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Brief paper

Distributed safety-critical control of nonlinear multi-agent systems ☆

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ABSTRACT

This paper considers the safety-critical control problem for nonlinear second-order multi-agent systems with constraints of each agent and inter-agent ones. We overcome the challenge of the time-varying and position-dependent communication network with limited sensing range by introducing a truncated function for the smooth addition and deletion of links in the edge set, and design a distributed and locally Lipschitz-continuous safety-critical control law, composed of a nominal controller for the objectives such as consensus, formation, and position swapping, etc., and a safety controller, which only takes effect when some neighboring agent enters the custom-designed boundary set. Meanwhile, to rigorously verify the safety of the whole multi-agent system, a continuously differentiable control barrier function is proposed under a relaxed feasibility condition in the sense that it is imposed on each subsystem and only needed in the boundary area.

1. Introduction

Practical systems are usually subject to safety requirements including state/input constraints and collision-free with static and dynamic obstacles, especially in the scenarios of autonomous driving (Cosner, Chen, Leung, & Pavone, 2023; Xiao, Ge, Han, & Zhang, 2022), safe navigation (Lafmejani & Berman, 2021; Verginis & Dimarogonas, 2021), multi-robot formation (Panagou, Stipanović, & Voulgaris, 2016), etc. In the literature, the control problem for systems with safety constraints is known as safety-critical control (Ames, Xu, Grizzle, & Tabuada, 2017; Cohen, Molnar, & Ames, 2024; Molnar, Cosner, Singletary, Ubellacker, & Ames, 2022; Nguyen & Sreenath, 2022; Wang & Xu, 2024; Wu, Liu, Egerstedt, & Jiang, 2023; Xu, Tabuada, Grizzle, & Ames, 2015).

The research on safety-critical control problems originates with a single system. By combining the barrier certificate and control Lyapunov function, Peter and Allgöwer (2007) constructed a universal safety feedback design for a first-order nonlinear affine system. Ames et al. (2017) introduced control barrier function(CBF) and control Lyapunov function based quadratic program framework to ensure the safety as a priority and simultaneously achieve performance objectives, while (Breeden & Panagou, 2023) further considered the input constraint, and by developing a robust CBF, the constrained stabilization problem was solved by a point-wise minimum norm controller. For Euler–Lagrange system, Molnar et al. (2022) presented a model-free controller for maintaining the safety of the robotic system and achieving the tracking purpose. Then the safety-critical control scheme is extended to uncertain systems. Xu et al. (2015) studied the nonlinear system under perturbations in the vector field and analyzed the robustness of the safety design, while an optimal robust control was provided in Nguyen and Sreenath (2022) for the safety and tracking problem of dynamic robotics. Wang and Xu (2024) presented an adaptive safety-critical control law through a nonlinear program to deal with the system with parametric uncertainties in drift terms and control-input matrices. Recently, nonlinear systems with high-relative-degree safety constraints have also been investigated. Xu (2018) generated the safety and tracking control for input–output linearizable systems by quadratic program, whose feasibility was guaranteed by establishing the control-sharing property of CBFs. Tan, Cortez, and Dimarogonas (2022) relaxed the CBF condition and designed a locally Lipschitz-continuous singularity-free control for enforcing the safety constraints.

When generalizing the safety-critical control to multi-agent systems, the basic safety requirement is collision avoidance, and potential/barrier function-based methods prove to be very effective. For example, Atınç, Stipanović, and Voulgaris (2020), Dong and Huang (2015), Fu, Wen, Yu, and Wu (2022), Panagou et al.

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(2016), Su, Chen, Wang, and Lin (2011) designed collision avoidance functions to generate repulsive force for keeping neighboring agents apart while utilizing proximity functions for attraction in order to maintain connectivity. With the aid of the properties of potential functions, Verginis and Dimarogonas (2021) proposed a 2nd-order navigation function to guide robots to a predefined goal without collisions. Model predictive control offers another train of thought for collision-free tracking by additionally addressing corner cases or deadlock issues. Lafmejani and Berman (2021) developed an online nonlinear model predictive control method to achieve collision-free and deadlock-free navigation for multiple nonholonomic robots, and Chen, Guo, and Li (2024) presented a fully distributed infinite-horizon model predictive control-based trajectories generation framework with deadlock resolution.

CBF-based methods have also been widely applied to the safety-critical control of multi-agent systems where not only the constraints of each agent but also the inter-agent ones have been taken into consideration. Pickem et al. (2017), Wang, Ames, and Egerstedt (2017) utilized CBF-based quadratic program to minimize the difference between the actual and the nominal controller with the collision-free requirement, while (Glotfelter, Cortés, & Egerstedt, 2017) provided a framework that permitted nonsmooth barrier functions. In order to guarantee the feasibility of quadratic program, Wu et al. (2023) developed a reshaped constraints technique and proposed a robust and locally Lipschitz-continuous safety-critical controller along with a nonlinear small-gain analysis for certificating the safety of the whole multi-agent system. There is growing interest in designing learning-based safety control methods. Cai, Cao, Lu, Zhang, and Xiong (2021) combined multi-agent reinforcement learning algorithms with decentralized CBF to ensure collision-free behaviors, and Cosner et al. (2023) introduced responsibility-aware CBFs and presented a method to learn responsibility allocations from data for safe autonomous vehicle interaction.

Motivated by the collision avoidance requirement of multi-agent systems in real applications, this paper proposes a distributed safety-critical control for a second-order nonlinear multi-agent system. The contribution is threefold. First, we provide a general definition of CBF and propose a specific form for the multi-agent system with individual and inter-agent constraints. It can be used to certificate the safety of the whole system, and relax existing CBF conditions (Ames et al., 2017; Cohen et al., 2024; Dong, Wang, & Hong, 2024; Pickem et al., 2017; Xu et al., 2015) in the sense that it depends on the information of each agent and its neighboring ones and is only needed if some pair enters the boundary area. Second, we design a distributed locally Lipschitz-continuous safety-critical control under the time-varying and position-dependent communication network. To overcome the technical challenge of such a network, the truncated function is introduced for the smooth addition and deletion of links, and another distinguished feature is that control parameters are elaborately designed to customize the boundary and internal sets. The safety controller is only active in the boundary area and the other control performance can be achieved by the nominal controller in the inner area. Finally, our CBF-based framework is different from potential/barrier functions-based methods in Atınç et al. (2020), Dong and Huang (2015), Fu et al. (2022), Panagou et al. (2016), Su et al. (2011), Verginis and Dimarogonas (2021) since we are able to additionally accommodate the safety constraint with a general nonlinear function for each agent and avoid large forces when the constraint is about to be violated. Multiple control objectives are achieved and demonstrated in the position swapping and flocking experiments of multi-robot systems in harsh environments with static and dynamic obstacles.

2. Problem formulation

Consider the nonlinear multi-agent system

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \ i = 1, \dots, N, \tag{1}$$

where $x_i \in \mathbb{R}^{2n}$ is the state and $u_i(t) \in \mathbb{R}^m$ is the control input. Motivated by the distributed control of UAVs (Wang et al., 2020), planar mobile robots (Fu et al., 2022; Verginis & Dimarogonas, 2021; Wang et al., 2017), and automated vehicles (Wang & Xu, 2024; Xiao et al., 2022), we focus on a group of second-order nonlinear systems,

$$f_i(x_i) = \begin{bmatrix} q_i \\ \hat{f}_i(p_i, q_i) \end{bmatrix}, \ g_i(x_i) = \begin{bmatrix} 0 \\ \hat{g}_i(p_i, q_i) \end{bmatrix}, \tag{2}$$

where $x_i = \text{col}(p_i, q_i)$ with $p_i \in \mathbb{R}^n$ and $q_i \in \mathbb{R}^n$ being the position and velocity of the ith agent, and functions $\hat{f}_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{g}_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are globally defined and continuously differentiable.

