sum_{t=1}^{T}\sum_{i=1}^{n}\left(\frac{\tilde{\varepsilon}_2^2 G_1^2\alpha_t}{\sigma} + \frac{\sigma\|\epsilon^z_{i,t}\|^2}{4\alpha_t}\right).
\end{aligned} \tag{54}$$

Combining (18a), (53), and (54) yields (22a).

(ii) Note that

$$\begin{aligned}
&\|[g_{i,t}(x_{i,t})]_+\|^2\\
&\geq \frac{1}{2}\|[g_{i,t}(x_{j,t})]_+\|^2 - \|[g_{i,t}(x_{i,t})]_+ - [g_{i,t}(x_{j,t})]_+\|^2\\
&\geq \frac{1}{2}\|[g_{i,t}(x_{j,t})]_+\|^2 - \|g_{i,t}(x_{i,t}) - g_{i,t}(x_{j,t})\|^2\\
&\geq \frac{1}{2}\|[g_{i,t}(x_{j,t})]_+\|^2 - G_2^2\|x_{i,t} - x_{j,t}\|^2
\end{aligned} \tag{55}$$

where the second and the third inequalities hold due to the nonexpansive property of the projection $[\,\cdot\,]_+$ and (11b), respectively.

It follows $g_t(x) = \mathrm{col}(g_{1,t}(x), \ldots, g_{n,t}(x))$ that

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\|[g_{i,t}(x_{j,t})]_+\|^2 = \sum_{t=1}^{T}\sum_{j=1}^{n}\|[g_t(x_{j,t})]_+\|^2. \tag{56}$$

Summing (55) over $i, j \in [n]$, $t \in [T]$, dividing by n, and using (56) and (18d) yield

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{n}\|[g_t(x_{j,t})]_+\|^2\\
&\leq \varepsilon_3 + \sum_{t=1}^{T}\sum_{i=1}^{n} 2(\|[g_{i,t}(x_{i,t})]_+\|^2 + G_2^2\tilde{\varepsilon}_4\|\epsilon^z_{i,t}\|^2).
\end{aligned} \tag{57}$$

From $g_{i,t}(y) \leq \mathbf{0}_{m_i}$ $\forall i \in [n]$ $\forall t \in \mathbb{N}_+$ when $y \in \mathcal{X}_T$, we have

$$h_T(y) \leq 0. \tag{58}$$

Combining (18b), (57), and (58) and yields

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\|^2 \leq \varepsilon_3 + \varepsilon_4\tilde{h}_T(y) \quad \forall y \in \mathcal{X}_T. \tag{59}$$

Combining (20) and (59) yields (22b).

(iii) From (18c) and (18e), it holds that

$$\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_{i,t}-x_{j,t}\| \le n\varepsilon_1+\tilde{\varepsilon}_2\sum_{t=1}^{T}\sum_{j=1}^{n}\|\epsilon^z_{j,t}\|$$

$$\le n\varepsilon_1+\frac{\tilde{\varepsilon}_2}{\sigma}\sum_{t=1}^{T}\sum_{j=1}^{n}(G_2\gamma_0\|[g_{j,t}(x_{j,t})]_+\|+G_1\alpha_t). \quad (60)$$

It follows (11b) that

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{j,t})]_+\| \le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\|[g_{i,t}(x_{j,t})]_+\|$$

$$\le \frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}(\|[g_{i,t}(x_{i,t})]_+\|+G_2\|x_{i,t}-x_{j,t}\|). \quad (61)$$

Combining (60) and (61) yields (22c).

D. Proof of Theorem 1

(i) It follows (23) that

$$\sum_{t=1}^{T}\alpha_t=\sum_{t=2}^{T}\frac{1}{t^c}+1\le\int_1^T\frac{1}{t^c}dt+1\le\frac{T^{1-c}}{1-c}. \quad (62)$$

Let $\alpha_0=\alpha_1$. From that $\{\alpha_t\}$ is nonincreasing and (12), we have

$$\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\Delta_{i,t}(y)$$
$$\le\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\alpha_0}\mathcal{D}_\psi(y,x_{i,1})-\frac{1}{\alpha_T}\mathcal{D}_\psi(y,x_{i,T+1})\right)$$
$$+\left(\frac{1}{\alpha_T}-\frac{1}{\alpha_0}\right)K$$
$$\le\frac{K}{\alpha_T}\quad\forall y\in\mathbb{X}. \quad (63)$$

Choosing $y=x_T^*$ and combining (22a), (62), and (63) yield

$$\text{Net-Reg}(T)\le\varepsilon_1G_1+\frac{\varepsilon_2}{1-c}T^{1-c}+KT^c \quad (64)$$

which gives (24a).

