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Brief Paper

Distributed Nash equilibrium seeking with stochastic event-triggered mechanism[☆]

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ABSTRACT

In this paper, we study the problem of consensus-based distributed Nash equilibrium (NE) seeking in a network of players represented as a directed graph, where each player aims to minimize their own local cost functions non-cooperatively. To address bandwidth constraints and limited energy, we propose a stochastic event-triggered algorithm that triggers individual players with a probability depending on certain events, thus enhancing communication efficiency through reduced continuous communication. We prove that our developed event-triggered algorithm achieves exponential convergence to the exact NE when the underlying communication graph is strongly connected. Furthermore, we establish that our proposed event-triggered communication scheme does not exhibit Zeno behavior. Finally, through numerical simulations of a spectrum access game and comparisons with existing event-triggered methods, we demonstrate the effectiveness of our proposed algorithm.

1. Introduction

Game theory has diverse applications including power grids (Wang et al., 2021), mobile ad-hoc networks (Stankovic et al., 2011), resource allocation (Rahman et al., 2019) and social networks (Ghaderi & Srikant, 2014), capturing competitive features among different entities. In non-cooperative games, each self-interested player aims to maximize or minimize its local objective function, often in conflict with other players. A Nash equilibrium (NE) in such games presents a rigorous mathematical characterization of stable and desirable solutions and has attracted considerable interest in the past few decades.

The rapid development of large-scale networks has made traditional NE seeking algorithms with centralized frameworks impractical due to limited scalability and high computation cost (Frihauf et al., 2011; Govindan & Wilson, 2003; Kannan & Shanbhag, 2012). In view of this, distributed NE seeking in non-cooperative games where players only communicate with their neighbors has shown theoretical significance and practical relevance in recent years. For discrete-time settings, Salehisadaghiani and Pavel (2016) developed an asynchronous gossip-based method that achieved almost sure convergence for seeking a NE, but the convergence was slowed down by diminishing step sizes. Later, Salehisadaghiani et al. (2019) utilized an alternating direction method of multipliers approach to achieve the NE with constant step sizes. For continuous-time cases, Gadjov and Pavel (2018) presented a passivity-based algorithm that leveraged incremental passivity properties of the pseudo-gradient to obtain the NE over networks. Ye and Hu (2017) proposed a consensus-based approach to exponentially converge the NE.

However, conventional distributed NE seeking algorithms mentioned above require continuous communication and result in a high communication burden, making them impractical in physical applications. This can be especially problematic for some embedding networks equipped with energy harvesting, where each player's energy is a scarce resource that needs careful monitoring and control. One motivating application is the spectrum access game in energy-harvesting body sensor networks (BSNs) (Niyato & Hossain, 2007), where multiple BSNs compete for bandwidth in a cognitive radio network. They use the allocated spectrum to transmit physiological data to a remote healthcare center, aiming to minimize their own transmission cost while receiving the best health service by selecting an appropriate spectrum size. To achieve the NE in a distributed manner, BSNs need to interact with their neighbors to compensate for the lack of global information on others' strategies. However, this continuous communication can excessively consume the scarce energy harvested from the ambient environment. Thus, there is a need for novel communication-efficient algorithms that seek the NE while conserving the harvested energy.

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The event-triggered mechanism has gained popularity in the control community since it reduces the communication burden by filtering out unnecessary information transmission (Wu et al., 2012). In the field of non-cooperative games, Shi and Yang (2019) proposed an edge-based event-triggering law for discrete-time aggregative games. However, the convergence speed is slow due to the diminishing step size. Recently, Yu et al. (2022) designed a static event-triggering law with a decaying threshold, and Xu et al. (2022) proposed a fully distributed edge-based adaptive dynamic event-triggered scheme for undirected networks. Nonetheless, these algorithms only converge to a neighborhood of the NE instead of the exact NE (Xu et al., 2022; Yu et al., 2022). To achieve predefined-time convergence with an arbitrarily small error, Liu and Yi (2023) constructed an adaptive event trigger with a time-base generator. Zhang et al. (2021) successfully applied a dynamic event-triggered method from Yi et al. (2018) to distributed games and demonstrated that the algorithm converges to the exact NE. However, all existing works focus on deterministic event-triggered algorithms that precisely specify the triggering times for each player.