The sensing range of each agent is $R > 0$, and agent i can communicate with agent j if and only if $\|p_i - p_j\| < R$, $i, j = 1, \dots, N$. Then the communication network of the multi-agent system (1) is position-dependent, described by a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ where $\mathcal{V} = \{1, \dots, N\}$ is the node set and the edge set $\mathcal{E}(t) = \{(i, j)|i, j \in \mathcal{V}\}$ is defined such that for $i, j \in \mathcal{V}$,

- if $(i, j) \in \mathcal{E}(t^-)$ and $\|p_i - p_j\| < R$, then $(i, j) \in \mathcal{E}(t)$;
- if $(i, j) \notin \mathcal{E}(t^-)$ and $\|p_i - p_j\| < R$, then $(i, j) \in \mathcal{E}(t)$;
- if $(i, j) \in \mathcal{E}(t^-)$ and $\|p_i - p_j\| \geq R$, then $(i, j) \notin \mathcal{E}(t)$;
- if $(i, j) \notin \mathcal{E}(t^-)$ and $\|p_i - p_j\| \geq R$, then $(i, j) \notin \mathcal{E}(t)$.

For the safety of the multi-agent system, it is required to consider the constraint of each agent and collision avoidance with each other and obstacles in the environment. Suppose there are $N_o \geq 0$ obstacles, indexed by $N + 1, \dots, N + N_o$. Denote the distance between agent i and agent or obstacle j as

$$d_{ij} = \|p_i - p_j\|, \ i \in \mathcal{V}, \ j \in \mathcal{V}_o, \ i \neq j, \tag{3}$$

where $\mathcal{V}_o = \{1, \dots, N + N_o\}$, and p_j, $j = N + 1, \dots, N + N_o$, is a constant vector representing the position of the jth static obstacle. Let $x_j = \text{col}(p_j, q_j)$ with $q_j = 0$ for $j = N + 1, \dots, N + N_o$. Assume $i \neq j$ throughout the paper. Thus, the safety constraint of system (2) is given by, for $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$,

$$S_{ij} = \{(x_i, x_j) \in \mathbb{R}^{4n} : c_i(x_i) > 0 \cap d_{ij} > r\}, \tag{4}$$

where $c_i : \mathbb{R}^{2n} \to \mathbb{R}$ is continuously differentiable, representing the constraint of agent i, and $0 < r < R$ is the minimum collision distance.

We aim to design a distributed control for the safety and the cooperative control of system (2), which takes the following form,

$$u_i = \hat{u}_i(x_i, x_j) + \delta_i(x_i, x_j, \hat{u}_i), \ i \in \mathcal{V}, \ j \in \mathcal{N}_i(t), \tag{5}$$

where $\mathcal{N}_i(t) = \{j \in \mathcal{V}_o \mid d_{ij} < R\}$, $\hat{u}_i(x_i, x_j)$ is the nominal controller for the performance of consensus, formation and position swapping, etc., which is distributed and locally Lipschitz-continuous, and $\delta_i(x_i, x_j, \hat{u}_i)$ is a safety controller to adjust the trajectories of agents for the forward invariance of the set S_{ij}. Under (5), the closed-loop system is

$$\begin{aligned} \dot{p}_i &= q_i, \ i = 1, \dots, N, \\ \dot{q}_i &= \hat{f}_i(p_i, q_i) + \hat{g}_i(p_i, q_i)(\hat{u}_i(x_i, x_j) + \delta_i(x_i, x_j, \hat{u}_i)). \end{aligned} \tag{6}$$

Then our problem can be formally formulated as follows.

Problem 1. Consider the multi-agent system (2) with the safety constraint (4). For all $(x_i(0), x_j(0)) \in S_{ij}$ with $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$, design the nominal controller $\hat{u}_i(x_i, x_j)$ and the safety controller $\delta_i(x_i, x_j, \hat{u}_i)$ in (5) such that the trajectories of all agents exist and are safe, i.e., $(x_i(t), x_j(t)) \in S_{ij}$ for all $t \geq 0$, and the other performance can be achieved if corner cases do not occur.

Remark 1. The safety-critical control problem of the multi-agent system (2) is different from those for a single system (Ames et al., 2017; Dong et al., 2024; Nguyen & Sreenath, 2022; Tan et al., 2022) in the sense that (4) describes not only the constraints of each agent but also the minimum clearance distances from other agents and obstacles. Problem 1 also involves a position-dependent communication network denoted by $\mathcal{G}(t)$, and thus, we essentially need to deal with the time-varying safety constraint in (4). Technically, for the safety purpose, it is required to design a distributed and locally Lipschitz-continuous control law, depending on the information from on-board sensors with limited sensing range R, and provide an efficient way to certificate that all the constraints of the whole multi-agent system are satisfied for all $t \geq 0$. The other performance is also required through the design of the nominal controller $\hat{u}_i(x_i, x_j)$ on the promise that corner cases do not happen. Otherwise, due to the emergence of stable spurious equilibrium points near the surface of the obstacle, some agents may get stuck in the corner case since the controller cannot force the agent to circulate the obstacle and escape from it (Gonçalves, Krishnamurthy, Tzes, & Khorrami, 2024).

3. CBF-based safety-critical control design

3.1. Definition of CBF for multi-agent systems (1)

We first provide the definition of CBF for the multi-agent system (1) with constraints $h_i^0(x_i) > 0$ and $z_{ij}^0(x_i, x_j) > 0$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, where $h_i^0 : \mathbb{R}^{2n} \to \mathbb{R}$ and $z_{ij}^0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ are sufficiently smooth functions. Assume the relative degrees[1] of $h_i^0(x_i)$ and $z_{ij}^0(x_i, x_j)$ with respect to subsystem i are $k \geq 1$ and $l \geq 1$, respectively. Define

$$\begin{aligned} h_i^m(x_i) &= (\frac{d}{dt} + \alpha_h^m) h_i^{m-1}(x_i), \ m = 1, \ldots, k, \\ z_{ij}^\sigma(x_i, x_j) &= (\frac{d}{dt} + \alpha_z^\sigma) z_{ij}^{\sigma-1}(x_i, x_j), \ \sigma = 1, \ldots, l, \end{aligned} \tag{7}$$

where $\alpha_h^m(\cdot)$, $m = 1, \ldots, k$, and $\alpha_z^\sigma(\cdot)$, $\sigma = 1, \ldots, l$, are extended class $\mathcal{K}$ functions. Also define

$$\begin{aligned} \Phi_i^{m-1} &= \{x_i \in \mathbb{R}^{2n} : h_i^{m-1}(x_i) > 0\}, \ m = 1, \ldots, k, \\ \Psi_{ij}^{\sigma-1} &= \{(x_i, x_j) \in \mathbb{R}^{4n} : z_{ij}^{\sigma-1}(x_i, x_j) > 0\}, \sigma = 1, \ldots, l, \\ \Phi_i &= \bigcap_{m=1}^{k} \Phi_i^{m-1}, \ \Psi_{ij} = \bigcap_{\sigma=1}^{l} \Psi_{ij}^{\sigma-1}. \end{aligned} \tag{8}$$