(ii) It follows (23) that

$$\sum_{t=1}^{T}\frac{\gamma_0}{\gamma_t^2}\le\sum_{t=1}^{T}\frac{1}{\gamma_t}=\frac{1}{\gamma_0}\sum_{t=1}^{T}\frac{1}{t^c}\le\frac{1}{\gamma_0(1-c)}T^{1-c}. \quad (65)$$

It follows (12) that

$$\sum_{i=1}^{n}\frac{\mathcal{D}_\psi(y,x_{i,1})}{\gamma_0}\le\frac{nK}{\gamma_0}\quad\forall y\in\mathbb{X}. \quad (66)$$

Combining (22b), (65), and (66) yields

$$\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\|\right)^2$$
$$\le\varepsilon_3T+\frac{n\varepsilon_4KT}{\gamma_0}+\frac{n\varepsilon_4(F\sigma+2G_1^2)}{\sigma\gamma_0(1-c)}T^{2-c}. \quad (67)$$

Thus, (24b) holds.

E. Proof of Theorem 2

(i) From (64), (25a) holds.

(ii) Choosing $y=x_s$ in (18b) and using (17a) and (21) yield

$$\epsilon_s\sum_{t=1}^{T}\sum_{i=1}^{n}\|[g_{i,t}(x_{i,t})]_+\|\le\tilde{h}_T(x_s). \quad (68)$$

Combining (23), (65), (66), and (68) yields

$$\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_{i,t}(x_{i,t})]_+\|\le n\varepsilon_7T^{1-c} \quad (69)$$

where

$$\varepsilon_7=\frac{1}{\epsilon_s}\left(\frac{F\sigma+2G_1^2}{\sigma\gamma_0(1-c)}+\frac{K}{\gamma_0}\right).$$

From (22c), (23), (62), and (69), we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{j,t})]_+\|\le n\varepsilon_1G_2+\left(\frac{\varepsilon_5}{1-c}+n\varepsilon_6\varepsilon_7\right)T^{1-c} \quad (70)$$

which yields (25b).

F. Proof of Theorem 3

Since Assumption 8 holds, (36) can be replaced by

$$l_{i,t}(x_{i,t})-l_{i,t}(y)$$
$$\le G_1\|\epsilon^z_{i,t}\|-\mu\mathcal{D}_\psi(y,x_{i,t})$$
$$+\langle\nabla f_{i,t}(x_{i,t})+\nabla r_t(z_{i,t+1}),z_{i,t+1}-y\rangle. \quad (71)$$

Note that compared with (36), (71) has an extra term $-\mu\mathcal{D}_\psi(y,x_{i,t})$. Then, (22a) can be replaced by

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}l_t(x_{i,t})-\sum_{t=1}^{T}l_t(y)$$
$$\le\varepsilon_1G_1+\varepsilon_2\sum_{t=1}^{T}\alpha_t$$
$$+\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Delta_{i,t}(y)-\mu\mathcal{D}_\psi(y,x_{i,t}))\quad\forall y\in\mathcal{X}_T. \quad (72)$$

Moreover, (18b) can be replaced by

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{1}{2}\left(\frac{q_{i,t+1}^\top g_{i,t}(x_{i,t})}{\gamma_t}+\frac{\sigma\|\epsilon^z_{i,t}\|^2}{2\gamma_0}\right)$$
$$\le h_T(y)+\tilde{h}_T(y)+\hat{h}_T(y)\quad\forall y\in\mathcal{X}_T \quad (73)$$

where

$$\hat{h}_T(y)=-\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\mu\mathcal{D}_\psi(y,x_{i,t})}{\gamma_t}.$$

As a result, (22b) can be replaced by

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\|$$
$$\le\sqrt{\varepsilon_3T+\varepsilon_4T(\tilde{h}_T(y)+\hat{h}_T(y))}\quad\forall y\in\mathcal{X}_T. \quad (74)$$

(i) Clearly

$$\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}(\Delta_{i,t}(y)-\mu\mathcal{D}_\psi(y,x_{i,t}))$$
$$=\frac{1}{n}\sum_{i=1}^{n}\Bigg(\frac{1}{\alpha_0}\mathcal{D}_\psi(y,x_{i,1})-\frac{1}{\alpha_T}\mathcal{D}_\psi(y,x_{i,T+1})$$
$$+\sum_{t=1}^{T}\left(\frac{1}{\alpha_t}-\frac{1}{\alpha_{t-1}}-\mu\right)\mathcal{D}_\psi(y,x_{i,t})\Bigg). \tag{75}$$

Denote

$$\varepsilon_8=\left\lceil\left(\frac{1}{\mu}\right)^{\frac{1}{1-c}}\right\rceil.$$

Then, it follows (23) that

$$\frac{1}{\alpha_{t+1}}-\frac{1}{\alpha_t}-\mu=\frac{t+1}{(t+1)^{1-c}}-\frac{t}{t^{1-c}}-\mu$$
$$<\frac{1}{t^{1-c}}-\mu\le 0\quad\forall t\ge\varepsilon_8. \tag{76}$$