Recently, Tsang et al. (2019, 2020) extended deterministic event triggers to stochastic versions by defining the triggering time more loosely. This extension achieved a better trade-off between the communication effort and convergence performance in multi-agent consensus and decentralized unconstrained optimization over undirected networks. Nonetheless, the existing stochastic event-triggering laws cannot be directly applied to distributed NE seeking problems since each player's cost function in a non-cooperative game is coupled with the actions of other players. Due to the complex information exchange setting in distributed NE seeking, the design of the stochastic event-triggered mechanism and the convergence analysis encounter more difficulties. Additionally, constrained action sets should also be considered. To the best of the authors' knowledge, no stochastic event-triggered mechanism has been designed for distributed constrained NE seeking problems.

All of the above motivates us to develop a stochastic event-triggered algorithm for a multi-agent system to seek the NE in a distributed constrained game. The main contributions of this paper are summarized below:

(1) We propose a novel stochastic event-triggered distributed NE seeking algorithm for constrained non-cooperative games in directed networks.
(2) We prove that the developed algorithm converges exponentially to the exact NE. Furthermore, we demonstrate that the algorithm is free of Zeno behavior, validating its feasibility.
(3) Simulation results for the spectrum access game in BSNs indicate that the proposed algorithm better balances the communication consumption and convergence properties than deterministic ones.

The remainder of this paper is organized as follows. In Section 2, some preliminaries are provided. Then the problem formulation about the distributed NE seeking under an event-triggered mechanism is presented in Section 3. In Section 4, a stochastic event-triggered algorithm is proposed first, and then the convergence, together with a guarantee on the exclusion of Zeno behavior is analyzed. Simulations are given in Section 5 to illustrate the effectiveness of the proposed algorithm. Finally, conclusions are offered in Section 6.

Notations. In this paper, we define $\mathbb{R}$ as the set of real numbers and $\mathbb{R}^N$ as the set of N-dimensional real vectors. We use $X \succ 0$ to indicate that the matrix X is positive definite. The notation $\mathrm{diag}\{a_1, a_2, \ldots, a_N\}$ denotes a diagonal matrix with elements $a_1, a_2, \ldots, a_N$. The matrix $I_N \in \mathbb{R}^{N\times N}$ represents the identity matrix and $\mathbf{1}_N \in \mathbb{R}^N$ denotes a vector with all elements being 1. The operator $\|\cdot\|$ refers to the induced 2-norm for matrices and the Euclidean norm for vectors. For any vector $\mathbf{v} \in \mathbb{R}^N$, $\mathbf{v}^T$ represents its transpose. The meaning of $P(E)$ is the probability of the event E happening. For any two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, $A \otimes B \in \mathbb{R}^{np\times mq}$ is the Kronecker product of A by B.

2. Preliminaries

2.1. Game theory

Definition 1. A game is defined as a tuple $\boldsymbol{\Gamma} = \{\mathcal{P}, \mathcal{X}, f\}$, where $\mathcal{P} = \{1, 2, \ldots, N\}$ is the set of players, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N$, $\mathcal{X}_i \subseteq \mathbb{R}$ is the action set of the ith player, and $f = \{f_1, f_2, \ldots, f_N\}$ is the set of cost functions for all players, with each $f_i : \mathbb{R}^N \to \mathbb{R}$ mapping the joint action of all players to the cost of player i.

Definition 2. An NE is an action profile $\mathbf{x}^* = [x_1^*, x_2^*, \ldots, x_N^*]^T \in \mathcal{X}$ such that for each player $i \in \mathcal{P}$, $f_i(x_i^*, \mathbf{x}_{-i}^*) \le f_i(x_i, \mathbf{x}_{-i}^*)$, where $x_i \in \mathcal{X}_i$ and $\mathbf{x}_{-i}^* = [x_1^*, x_2^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_N^*]^T$.

2.2. Graph theory

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and a set of edges $\mathcal{E}$, where each edge $(i, j) \in \mathcal{E}$ denotes a communication link from player j to player i.

The underlying topology of $\mathcal{G}$ is represented by the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$, if $(i, j) \notin \mathcal{E}$. The degree matrix is defined as $D = \mathrm{diag}\{d_1^{\mathrm{in}}, d_2^{\mathrm{in}}, \ldots, d_N^{\mathrm{in}}\}$, where $d_i^{\mathrm{in}} = \sum_{j=1}^N a_{ij}$. The Laplacian matrix L is then defined as $L = D - A$. The directed graph $\mathcal{G}$ is said to be strongly connected if, for any node, there exists a directed path to every other node.