Definition 1. Let C_{ij} for all $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$ be a set of continuously differentiable functions $h_i^m(x_i)$, $m = 1, \ldots, k$ and $z_{ij}^\sigma(x_i, x_j)$, $\sigma = 1, \ldots, l$, where

$$C_{ij} = \Phi_i \cap \Psi_{ij}. \tag{9}$$

Then C^1 function $H : \mathbb{R}^{2n(N+N_o)} \to \mathbb{R}$ in the form of

$$\begin{aligned} H(x) &= F(h_i^{m-1}(x_i), z_{ij}^{\sigma-1}(x_i, x_j)), \\ &\quad m = 1, \ldots, k, \ \sigma = 1, \ldots, l, \end{aligned} \tag{10}$$

for $x = \mathrm{col}(x_1, \ldots, x_{N+N_o})$ is a CBF for multi-agent system (1) if there exists an extended class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$\dot{H}(x) > -\alpha(H(x)). \tag{11}$$

Note that for a single system (1) with $N = 1$, the set C_{ij} reduces to $C = \bigcap_{m=1}^{k}\{x_1 \in \mathbb{R}^{2n} : h_1^{m-1}(x_1) > 0\}$ and u_1 is designed such that

$$\frac{\partial h_1^{k-1}(x_1)}{\partial x_1} f_1(x_1) + \frac{\partial h_1^{k-1}(x_1)}{\partial x_1} g_1(x_1) u_1 > -\alpha(h_1^{k-1}(x_1)). \tag{12}$$

However, the control u_1 takes no effect when $\frac{\partial h_1^{k-1}(x_1)}{\partial x_1} g_1(x_1) = 0$. Thus, in the literature of safety-critical control, it is assumed that for the case of relative degree $k = 1$, $\frac{\partial h_1^0(x_1)}{\partial x_1} g_1(x_1) \neq 0$ in Ames et al. (2017), Xu et al. (2015) or $\frac{\partial h_1^0(x_1)}{\partial x_1} f_1(x_1) \geq -\alpha(h_1^0(x_1))$ if $\frac{\partial h_1^0(x_1)}{\partial x_1} g_1(x_1) = 0$ in Cohen et al. (2024).

For the multi-agent system (1), it is tricky to design distributed safety-critical control $u_i(x_i, x_j, j \in \mathcal{N}_i(t))$ such that CBF $H(x)$ satisfies (11), and imposing a condition such as

$$\frac{\partial H(x)}{\partial x} f(x) > -\alpha(H(x)) \text{ if } \frac{\partial H(x)}{\partial x} g(x) = 0 \tag{13}$$

with $f(x) = \mathrm{col}(f_1(x_1), \ldots, f_N(x_N))$ and $g(x) = \mathrm{col}(g_1(x_1), \ldots, g_N(x_N))$ on all agents is impractical or even impossible due to the information coupling. It is desirable to find a relaxed condition for each agent $i \in \mathcal{V}$, which only depends on the information of itself and its neighbors. Thus, for subsystem $i \in \mathcal{V}$, we define

$$\begin{aligned} B_i(x_i, x_j) &= F_i(h_i^{m-1}(x_i), z_{ij}^{\sigma-1}(x_i, x_j)), \\ &\quad m = 1, \ldots, k, \ \sigma = 1, \ldots, l, \ j \in \mathcal{N}_i(t), \end{aligned} \tag{14}$$

where $F_i(\cdot)$ is some continuously differentiable function. Also divide C_{ij} into the boundary set, denoted by C_{ij}^b, and the internal set $C_{ij}^n = C_{ij} \backslash C_{ij}^b$, where for $i \in \mathcal{V}$ and $j \in \mathcal{V}_0$,

$$C_{ij}^b = \{(x_i, x_j) \in C_{ij} : 0 < h_i^0(x_i) < \epsilon_1 \cup 0 < z_{ij}^0(x_i, x_j) < \epsilon_1\}, \tag{15}$$

for some constant $\epsilon_1 > 0$. Then conditions similar to (13) can be relaxed into the following one.

Assumption 1. If agent $i \in \mathcal{V}$ has a neighbor $k \in \mathcal{V}_o$ such that $(x_i, x_k) \in C_{ik}^b$, then $B_i(x_i, x_j)$ satisfies for all $(x_i, x_j) \in C_{ij}$,

$$\frac{\partial B_i(x_i, x_j)}{\partial x_i} f_i(x_i) > -\kappa B_i(x_i, x_j) \text{ if } \frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) = 0, \tag{16}$$

where $\kappa > 0$ is some constant.

Remark 2. Physically, Assumption 1 means (16) is only needed if there exists some pair $(x_i, x_k) \in C_{ik}^b$, i.e., when agent i and its neighbors are far away from the boundary of C_{ij}, $i \in \mathcal{V}, j \in \mathcal{V}_o$, with elaborately designed $B_i(x_i, x_j)$, it is possible to guarantee the safety of the whole system without the condition (16). The verification procedure of Assumption 1 is summarized as follows. First, check whether there exists $k \in \mathcal{V}_o$ such that $(x_i, x_k) \in C_{ik}^b$. If not, Assumption 1 is not necessary, and if otherwise, further check whether there exists $(x_i, x_j) \in C_{ij}$ such that $\frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) = 0$. Finally, use the optimization solver to estimate the minimum of $\frac{\frac{\partial B_i(x_i, x_j)}{\partial x_i} f_i(x_i)}{B_i(x_i, x_j)}$ subject to $\frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) = 0$ for all $(x_i, x_j) \in C_{ij}$, and then κ can be chosen such that (16) holds.

3.2. Safety-critical control design for second-order multi-agent systems

For the constraint $c_i(x_i) > 0$, $i \in \mathcal{V}$, and the collision avoidance constraint $d_{ij} > r$, $j \in \mathcal{V}_o$, in (4), we first specify the constraint

[1] The relative degree of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ with respect to system $\dot{x} = f(x) + g(x)u$ is the number of times we need to differentiate it along the dynamics of $\dot{x} = f(x) + g(x)u$ until control u explicitly shows (Xiao & Belta, 2019).

functions $h_i^m(x_i)$, $m = 1, \ldots, k$, and $z_{ij}^\sigma(x_i, x_j)$, $\sigma = 1, \ldots, l$, in (7). The relative degree of the constraint $c_i(x_i) > 0$ with respect to (2) is $k = 1$, and then define the constraint function $h_i^0(x_i)$ as follows,

$$h_i^0(x_i) = \chi\Big(\frac{c_i(x_i)}{\epsilon}\Big), \tag{17}$$

where constant $\epsilon > 0$ and function $\chi : \mathbb{R} \to \mathbb{R}$ is C^2 satisfying

$$\begin{cases} \chi(\tau) = \epsilon_1, \ \tau \geq 1, \\ \frac{\partial \chi(\tau)}{\partial \tau} > 0, \ \tau < 1, \\ \chi(0) = 0. \end{cases} \tag{18}$$

For the inter-agent constraint $d_{ij} > r$ for the second-order system (2), we can obtain the relative degree $l = 2$, and then functions $z_{ij}^0(x_i, x_j)$ and $z_{ij}^1(x_i, x_j)$ are designed as,

$$z_{ij}^0(x_i, x_j) = \chi\Big(\frac{d_{ij}^2 - r^2}{r_0^2 - r^2}\Big), \quad i \in \mathcal{V}, \ j \in \mathcal{V}_o, \tag{19a}$$

$$z_{ij}^1(x_i, x_j) = \Big(\frac{d}{dt} + \alpha_z^1\Big) z_{ij}^0(x_i, x_j), \tag{19b}$$

for some constant $r < r_0 < R$. For $i \in \mathcal{V}$ and $j \notin \mathcal{N}_i(t)$, $d_{ij} > R$, $z_{ij}^0(x_i, x_j) = \epsilon_1$, and $z_{ij}^1(x_i, x_j) = \alpha_z^1(\epsilon_1)$. Then for the second-order multi-agent system (2), design $B_i(x_i, x_j)$ in (14) in the following form,

$$B_i(x_i, x_j) = h_i^0(x_i) \prod_{j \in \mathcal{N}_i(t)} z_{ij}^1(x_i, x_j) \prod_{j \in \mathcal{V}_o \backslash \mathcal{N}_i(t)} \alpha_z^1(\epsilon_1). \tag{20}$$

Note that (20) is distributed, depending on x_i and x_j, $j \in \mathcal{N}_i(t)$, and the control u_i explicitly shows in $\frac{\partial B_i(x_i, x_j)}{\partial x_i} \dot{x}_i$.