Choosing $y=x_T^*\in\mathcal{X}_T$ and combining (12), (62), and (72)–(76) yield

$$\text{Net-Reg}(T)$$
$$\le\varepsilon_1G_1+\frac{\varepsilon_2}{1-c}T^{1-c}+\frac{1}{n\alpha_0}\sum_{i=1}^{n}\mathcal{D}_\psi(x^*,x_{i,1})$$
$$+\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{\varepsilon_8}\left(\frac{1}{\alpha_t}-\frac{1}{\alpha_{t-1}}-\mu\right)\mathcal{D}_\psi(x^*,x_{i,t})$$
$$\le\varepsilon_1G_1+\frac{\varepsilon_2}{1-c}T^{1-c}+\varepsilon_8[1-\mu]_+K. \tag{77}$$

Hence, (27a) holds.

(ii) From (67), (27b) holds.

(iii) From (70), (28) holds.

G. Proof of Theorem 4

We know that (72)–(74) still hold.

(i) From (29) and (75), we have

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Delta_{i,t}(y)-\mu\mathcal{D}_\psi(y,x_{i,t}))\le 0. \tag{78}$$

It follows (29) that

$$\sum_{t=1}^{T}\alpha_t=\sum_{t=1}^{T}\frac{1}{\mu t}=\sum_{t=2}^{T}\frac{1}{\mu t}+\frac{1}{\mu}$$
$$\le\int_{1}^{T}\frac{1}{\mu t}dt+\frac{1}{\mu}\le\frac{1}{\mu}(\log(T)+1). \tag{79}$$

Choosing $y=x^*\in\mathcal{X}_T$ in (72) and using (78) and (79) yield

$$\text{Net-Reg}(T)\le\varepsilon_1G_1+\frac{\varepsilon_2}{\mu}(\log(T)+1).$$

Hence, (30a) holds.

(ii) It follows (29) that

$$\sum_{t=1}^{T}\frac{1}{\gamma_{t+1}}=\frac{1}{\gamma_0\mu}\sum_{t=1}^{T}\frac{1}{t}\le\frac{1}{\gamma_0\mu}(\log(T)+1)$$

$$\sum_{t=1}^{T}\frac{\gamma_0}{\gamma_{t+1}^2}\le\frac{1}{\gamma_0\mu^2}\left(\int_{t=2}^{T}\frac{1}{t^2}dt+1\right)\le\frac{2}{\gamma_0\mu^2}$$

$$\sum_{i=1}^{n}\frac{\mathcal{D}_\psi(y,x_{i,1})}{\gamma_0}+\hat{h}_T(y)$$
$$\le\sum_{i=1}^{n}\frac{\mathcal{D}_\psi(y,x_{i,1})}{\gamma_0}-\sum_{i=1}^{n}\frac{\mu\mathcal{D}_\psi(y,x_{i,t})}{\gamma_1}=0\quad\forall y\in\mathbb{X}$$

which yield

$$\tilde{h}_T(y)+\hat{h}_T(y)\le\varepsilon_9+\frac{nF\log(T)}{\gamma_0\mu} \tag{80}$$

where

$$\varepsilon_9=\frac{nF}{\gamma_0\mu}+\frac{4nG_1^2}{\sigma\gamma_0\mu^2}.$$

From (74) and (80), we have

$$\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{i,t})]_+\|\right)^2\le\varepsilon_3T+\varepsilon_4T\left(\varepsilon_9+\frac{nF\log(T)}{\gamma_0\mu}\right)$$

which yields (30b).

(iii) Choosing $y=x_s$ in (73) and using (17a) and (21) yield

$$\epsilon_s\sum_{t=1}^{T}\sum_{i=1}^{n}\|[g_{i,t}(x_{i,t})]_+\|\le\tilde{h}_T(x_s)+\hat{h}_T(x_s). \tag{81}$$

From (22c), (29), and (79)–(81), we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\|[g_t(x_{j,t})]_+\|$$
$$\le n\varepsilon_1G_2+\frac{\varepsilon_5}{\mu}(\log(T)+1)+\frac{\varepsilon_6}{\epsilon_s}\left(\varepsilon_9+\frac{nF\log(T)}{\gamma_0\mu}\right).$$

Hence, (31) holds.

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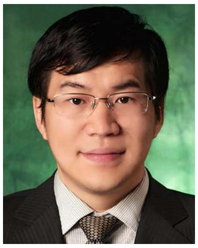

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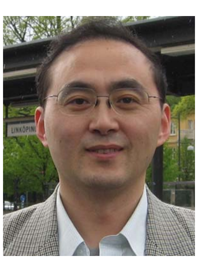

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