We present the following lemma regarding strongly connected directed graphs (Zhang et al., 2021):

Lemma 3. *$(L \otimes I_N + B_0)$ is a non-singular M-matrix if and only if $\mathcal{G}$ is a directed and strongly connected graph, where $B_0 = \mathrm{diag}\{a_{11}, \ldots, a_{1N}, a_{21}, \ldots, a_{2N}, \ldots, a_{N1}, \ldots, a_{NN}\}$. Moreover, there exist positive definite matrices P and Q such that*

$$(L \otimes I_N + B_0)^T P + P (L \otimes I_N + B_0) = Q. \tag{1}$$

2.3. Projection operator

A set $\mathcal{X} \subseteq \mathbb{R}^N$ is convex if $c\mathbf{v}_1 + (1 - c)\mathbf{v}_2 \in \mathcal{X}$, for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{X}$ and any $c \in [0, 1]$. For a closed and convex set $\mathcal{X}$, the projection operator $\mathbb{P}_{\mathcal{X}}(\cdot) : \mathbb{R}^N \to \mathcal{X}$ is defined as $\mathbb{P}_{\mathcal{X}}(\mathbf{v}) = \arg\min_{\mathbf{z}\in\mathcal{X}} \|\mathbf{v} - \mathbf{z}\|$.

Lemma 4 (*Facchinei and Pang (2003)*)**.** *For a closed and convex set $\mathcal{X} \subseteq \mathbb{R}^N$, the projector $\mathbb{P}_{\mathcal{X}}(\cdot)$ is non-expansive, i.e., for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{X}$, $\|\mathbb{P}_{\mathcal{X}}(\mathbf{v}_1) - \mathbb{P}_{\mathcal{X}}(\mathbf{v}_2)\| \le \|\mathbf{v}_1 - \mathbf{v}_2\|$.*

3. Problem formulation

Consider a non-cooperative multi-agent system with $N > 1$ players represented by a strongly connected directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each selfish player i intends to minimize its own cost function,

$$\min_{x_i\in\mathcal{X}_i} f_i(x_i, \mathbf{x}_{-i}), \tag{2}$$

where $\mathcal{X}_i$ is a closed convex set, and f_i is the convex cost function of player i satisfying the following assumptions:

Assumption 5. $f_i(\mathbf{x})$ is twice continuously differentiable and $\frac{\partial f_i}{\partial x_i}(\mathbf{x})$ is globally Lipschitz for all $i \in \mathcal{V}$, that is, there exists a constant $l_i > 0$ such that $\left\| \frac{\partial f_i}{\partial x_i}(\mathbf{x}) - \frac{\partial f_i}{\partial x_i}(\mathbf{y}) \right\| \le l_i \|\mathbf{x} - \mathbf{y}\|$.

Assumption 6. There exists a constant $\mu > 0$ such that $(\mathbf{x} - \mathbf{y})^T (F(\mathbf{x}) - F(\mathbf{y})) \ge \mu \|\mathbf{x} - \mathbf{y}\|^2$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, where $F(\mathbf{x}) = \left[\frac{\partial f_1}{\partial x_1}(\mathbf{x}), \frac{\partial f_2}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f_N}{\partial x_N}(\mathbf{x})\right]^T \in \mathbb{R}^N$ denotes the pseudo-gradient (the stacked vector of all players' partial gradients w.r.t. local cost functions).

Remark 7. Under Assumption 5, the NE of game (2), denoted as $\mathbf{x}^*$, can be found by solving the variational inequality $VI(\mathcal{X}, F)$, where $(\mathbf{x} - \mathbf{x}^*)^T F(\mathbf{x}^*) > 0$ holds for all $\mathbf{x} \in \mathcal{X}$ (Facchinei & Pang, 2003). Assumption 6 implies that $VI(\mathcal{X}, F)$ has at most one solution (Facchinei & Kanzow, 2007). Thus, the existence and uniqueness of the NE of (2) follows and $\mathbf{x}^*$ satisfies

$$\mathbf{x}^* = \mathbb{P}_{\mathcal{X}}(\mathbf{x}^* - \tilde{\alpha} F(\mathbf{x}^*)), \ \forall \tilde{\alpha} > 0. \tag{3}$$