Based on $B_i(x_i, x_j)$ in (20), design the distributed safety controller $\delta_i(x_i, x_j, \hat{u}_i)$, $i \in \mathcal{V}$, $j \in \mathcal{N}_i(t)$, in the following form,

$$\delta_i(x_i, x_j, \hat{u}_i) = \begin{cases} -\frac{a_i + \sqrt{a_i^2 + \gamma \|b_i\|^4}}{\|b_i\|^2} b_i, & b_i \neq 0 \\ 0, & b_i = 0 \end{cases} \tag{21}$$

where constant $\gamma > 0$ and

$$b_i = -\Big(\frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i)\Big)^T \tag{22a}$$

$$a_i = -\frac{\partial B_i(x_i, x_j)}{\partial x_i} f_i(x_i) - \frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) \hat{u}_i(x_i, x_j) - \kappa B_i(x_i, x_j) \tag{22b}$$

Remark 3. The design of function $B_i(x_i, x_j)$ is fundamentally different from reciprocal CBFs in Ames et al. (2017), Dong et al. (2024), Nguyen and Sreenath (2022) or potential/barrier functions in Atınç et al. (2020), Dong and Huang (2015), Fu et al. (2022), Panagou et al. (2016), Su et al. (2011), Verginis and Dimarogonas (2021). First, the design ideas are different. Potential function $\phi(\cdot)$ usually has the property of $\phi(d_{ij}) \to \infty$ as $d_{ij} \to r$ to generate large repulsive force for collision avoidance, while we use $B_i(x_i, x_j) = 0$ to indicate the violation of the constraints. Second, in our design (20), we can additionally handle the constraint $c_i(x_i) > 0$ for $c_i(\cdot)$ being any continuously differentiable function. Finally, the working mechanism is fundamentally different. Based on $B_i(x_i, x_j)$, the safety controller $\delta_i(x_i, x_j, \hat{u}_i)$ takes effects only when the constraint is about to be violated. Specifically, if some pair (x_i, x_j) enters the boundary area C_{ij}^b, $B_i(x_i, x_j)$ decreases smoothly, resulting in $\frac{\partial B_i(x_i, x_j)}{\partial x_i} \neq 0$, and then $\delta_i(x_i, x_j, \hat{u}_i) \neq 0$ drives the pair away from the boundary. In the internal area C_{ij}^n, $B_i(x_i, x_j)$ becomes a constant and $\delta_i(x_i, x_j, \hat{u}_i)$ generates zero force. Such a mechanism allows to save energy in C_{ij}^n and customize C_{ij}^b by designing parameters ϵ_1, ϵ and r_0 in (15), (17) and (19a).

Remark 4. It is technically challenging to design the distributed control u_i in (5) under the time-varying communication network $\mathcal{G}(t)$. First, it is required to design a locally Lipschitz-continuous safety controller $\delta_i(x_i, x_j, \hat{u}_i)$, whereas the addition and deletion of links in $\mathcal{G}(t)$ may result in discontinuity of the design. We introduce the truncated function $\chi(\cdot)$ in (17) and (19a) and hence design the continuously differentiable function $B_i(x_i, x_j)$ for the smooth addition and deletion of links. Second, in the position-dependent graph $\mathcal{G}(t)$, it is impossible to impose assumptions such as $\mathcal{G}(t)$ is connected or jointly connected for $t \geq 0$ as in consensus problems since the connectivity of $\mathcal{G}(t)$ is dynamically changing with the relative distances of all agents. Thus, in the tasks involving consensus, e.g., consensus of the velocity in Section 5.2, it is necessary to enforce another constraint to make sure the graph is connected for all the time, and design the distributed nominal controller $\hat{u}_i(x_i, x_j)$ under the condition that the graph is initially connected.

4. Main results

In this section, we prove that Problem 1 can be solved by the distributed safety-critical control law u_i in (5) with $\delta_i(x_i, x_j, \hat{u}_i)$ given by (21). For verifying the safety of the whole multi-agent system, construct $H(\cdot)$ as follows,

$$H(x) = \prod_{i \in \mathcal{V}} \Big(h_i^0(x_i) \prod_{j \in \mathcal{V}_o, i < j} z_{ij}^1(x_i, x_j) \Big). \tag{23}$$

According to Definition 1, we need to check whether $H(x)$ in (23) satisfies (11). To this end, we first study the property of $B_i(x_i, x_j)$ in (20).

Lemma 1. *Under Assumption 1, the function $B_i(x_i, x_j)$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, is continuously differentiable and satisfies*

$$\frac{\partial B_i(x_i, x_j)}{\partial x_i} \dot{x}_i > -\kappa B_i(x_i, x_j), \ \ \forall (x_i, x_j) \in C_{ij}. \tag{24}$$

Proof. Note that $c_i(\cdot)$, $i \in \mathcal{V}$, is continuously differentiable and $\chi(\cdot)$ is C^2, and then $h_i^0(x_i)$ in (17) is continuously differentiable. Since $z_{ij}^1(x_i, x_j)$, $j \in \mathcal{V}_o$, in (19b) is continuously differentiable, from (20), $B_i(x_i, x_j)$ is continuously differentiable.

Along the closed-loop system (6), the time derivative of $B_i(x_i, x_j)$ satisfies, for $i \in \mathcal{V}$ and $j \in \mathcal{N}_i(t)$,

$$\begin{aligned} \frac{\partial B_i(x_i, x_j)}{\partial x_i} \dot{x}_i &= \frac{\partial B_i(x_i, x_j)}{\partial x_i} \Big(f_i(x_i) + g_i(x_i) u_i \Big) \\ &= \frac{\partial B_i(x_i, x_j)}{\partial x_i} f_i(x_i) + \frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) (\hat{u}_i(x_i, x_j) \\ &\quad + \delta_i(x_i, x_j, \hat{u}_i)) \\ &= -a_i - \kappa B_i(x_i, x_j) - b_i^T \delta_i(x_i, x_j, \hat{u}_i). \end{aligned} \tag{25}$$

If $b_i \neq 0$, from (21), $\delta_i(x_i, x_j, \hat{u}_i) = -\frac{a_i + \sqrt{a_i^2 + \gamma \|b_i\|^4}}{\|b_i\|^2} b_i$. Then from (25),

$$\begin{aligned} \frac{\partial B_i(x_i, x_j)}{\partial x_i} \dot{x}_i &= -a_i - \kappa B_i(x_i, x_j) \\ &\quad - b_i^T \Big(-\frac{a_i + \sqrt{a_i^2 + \gamma \|b_i\|^4}}{\|b_i\|^2} b_i \Big) \\ &= -\kappa B_i(x_i, x_j) + \sqrt{a_i^2 + \gamma \|b_i\|^4} > -\kappa B_i(x_i, x_j) \end{aligned} \tag{26}$$