In a distributed setting, each player communicates with its neighbors to obtain partial information about the others' actions. We consider a leader–follower-based consensus control algorithm with projected gradient play dynamics (Liang et al., 2022; Ye & Hu, 2017):

$$\dot{x}_i(t) = \mathbb{P}_{\mathcal{X}_i}\left(x_i(t) - \alpha \frac{\partial f_i}{\partial x_i} (\mathbf{y}_i(t)) \right) - x_i(t), \tag{4}$$

$$\begin{aligned} \dot{y}_{ij}(t) = -\beta \Bigg[&\sum_{k=1}^{N} a_{ik} \left(y_{ij}(t) - y_{kj}(t) \right) \\ &+ a_{ij} \left(y_{ij}(t) - x_j(t) \right) \Bigg], \end{aligned} \tag{5}$$

where $\alpha, \beta > 0$ are step sizes, $\mathbf{y}_i = [y_{i1}, y_{i2}, \ldots, y_{iN}]^T \in \mathbb{R}^N$, y_{ij} is player i's estimate on player j, $y_{ii} = x_i$, and the initial actions are chosen as $x_i(0) \in \mathcal{X}_i$.

The algorithm composed of (4) and (5) requires continuous communication among players. We employ an event-triggered mechanism to reduce the communication times, i.e., a player only broadcasts its action and estimate when certain critical events occur. Thus, the update law of (5) becomes

$$\begin{aligned} \dot{y}_{ij}(t) = -\beta \Bigg[&\sum_{k=1}^{N} a_{ik} \left(\hat{y}_{ij}(t) - \hat{y}_{kj}(t) \right) \\ &+ a_{ij} \left(\hat{y}_{ij}(t) - \hat{x}_j(t) \right) \Bigg], \end{aligned} \tag{6}$$

where $\hat{y}_{ij}$ represents the latest estimate on player j broadcast by player i, and $\hat{x}_j$ is the latest state broadcast by player j. Suppose player i's triggering time instants are $\{t_1^i, t_2^i, \ldots, t_k^i, \ldots\}$, and then $\hat{y}_{ij}(t) = y_{ij}(t_k^i)$, $\hat{x}_i(t) = x_i(t_k^i)$ for $t \in \left[t_k^i, t_{k+1}^i\right)$.

Our objective is to develop an event-triggered mechanism such that the NE can be asymptotically achieved. Specifically, we aim to design a decision variable:

$$\gamma_i(t) = \begin{cases} 1, & \hat{x}_i(t) = x_i(t), \hat{y}_{ij}(t) = y_{ij}(t), \\ 0, & \text{otherwise}, \end{cases} \quad \forall i \in \mathcal{V},$$

so that the average communication rate

$$\Gamma(t) = \frac{1}{Nt} \sum_{i=1}^{N} \int_0^t \gamma_i(t) dt \tag{7}$$

with $\Gamma(0) = 0$ can be reduced. For the stochastic event-triggered mechanism, we consider its expected value $\mathbb{E}[\Gamma(t)]$ due to the randomness of $\gamma_i(t)$. Moreover, the event-triggered mechanism should not exhibit Zeno behavior which refers to the phenomenon that an infinite number of events occur in a finite time.

Remark 8. The problem formulation in this paper differs from the distributed optimization problem that Tsang et al. (2020) solved. In their scenario, N agents cooperatively minimize a global cost function, $F(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x)$ where agent i's cost function is *decoupled* with the other agents' actions, $x_j, j \neq i$. Although we can regard (2) as a set of parallel optimization problems, each player's cost function f_i depends on all other players' decisions $\mathbf{x}_{-i}$ and each player only has access to information about its neighbors. As a result, player i needs to keep an estimate of other players' strategies, $\mathbf{y}_i$, and communicates this information with neighbors to seek the NE. Due to this more complex information exchange setting, the stochastic event-triggering law design and convergence analysis become more complex compared to Tsang et al. (2020).

4. Main results

In this section, we propose a stochastic event-triggered algorithm for distributed NE seeking and prove that it converges exponentially without Zeno behavior.

4.1. Proposed stochastic event-triggering law

The compact form of (4) and (6) can be written as

$$\dot{\mathbf{x}}(t) = \mathbb{P}_{\mathcal{X}}\left(\mathbf{x}(t) - \alpha \frac{\partial f}{\partial \mathbf{x}}(\mathbf{y}(t)) \right) - \mathbf{x}(t), \tag{8}$$

$$\dot{\mathbf{y}}(t) = -\beta \left[(L \otimes I_N + B_0) \left(\hat{\mathbf{y}}(t) - \mathbf{1}_N \otimes \hat{\mathbf{x}}(t) \right) \right], \tag{9}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$, $\hat{\mathbf{x}} = \left[\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N\right]^T$, $\mathbf{y} = \left[\mathbf{y}_1^T, \mathbf{y}_2^T, \ldots, \mathbf{y}_N^T\right]^T$, $\hat{\mathbf{y}} = \left[\hat{\mathbf{y}}_1^T, \hat{\mathbf{y}}_2^T, \ldots, \hat{\mathbf{y}}_N^T\right]^T$, $\hat{\mathbf{y}}_i = \left[\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{iN}\right]^T$, and $\frac{\partial f}{\partial \mathbf{x}}(\mathbf{y}) = \left[\frac{\partial f_1}{\partial x_1}(\mathbf{y}_1), \frac{\partial f_1}{\partial x_2}(\mathbf{y}_2), \ldots, \frac{\partial f_1}{\partial x_N}(\mathbf{y}_N)\right]^T$. The equality (9) holds from $(L \otimes I_N)(\mathbf{1}_N \otimes \hat{\mathbf{x}}(t)) = 0$.

We define the event errors of player i as

$$e_{x_i}(t) = \hat{x}_i(t) - x_i(t), \tag{10}$$

$$\mathbf{e}_{\mathbf{y}_i}(t) = \hat{\mathbf{y}}_i(t) - \mathbf{y}_i(t), \tag{11}$$

and the consensus error between player i's estimate and j's estimate as

$$\Delta_{ij}(t) = \hat{\mathbf{y}}_i(t) - \hat{\mathbf{y}}_j(t). \tag{12}$$

We propose a stochastic event trigger:

$$\gamma_i(t) = \begin{cases} 1, & \xi_i(t) > \kappa \exp\left(-c_i \rho_i(t) / \delta_i(t)\right), \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

where $\kappa > 1$ is a parameter, $\xi_i(t) \in (a, 1)$ an arbitrary stationary ergodic random process with a constant $a > 0$, $c_i > 0$ a constant, and $\delta_i(t) > 0$ a decreasing function w.r.t. t. Inspired by Tsang et al. (2020) and Zhang et al. (2021), $\rho_i(t)$ and $\delta_i(t)$ are defined as

$$\rho_i(t) = e_{x_i}(t)^2 + \left\|\mathbf{e}_{\mathbf{y}_i}(t)\right\|^2 - \sigma_i \left\| \sum_{j=1}^{N} a_{ij} \Delta_{ij}(t) \right\|^2, \tag{14}$$

$$\dot{\delta}_i(t) = -\eta \delta_i(t), \tag{15}$$

where $\sigma_i > 0$ and $\eta > 0$. According to (13) and (14), we can infer the following condition when no trigger occurs, i.e., $\gamma_i(t) = 0$:

$$\begin{aligned} &e_{x_i}(t)^2 + \left\|\mathbf{e}_{\mathbf{y}_i}(t)\right\|^2 - \sigma_i \left\| \sum_{j=1}^{N} a_{ij} \Delta_{ij}(t) \right\|^2 \\ &\le \frac{\delta_i(t)}{c_i} \left(\ln \kappa - \ln \xi_i(t) \right). \end{aligned} \tag{16}$$