If $b_i = 0$, we first consider the case of $(x_i, x_j) \in C_{ij}^n$, $j \in \mathcal{V}_o$. Note that $C_{ij}^n = C_{ij} \backslash C_{ij}^b$. Thus, $(x_i, x_j) \notin C_{ij}^b$ and $(x_i, x_j) \in C_{ij}$, and then from (9) and (15), $z_{ij}^0(x_i, x_j) \geq \epsilon_1$ and $h_i^0(x_i) \geq \epsilon_1$. Note that the maximum of $\chi(\cdot)$ in (18) is ϵ_1. From (19a) and (19b), for all $(x_i, x_j) \in C_{ij}^n$, $j \in \mathcal{V}_o$,

$$z_{ij}^0(x_i, x_j) = \epsilon_1,\ z_{ij}^1(x_i, x_j) = \alpha_z^1(\epsilon_1),\ h_i^0(x_i) = \epsilon_1.$$

From (20), we have

$$B_i(x_i, x_j) = \prod_{j \in \mathcal{V}_o} \alpha_z^1(\epsilon_1),\ \frac{\partial B_i(x_i, x_j)}{\partial x_i} = 0. \tag{27}$$

Together with (21) and (22),

$$a_i = -\kappa \prod_{j \in \mathcal{V}_o} \alpha_z^1(\epsilon_1) < 0,\ b_i = 0, \tag{28}$$

$$\delta_i(x_i, x_j, \hat{u}_i) = 0,\ (x_i, x_j) \in C_{ij}^n,\ j \in \mathcal{V}_o.$$

Then from (25),

$$\frac{\partial B_i(x_i, x_j)}{\partial x_i}\dot{x}_i = -a_i - \kappa B_i(x_i, x_j) > -\kappa B_i(x_i, x_j). \tag{29}$$

Next, consider the case that there exists some pair $(x_i, x_j) \in C_{ij}^b$. If $b_i = 0$, from (22a), $\frac{\partial B_i(x_i,x_j)}{\partial x_i} g_i(x_i) = 0$. Under Assumption 1, $\frac{\partial B_i(x_i,x_j)}{\partial x_i} f_i(x_i) > -\kappa B_i(x_i, x_j)$. Then from (22),

$$a_i = -\frac{\partial B_i(x_i, x_j)}{\partial x_i} f_i(x_i) - \kappa B_i(x_i, x_j) < 0,\ b_i = 0, \tag{30}$$

and from (25), (29) holds for the case of $b_i = 0$. Together with (26) and (29), we can conclude that (24) is satisfied.

Based on Lemma 1, we establish the necessary and sufficient condition for the safety of the multi-agent system (2) based on $H(x)$ in (23).

Lemma 2. *Under Assumption 1, $(x_i(t), x_j(t)) \in C_{ij}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, for $t \in [0, t_1)$ with $0 < t_1 \le +\infty$, if and only if $H(x)$ in (23) satisfies (11) for all $t \in [0, t_1)$.*

Proof. Only if part

Since $d_{ij} = d_{ji}$ for all $i, j \in \mathcal{V}$, $z_{ij}^1(x_i, x_j) = z_{ji}^1(x_j, x_i)$ from (19). For some agent $k \in \mathcal{V}$, we have

$$\begin{aligned}
&\prod_{i\in\mathcal{V}}\prod_{j\in\mathcal{V}_o, i<j} z_{ij}^1(x_i, x_j) = \Big(\prod_{i=k}\prod_{j\in\mathcal{V}_o, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i\in\mathcal{V}}\\
&\prod_{j=k, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big)\\
=&\Big(\prod_{i=k}\prod_{j\in\mathcal{V}, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i\in\mathcal{V}}\prod_{j=k, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i=k}\\
&\prod_{j\in\mathcal{V}_o\backslash\mathcal{V}, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big)\\
=&\Big(\prod_{i=k}\prod_{j\in\mathcal{V}, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i\in\mathcal{V}}\prod_{j=k, i<j} z_{ji}^1(x_j, x_i)\Big)\Big(\prod_{i=k}\\
&\prod_{j\in\mathcal{V}_o\backslash\mathcal{V}, i<j} z_{ij}^1(x_i, x_j)\Big)\Big(\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big)\\
=&\prod_{j\in\mathcal{V}, k<j} z_{kj}^1(x_k, x_j)\prod_{i\in\mathcal{V}, i<k} z_{ki}^1(x_k, x_i)\\
&\prod_{j\in\mathcal{V}_o\backslash\mathcal{V}, k<j} z_{kj}^1(x_k, x_j)\Big(\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big)\\
=&\prod_{j\in\mathcal{V}\backslash\{k\}} z_{kj}^1(x_k, x_j)\prod_{j\in\mathcal{V}_o\backslash\mathcal{V}, k<j} z_{kj}^1(x_k, x_j)\\
&\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)
\end{aligned}$$

For all $j \in \mathcal{V}_o\backslash\mathcal{V}$ and $k \in \mathcal{V}$, we have $k < j$. Then $\prod_{j\in\mathcal{V}_o\backslash\mathcal{V}, k<j} z_{kj}^1(x_k, x_j) = \prod_{j\in\mathcal{V}_o\backslash\mathcal{V}} z_{kj}^1(x_k, x_j)$, and thus,

$$\begin{aligned}
&\prod_{i\in\mathcal{V}}\prod_{j\in\mathcal{V}_o, i<j} z_{ij}^1(x_i, x_j) = \prod_{j\in\mathcal{V}\backslash\{k\}} z_{kj}^1(x_k, x_j)\\
&\prod_{j\in\mathcal{V}_o\backslash\mathcal{V}} z_{kj}^1(x_k, x_j)\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\\
=&\prod_{j\in\mathcal{V}_o\backslash\{k\}} z_{kj}^1(x_k, x_j)\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j).
\end{aligned}$$

Then from (20) and (23), for some agent $k \in \mathcal{V}$,

$$\begin{aligned}
H(x) =& h_k^0(x_k)\prod_{i\in\mathcal{V}\backslash\{k\}} h_i^0(x_i)\prod_{j\in\mathcal{V}_o\backslash\{k\}} z_{kj}^1(x_k, x_j)\\
&\prod_{i\in\mathcal{V}\backslash\{k\}}\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\\
=&h_k^0(x_k)\prod_{j\in\mathcal{V}_o\backslash\{k\}} z_{kj}^1(x_k, x_j)\prod_{i\in\mathcal{V}\backslash\{k\}}\Big(h_i^0(x_i)\\
&\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big)\\
=&B_k(x_k, x_j)\prod_{i\in\mathcal{V}\backslash\{k\}}\Big(h_i^0(x_i)\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big).
\end{aligned} \tag{31}$$

Thus, for any $k \in \mathcal{V}$,

$$\frac{\partial H(x)}{\partial x_k} = \frac{\partial B_k(x_k, x_j)}{\partial x_k}\prod_{i\in\mathcal{V}\backslash\{k\}}\Big(h_i^0(x_i)\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big). \tag{32}$$

For $j \in \mathcal{V}_o\backslash\mathcal{V}$, $\dot{x}_j = 0$. Then the derivative of $H(x)$ satisfies

$$\dot{H}(x) = \sum_{k\in\mathcal{V}_o}\frac{\partial H(x)}{\partial x_k}\dot{x}_k = \sum_{k\in\mathcal{V}}\frac{\partial H(x)}{\partial x_k}\dot{x}_k. \tag{33}$$

Since $(x_k(t), x_j(t)) \in C_{kj}$, $k \in \mathcal{V}$, $j \in \mathcal{V}_o$, for all $t \in [0, t_1)$, from (9), $h_i^0(x_i) > 0$ and $z_{kj}^1(x_k, x_j) > 0$. Then from (20), $B_k(x_k, x_j) > 0$ for all $t \in [0, t_1)$. From (31), (32) and (33), for all $t \in [0, t_1)$,

$$\begin{aligned}
\dot{H}(x) =& \sum_{k\in\mathcal{V}}\frac{\partial B_k(x_k, x_j)}{\partial x_k}\dot{x}_k\prod_{i\in\mathcal{V}\backslash\{k\}}\Big(h_i^0(x_i)\\
&\prod_{j\in\mathcal{V}_o\backslash\{k\}, i<j} z_{ij}^1(x_i, x_j)\Big)\\
=& \sum_{k\in\mathcal{V}}\frac{\partial B_k(x_k, x_j)}{\partial x_k}\dot{x}_k\frac{H(x)}{B_k(x_k, x_j)}.
\end{aligned} \tag{34}$$

By Lemma 1, under Assumption 1, (24) holds. Then from (34), $\dot{H}(x) > \sum_{k\in\mathcal{V}}\frac{H(x)}{B_k(x_k,x_j)}(-\kappa B_k(x_k, x_j)) = -\kappa N H(x)$ for all $t \in [0, t_1)$, which satisfies (11) with $\alpha(\tau) = \kappa N\tau$.