Remark 9. In the literature, $\rho_i(t)$ is usually referred to as a triggering function that depends on event error, consensus error, and network parameters. Different triggering functions can result in different event-triggering laws with varying performances. Deterministic event-triggered mechanisms always trigger player i whenever $\rho_i(t) > 0$ (Xu et al., 2022; Yu et al., 2022; Zhang et al., 2021). However, stochastic event triggers are characterized by the fact that player i triggers with a certain probability that increases with $\rho_i(t)$. For example, when $a = \frac{1}{2}$ and $\xi_i(t)$ is a uniformly distributed random process, if $\rho_i(t) \le 0$, then it is impossible for player i to trigger due to $\kappa \exp\left(-c_i\rho_i(t)/\delta_i(t)\right) \ge 1 > \xi_i(t)$, which is consistent with the deterministic event-triggering law since $\gamma_i(t) = 0$ when $\rho_i(t) \le 0$. Conversely, if $\rho_i(t) > 0$, we can infer that $P\left[\gamma_i(t) = 1\right] = \frac{1}{2}\left[1 - \kappa\exp\left(-c_i\rho_i(t)/\delta_i(t)\right)\right]$ based on the distribution of $\xi_i(t)$, i.e., $P\left[\gamma_i(t) = 1\right]$ monotonically increases with the value of $\rho_i(t)$. When $\xi_i(t)$ is a strictly positive constant, the stochastic event trigger reduces to a deterministic one. Therefore, (13) can be viewed as a generalized version of the deterministic trigger, which can further reduce the communication burden and is more practical for networks with tighter communication requirements.

4.2. Convergence analysis

To simplify notation, the time index t is omitted in the following analysis.

Define the seeking errors of $\mathbf{x}$ and $\mathbf{y}$ as $\boldsymbol{\varepsilon}_\mathbf{x} = \mathbf{x} - \mathbf{x}^*$ and $\boldsymbol{\varepsilon}_\mathbf{y} = \mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}$. Then based on (8)–(11), the dynamics of $\boldsymbol{\varepsilon}_\mathbf{x}$ and $\boldsymbol{\varepsilon}_\mathbf{y}$ can be written as

$$
\begin{aligned}
\dot{\boldsymbol{\varepsilon}}_\mathbf{x} &= \dot{\mathbf{x}} = \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x})\right) - \mathbf{x},\\
\dot{\boldsymbol{\varepsilon}}_\mathbf{y} &= \dot{\mathbf{y}} - \mathbf{1}_N \otimes \dot{\mathbf{x}}\\
&= \beta\left(L \otimes I_N + B_0\right)\left(\mathbf{1}_N \otimes \mathbf{e}_\mathbf{x} - \boldsymbol{\varepsilon}_\mathbf{y} - \mathbf{e}_\mathbf{y}\right)\\
&\quad - \mathbf{1}_N \otimes \left\{\mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x})\right) - \mathbf{x}\right\},
\end{aligned}
$$

where $\mathbf{e}_\mathbf{x} = [e_{x_1}, e_{x_2}, \ldots, e_{x_N}]^T$ and $\mathbf{e}_\mathbf{y} = [\mathbf{e}_{\mathbf{y}_1}^T, \mathbf{e}_{\mathbf{y}_2}^T, \ldots, \mathbf{e}_{\mathbf{y}_N}^T]^T$.

Theorem 10. *For a multi-agent system, if Assumptions 5 and 6 hold, the distributed algorithm (4) and (6) under the stochastic event-triggering law (13) exponentially converges to the NE $\mathbf{x}^*$ with*

$$0 < \alpha < \frac{2\mu\beta\lambda - 8C_2C_3 - 2\mu C_4}{8C_1C_2C_3 + 4\mu C_2C_3 + \beta C_1^2\lambda - C_1^2C_4}, \tag{17}$$

$$\beta > \frac{4C_2C_3 + \mu C_4}{\mu\lambda}, \tag{18}$$

$$0 < \sigma \le \frac{N-1}{2N\left\|L\right\|^2}, \tag{19}$$

where $\sigma = \max_i\{\sigma_i\}$, λ is the minimum eigenvalue of Q, $C_1 = \bar{l}\sqrt{N}$, $C_2 = \bar{l}$, $C_3 = \sqrt{N}\left\|P\right\|$, $C_4 = 2\sqrt{2(N-1)}\left\|P(L \otimes I_N + B_0)\right\|$, and $\bar{l} = \max_i\{l_i\}$.