If part.

Since $(x_i(0), x_j(0)) \in C_{ij}$, $h_i^0(x_i(0)) > 0$ and $z_{ij}^1(x_i(0), x_j(0)) > 0$. From (23), $H(x(0)) > 0$. Let

$$\dot{y}(t) = -\alpha(y(t)),\ y(0) = H(x(0)).$$

Then by Lemma 4.4 in Khalil (2002), there exists a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that

$$y(t) = \beta(y(0), t) = \beta(H(x(0)), t),\ \forall t \in [0, t_1).$$

From (11) and by Lemma 3.4 in Khalil (2002),

$$H(x(t)) > y(t) = \beta(H(x(0)), t) > 0,\ \forall t \in [0, t_1). \tag{35}$$

Assume that there exists some $t_1^* \in (0, t_1)$ such that $h_i^0(x_i(t_1^*)) \le 0$ or $z_{ij}^1(x_i(t_1^*), x_j(t_1^*)) \le 0$ for some $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$. Since $z_{ij}^1(x_i, x_j)$, $h_i^0(x_i)$, $x_i(t)$, and $x_j(t)$ are continuous, in order to achieve $h_i^0(x_i(t_1^*)) \le 0$ or $z_{ij}^1(x_i(t_1^*), x_j(t_1^*)) \le 0$, there must exist some

$t_2^* \in (0, t_1^*]$ such that $h_i^0(x_i(t_2^*)) = 0$ or $z_{ij}^1(x_i(t_2^*), x_j(t_2^*)) = 0$. Then from (23), $H(x(t_2^*)) = 0$, contradicting (35). Therefore, $h_i^0(x_i) > 0$ and $z_{ij}^1(x_i, x_j) > 0$ for all $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$, $t \in [0, t_1)$. Since $(x_i(0), x_j(0)) \in C_{ij}$, by Lemma 2 in Glotfelter et al. (2017), $z_{ij}^0(x_i, x_j) > 0$, and thus, for all $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$,

$$(x_i(t), x_j(t)) \in C_{ij},\ d_{ij} > r,\ c_i(x_i) > 0. \quad \square \tag{36}$$

Based on Lemmas 1 and 2, we present the main theorem.

Theorem 1. *Under Assumption 1, consider the multi-agent system (2) with the safety constraint (4). For all $(x_i(0), x_j(0)) \in C_{ij}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, Problem 1 is solvable by the distributed safety-critical controller u_i in (5) based on $B_i(x_i, x_j)$ in (20).*

Proof. By Lemma 1, $B_i(x_i, x_j)$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, is distributed and continuously differentiable. Since $f_i(x_i)$ and $g_i(x_i)$ are continuously differentiable and $\hat{u}_i(x_i, x_j)$ is locally Lipschitz-continuous, from (22a) and (22b), a_i and b_i are locally Lipschitz-continuous. For all $(x_i, x_j) \in C_{ij}$, if $b_i = 0$, it follows from (28) and (30) that $a_i < 0$. Note that $\delta_i(x_i, x_j, \hat{u}_i)$, $i \in \mathcal{V}$, $j \in \mathcal{N}_i(t)$, given by (21), is in the form of (9) in Peter and Allgöwer (2007). Thus, from the proof of Theorem 7 in Peter and Allgöwer (2007), $\delta_i(x_i, x_j, \hat{u}_i)$ is analytic for all $b_i \neq 0 \cup a_i < 0$. Together with the locally Lipschitz-continuity of a_i and b_i, $\delta_i(x_i, x_j, \hat{u}_i)$ is locally Lipschitz-continuous for all $(x_i, x_j) \in C_{ij}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$. Thus, the solution of the closed-loop system (6) exists.

For any $(x_i(0), x_j(0)) \in C_{ij}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, by the continuity of $x_i(t)$, there exists $0 < t_1 \leq +\infty$ such that

$$(x_i(t), x_j(t)) \in C_{ij},\ \forall t \in [0, t_1), \tag{37}$$

which implies $h_i^0(x_i) > 0$ and $z_{ij}^1(x_i, x_j) > 0$ for all $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$, $t \in [0, t_1)$. Also by Lemma 2, (11) holds, and thus, from (36), $c_i(x_i) > 0$ and $d_{ij} > r$, $i \in \mathcal{V}, j \in \mathcal{V}_o$, $t \in [0, t_1)$. Thus,

$$(x_i(t), x_j(t)) \in S_{ij},\ \forall t \in [0, t_1). \tag{38}$$

Next is to show $t_1 = +\infty$ by contradiction. Assume that there exits some time $t_2 \geq t_1$ such that $(x_i(t_2), x_j(t_2)) \notin C_{ij}$, and then $(x_i(t_2), x_j(t_2)) \notin \Phi_i$ or $(x_i(t_2), x_j(t_2)) \notin \Psi_{ij}$ for some $i \in \mathcal{V}$ and $j \in \mathcal{V}_o$.

For the case of $(x_i(t_2), x_j(t_2)) \notin \Phi_i$, $h_i^0(x_i(t_2)) \leq 0$. Since $x_i(t)$ and $h_i^0(x_i)$ are continuous, in order to achieve $h_i^0(x_i(t_2)) \leq 0$, it must be $h_i^0(x_i(t)) \to 0$ as $t \to t_1^-$, which implies $H(x(t_1^-)) \to 0$, contradicting (35). Therefore, we can obtain

$$h_i^0(x_i(t)) > 0,\ \forall t \geq 0. \tag{39}$$

Similarly, we can prove $z_{ij}^1(x_i(t), x_j(t)) > 0$ for all $t \geq 0$. By Lemma 2 in Glotfelter et al. (2017), for all $(x_i(0), x_j(0)) \in C_{ij}$,

$$z_{ij}^0(x_i(t), x_j(t)) > 0,\ i \in \mathcal{V},\ j \in \mathcal{V}_o,\ t \geq 0. \tag{40}$$

Thus, $(x_i(t_2), x_j(t_2)) \notin \Psi_{ij}$ cannot happen. Then from (39) and (40), we have $c_i(x_i) > 0$ and $d_{ij} > r$ for all $t \geq 0$, and thus, $(x_i(t), x_j(t)) \in S_{ij}$ for all $t \geq 0$. $\square$

Remark 5. There are five key parameters $\gamma, \kappa, \epsilon_1, r_0$, and ϵ in the design (5) and we establish their relationship with the control performance. First, the parameter γ determines the magnitude of $\delta_i(x_i, x_j, \hat{u}_i)$. As γ increases, $\delta_i(x_i, x_j, \hat{u}_i)$ also increases, and thus, the agent can move faster towards the interior of the safe set. Second, κ influences the applicability of the controller $\delta_i(x_i, x_j, \hat{u}_i)$. Specifically, as κ increases, (16) in Assumption 1 is easier to satisfy, but more force of the safety controller is needed. Finally, ϵ_1, r_0, and ϵ can determine the range of the boundary area C_{ij}^b. Note that corner cases can only happen in C_{ij}^b since $\delta_i(x_i, x_j, \hat{u}_i) = 0$ in C_{ij}^n and is active only if there exists some neighboring agent k such that $(x_i, x_k) \in C_{ik}^b$. Thus, decreasing ϵ_1, ϵ, and r_0 may reduce the range of C_{ij}^b and hence the probability of the occurrence of corner cases.

5. Illustrated examples

In this section, we apply the safety-critical control (5) to solve the position swapping and flocking problems of multi-robot systems.

5.1. Multi-robot position swapping

Consider a multi-agent system consisting of eight robots moving on 2-dimensional Euclidean space as in Wang et al. (2017),

$$\begin{aligned} \dot{p}_i &= q_i, \\ \dot{q}_i &= u_i, \quad i = 1, 2, \ldots, 8, \end{aligned} \tag{41}$$

where $p_i = \mathrm{col}(p_{i1}, p_{i2}) \in \mathbb{R}^2$, $q_i = \mathrm{col}(q_{i1}, q_{i2}) \in \mathbb{R}^2$, and $u_i = \mathrm{col}(u_{i1}, u_{i2}) \in \mathbb{R}^2$ denote the position, velocity, and control vectors, respectively.

The communication radius is $R = 2$ and the minimum collision distance is $r = 0.3$. There are 3 obstacles located at $\mathrm{col}(-0.5, -0.5)$, $\mathrm{col}(0.3, 0.3)$ and $\mathrm{col}(0, -1)$. For each agent, also consider the constraint

$$c_i(x_i) = -2p_i^T q_i + (4 - p_i^T p_i) > 0,\ i = 1, \ldots, 8. \tag{42}$$

For the task of swapping positions, design the nominal controller

$$\hat{u}_i(x_i, x_j) = -k_1(p_i - p_j) - k_2 q_i,\ i = 1, \ldots, 8, \tag{43}$$

where $k_1 = 1$, $k_2 = 1$, and p_j is the position of the target agent to be swapped. For satisfying the constraint (42) and avoiding collisions, define $h_i^0(x_i)$ in (17) with $\epsilon = 1$, $z_{ij}^0(x_i, x_j)$ in (19a) with $r_0 = 1$ and

$$\chi(\tau) = \begin{cases} \epsilon_1(\tau - 1)^3 + \epsilon_1, & \tau < 1, \\ \epsilon_1, & \tau \geq 1, \end{cases}$$

Then design $z_{ij}^1(x_i, x_j)$ in (19b) with $\alpha_z^1(\tau) = \tau$ and C_{ij}^b in (15) with $\epsilon_1 = 1$. Thus, the safety-critical control in (5) is composed of the nominal controller $\hat{u}_i(x_i, x_j)$ in (43) and the safety controller $\delta_i(x_i, x_j, \hat{u}_i)$ with $\gamma = 0.1$ and $B_i(x_i, x_j)$ given by (20).

Next, we verify Assumption 1. Note that $(x_i, x_k) \in C_{ik}^b$ holds if $d_{ik} < r_0 = 1$ or $c_i(x_i) < \epsilon = 1$, for all $k \in \mathcal{V}_o$, and then verify if there exists $(x_i, x_j) \in C_{ij}$ such that $\frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) = 0$. Note that $B_i(x_i, x_j) > 0$ for all $(x_i, x_j) \in C_{ij}$. When $\frac{\partial B_i(x_i, x_j)}{\partial x_i} g_i(x_i) = 0$, utilize the optimization solver in MATLAB to estimate the minimum of $\frac{\frac{\partial B_i}{\partial x_i} f_i(x_i)}{B_i(x_i, x_j)}$, and in this example, (16) holds for $\kappa > 5.904$. Let $\kappa = 8$.

Now conduct the simulation with the same initial conditions as Experiment A in Wang et al. (2017), which satisfy $(x_i(0), x_i(0)) \in C_{ij}$. The difference of our method from existing collision avoidance controllers is that (5) is designed under the relaxed condition (16), and the constraint (42) for each agent can also be satisfied for all $t \geq 0$. Fig. 1 demonstrates $c_i(x_i) > 0$ and $d_{ij} > 0.3$, $i = 1, \ldots, 8$, $j = 1, \ldots, 11$, and the trajectories of all agents can be found in Fig. 2.

5.2. Multi-robot flocking

Our design (5) can also be applied to the flocking with connectivity preservation problem, which requires to preserve existing links, achieve velocity consensus, and avoid collisions. Consider the multi-agent system (41) with $R = 3.5$ and $r = 0.5$. Also consider a virtual leader as in Su et al. (2011), described by

$$\dot{p}_0 = q_0,\ \dot{q}_0 = 0, \tag{44}$$

where $p_0 = \mathrm{col}(p_{01}, p_{02}) \in \mathbb{R}^2$ and $q_0 = \mathrm{col}(q_{01}, q_{02}) \in \mathbb{R}^2$ are respectively the position and velocity of the virtual leader. Let

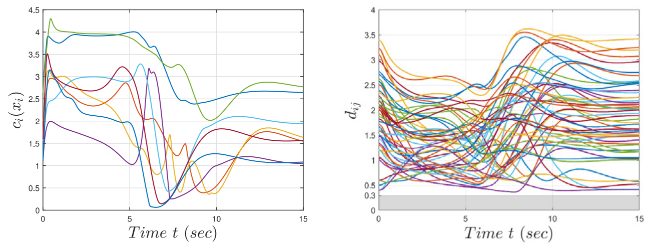

Fig. 1. The profiles of $c_i(x_i)$ and d_{ij}, $i \in \mathcal{V}$, $j \in \mathcal{V}_o$, under the safety-critical control law u_i in (5) based on (21) and (43).

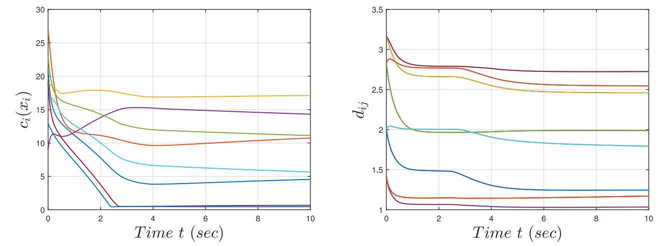

Fig. 3. The profiles of $c_i(x_i)$ and d_{ij}, $(i,j) \in \mathcal{E}(0)$, under the safety-critical control law u_i in (5) based on (21) and (46).

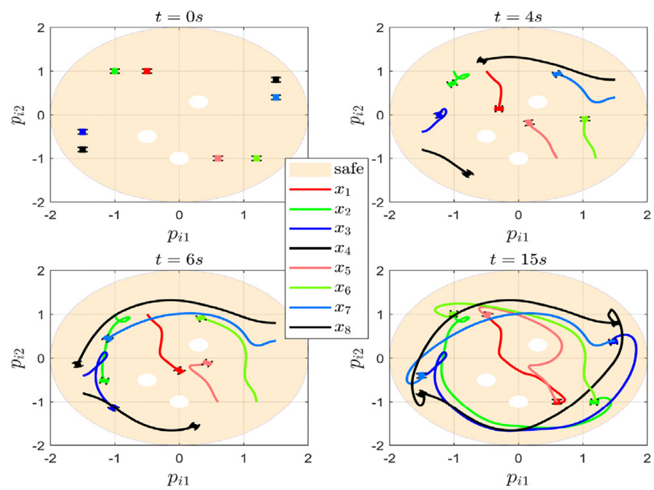

Fig. 2. The trajectories of all agents at 0 s, 4 s, 6 s, and 15 s.