Proof. Inspired by Liang et al. (2022), we consider the Lyapunov candidate

$$V = \phi_2V_1 + \phi_1V_2 + \zeta\sum_{i=1}^{N}\delta_i, \tag{20}$$

where $V_1 = \boldsymbol{\varepsilon}_\mathbf{x}^T\boldsymbol{\varepsilon}_\mathbf{x}$, $V_2 = \boldsymbol{\varepsilon}_\mathbf{y}^TP\boldsymbol{\varepsilon}_\mathbf{y}$, $\phi_1 = 2\alpha C_2$, $\phi_2 = 2C_3(2 + \alpha C_1)$, $\zeta = \frac{2\phi_1C_5}{\eta\min_i\{c_i\}}(\ln\kappa - \ln a)$, and $C_5 = N\sqrt{\frac{2}{N-1}}\left\|P\left(L \otimes I_N + B_0\right)\right\|$.

For the time derivative of V_1, we have

$$
\begin{aligned}
\dot{V}_1(t) &= 2\boldsymbol{\varepsilon}_\mathbf{x}(t)^T\dot{\boldsymbol{\varepsilon}}_\mathbf{x}(t)\\
&= -2\boldsymbol{\varepsilon}_\mathbf{x}^T\left\{\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right)\right\}\\
&\quad - 2\boldsymbol{\varepsilon}_\mathbf{x}^T\left\{\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x})\right)\right.\\
&\quad \left. - \left[\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right)\right]\right\}.
\end{aligned} \tag{21}
$$

For the first term of (21),

$$
\begin{aligned}
&2\boldsymbol{\varepsilon}_\mathbf{x}^T\left\{\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right)\right\}\\
=&2\boldsymbol{\varepsilon}_\mathbf{x}^T\left\{\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right)\right.\\
&\left. - \left[\mathbf{x}^* - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x}^* - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x}^*)\right)\right]\right\}\\
=&2\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|^2 - 2\boldsymbol{\varepsilon}_\mathbf{x}^T\left[\mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right)\right.\\
&\left. - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x}^* - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x}^*)\right)\right]\\
\ge&2\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|\left(\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\| - \left\|\boldsymbol{\varepsilon}_\mathbf{x} - \alpha d(\mathbf{x})\right\|\right),
\end{aligned} \tag{22}
$$

where $d(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x}) - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x}^*)$, and the first equation holds from (3). Then

$$
\begin{aligned}
\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\| - \left\|\boldsymbol{\varepsilon}_\mathbf{x} - \alpha d(\mathbf{x})\right\| &= \frac{\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|^2 - \left\|\boldsymbol{\varepsilon}_\mathbf{x} - \alpha d(\mathbf{x})\right\|^2}{\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\| + \left\|\boldsymbol{\varepsilon}_\mathbf{x} - \alpha d(\mathbf{x})\right\|}\\
&\ge \frac{2\alpha\boldsymbol{\varepsilon}_\mathbf{x}^Td(\mathbf{x}) - \alpha^2\left\|d(\mathbf{x})\right\|^2}{\left(2 + \alpha\sqrt{N}l\right)\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|}\\
&\ge \frac{2\alpha\mu - \alpha^2\bar{l}^2N}{2 + \alpha\bar{l}\sqrt{N}}\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|.
\end{aligned} \tag{23}
$$

For the last two terms of (21),

$$
\begin{aligned}
&- 2\boldsymbol{\varepsilon}_\mathbf{x}^T\left\{\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x})\right)\right.\\
&\left. - \left[\mathbf{x} - \mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right)\right]\right\}\\
\le&2\alpha\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|\left\|\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x}) - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x})\right\|\\
\le&2\alpha\bar{l}\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|\left\|\boldsymbol{\varepsilon}_\mathbf{y}\right\|.
\end{aligned} \tag{24}
$$

Combining (22), (23), (24) into (21), we get

$$\dot{V}_1 \le -\omega_1\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|^2 + \phi_1\left\|\boldsymbol{\varepsilon}_\mathbf{x}\right\|\left\|\boldsymbol{\varepsilon}_\mathbf{y}\right\|, \tag{25}$$

where $\omega_1 = \frac{2(2\alpha\mu - \alpha^2C_1^2)}{2 + \alpha C_1}$.