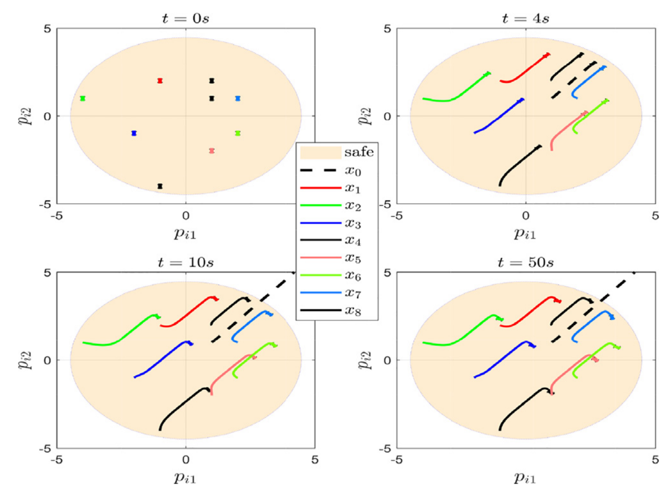

Fig. 4. The trajectories of all agents under the safety-critical control law (5) at 0s, 5s, 10s, and 50s.

$x_0 = \mathrm{col}(p_0, q_0)$. The initial conditions of systems (41) and (44) are given by $x_1(0) = \mathrm{col}(-1, 2, 2, -1)$, $x_2(0) = \mathrm{col}(-4, 1, 3, -1)$, $x_3(0) = \mathrm{col}(-2, -1, 1, 1)$, $x_4(0) = \mathrm{col}(-1, -4, 0, 1)$, $x_5(0) = \mathrm{col}(1, -2, 0, 2.5)$, $x_6(0) = \mathrm{col}(2, -1, -1, 0)$, $x_7(0) = \mathrm{col}(2, 1, -2, 1)$, $x_8(0) = \mathrm{col}(1, 2, 0, 0)$. The initial edge set is $\mathcal{E}(0) = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 1)\}$, and thus, the initial graph is connected. Let agent 8 be the informed agent.

In the flocking with connectivity preservation problem, for the purpose of preserving initial links, the safety constraint in (4) are strengthened into

$$c_i(x_i) = -2p_i^T q_i + (18 - p_i^T p_i) > 0,\ \ d_{ij} > r, i \in \mathcal{V}, j \in \mathcal{V}_o,$$
$$d_{ij} < R,\ \ (i,j) \in \mathcal{E}(0). \tag{45}$$

We first design the distributed nominal controller for the consensus of the velocity as follows,

$$\hat{u}_i(x_i, x_j) = \sum_{j \in \mathcal{N}_i(t)} 2(q_j - q_i) - 2a_{i0}(q_i - q_0), \tag{46}$$

where $a_{i0} = 1$, $i = 1, \ldots, 8$, if agent i can receive the information from virtual leader, and $a_{i0} = 0$ if otherwise. Since agent 8 is the informed agent, $a_{80} = 1$ and $a_{i0} = 0$ for $i = 1, \ldots, 7$.

In order to preserve the initial links in $\mathcal{E}(0)$, i.e., satisfy the additional constraint $d_{ij} < R$, $(i,j) \in \mathcal{E}(0)$, in (45), define constraint functions $\bar{z}_{ij}^0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ and $\bar{z}_{ij}^1 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ in the following form,

$$\bar{z}_{ij}^0(x_i, x_j) = \chi\Big(\frac{R^2 - d_{ij}^2}{R^2 - r_1^2}\Big),\ \ (i,j) \in \mathcal{E}(0),$$
$$\bar{z}_{ij}^1(x_i, x_j) = \Big(\frac{d}{dt} + \alpha_z^1\Big)\bar{z}_{ij}^0(x_i, x_j),$$

for $r_1 = 2.5$. Then design the safety controller $\delta_i(x_i, x_j, \hat{u}_i)$ with $\gamma = 0.1$, $\kappa = 8$ and $B_i(x_i, x_j) = h_i^0(x_i) \prod_{j \in \mathcal{N}_i(t)} z_{ij}^1(x_i, x_j) \prod_{j \in \mathcal{V}_o \setminus \mathcal{N}_i(t)} \alpha_z^1(\epsilon_1) \prod_{(i,j) \in \mathcal{E}(0)} \bar{z}_{ij}^1(x_i, x_j)$.

In the literature, the flocking problem is considered within the potential/barrier function-based framework. See references in Atınç et al. (2020), Dong and Huang (2015), Su et al. (2011). The difference of our design (5) is mainly twofold. First, (5) can additionally satisfy the constraint $c_i(x_i) > 0$ in (45). From Fig. 3, one can observe $c_i(x_i) > 0$ and $0.5 < d_{ij} < 3.5$, $(i,j) \in \mathcal{E}(0)$, and even if the tracking purpose conflicts with the safety constraints, i.e., the virtual leader goes outside of the safe area, the safety requirements in (45) are still satisfied as the priority task, demonstrated in Fig. 4. Second, the property of potential or barrier function promises a tremendous force as $d_{ij} \to r$ or $d_{ij} \to R$, while due to the different working mechanism of CBF-based control, the control force from (5) is finite, shown in Fig. 5.

6. Conclusion

This paper has proposed a safety-critical control for second-order nonlinear multi-agent systems in the time-varying and position-dependent communication network. To overcome the technical difficulties of time-varying constraints due to the addition and deletion of links, we have introduced a truncated function with relaxation parameters, and designed a locally Lipschitz-continuous distributed safety-critical controller, which only takes effect if there exists a pair entering the boundary of the safe set. To demonstrate the forward invariance of the safe set with both individual and inter-agent constraints, we have provided the definition of CBF of the multi-agent system and proposed a specific one. A direct relationship between its property and the safety of the whole system has also been established. In the future, we

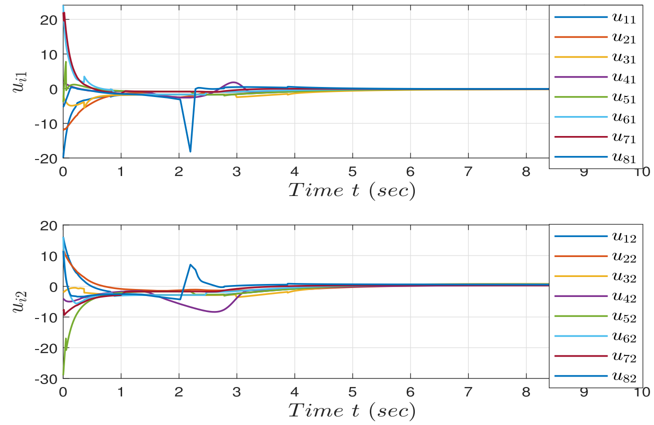

Fig. 5. The profile of the safety-critical control law u_i, $i = 1, \ldots, 8$, in (5).

will further combine model predictive control technique to guide agents through corner cases.

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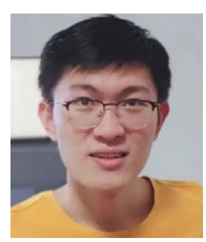

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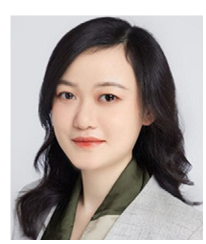

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