Moreover, the time derivative of V_2 is

$$
\begin{aligned}
&\dot{V}_2(t)\\
&= 2\boldsymbol{\varepsilon}_\mathbf{y}^TP\dot{\boldsymbol{\varepsilon}}_\mathbf{y}\\
&= -2\boldsymbol{\varepsilon}_\mathbf{y}^TP\left\{\mathbf{1}_N \otimes \left[\mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x})\right) - \mathbf{x}\right]\right\}\\
&\quad + 2\beta\boldsymbol{\varepsilon}_\mathbf{y}^TP\left(L \otimes I_N + B_0\right)\left(\mathbf{1}_N \otimes \mathbf{e}_\mathbf{x} - \boldsymbol{\varepsilon}_\mathbf{y} - \mathbf{e}_\mathbf{y}\right)\\
&= -2\boldsymbol{\varepsilon}_\mathbf{y}^TP\left\{\mathbf{1}_N \otimes \left[\mathbb{P}_{\mathcal{X}}\left(\mathbf{x} - \alpha\frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_\mathbf{y} + \mathbf{1}_N \otimes \mathbf{x})\right) - \mathbf{x}\right]\right\}\\
&\quad + 2\beta\boldsymbol{\varepsilon}_\mathbf{y}^TP\left(L \otimes I_N + B_0\right)\left(\mathbf{1}_N \otimes \mathbf{e}_\mathbf{x} - \mathbf{e}_\mathbf{y}\right)\\
&\quad - \beta\boldsymbol{\varepsilon}_\mathbf{y}^TQ\boldsymbol{\varepsilon}_\mathbf{y}.
\end{aligned} \tag{26}
$$

Similar to (22), the first term of (26) satisfies

$$
\begin{aligned}
& -2\boldsymbol{\varepsilon}_{\mathbf{y}}^T P\left\{\mathbf{1}_N \otimes\left[\mathbb{P}_{\mathcal{X}}\left(\mathbf{x}-\alpha \frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_{\mathbf{y}}+\mathbf{1}_N \otimes \mathbf{x})\right)-\mathbf{x}\right]\right\} \\
\leq & 2\sqrt{N}\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|\|P\|\left\|\mathbb{P}_{\mathcal{X}}\left(\mathbf{x}-\alpha \frac{\partial f}{\partial \mathbf{x}}(\boldsymbol{\varepsilon}_{\mathbf{y}}+\mathbf{1}_N \otimes \mathbf{x})\right)-\mathbf{x}\right. \\
& \left.-\left[\mathbb{P}_{\mathcal{X}}\left(\mathbf{x}^*-\alpha \frac{\partial f}{\partial \mathbf{x}}(\mathbf{1}_N \otimes \mathbf{x}^*)\right)-\mathbf{x}^*\right]\right\| \\
\leq & 2\sqrt{N}\|P\|\left(2+\alpha \bar{l}\sqrt{N}\right)\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|+2\alpha \bar{l}\sqrt{N}\|P\|\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2, \qquad (27)
\end{aligned}
$$

and the second term of (26),

$$
\begin{aligned}
& 2\boldsymbol{\varepsilon}_{\mathbf{y}}^T P\left(L \otimes I_N+B_0\right)\left(\mathbf{1}_N \otimes \mathbf{e}_{\mathbf{x}}\right) \\
\leq & \frac{1}{\nu}\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2+\nu N\left\|P\left(L \otimes I_N+B_0\right)\right\|^2\left\|\mathbf{e}_{\mathbf{x}}\right\|^2, \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
& -2\boldsymbol{\varepsilon}_{\mathbf{y}}^T P\left(L \otimes I_N+B_0\right) \mathbf{e}_{\mathbf{y}} \\
\leq & \frac{1}{\nu}\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2+\nu\left\|P\left(L \otimes I_N+B_0\right)\right\|^2\left\|\mathbf{e}_{\mathbf{y}}\right\|^2, \qquad (29)
\end{aligned}
$$

for any $\nu > 0$ according to Young's inequality.

According to (16),

$$
\begin{aligned}
& \|\mathbf{e}_{\mathbf{x}}\|^2+\left\|\mathbf{e}_{\mathbf{y}}\right\|^2 \\
= & \sum_{i=1}^N e_{x_i}^2+\sum_{i=1}^N\left\|\mathbf{e}_{\mathbf{y}_i}\right\|^2 \\
\leq & \sum_{i=1}^N \frac{\delta_i}{c_i}(\ln \kappa-\ln \xi_i)+\sum_{i=1}^N \sigma_i\left\|\sum_{j=1}^N a_{ij}\Delta_{ij}(t)\right\|^2 \\
\leq & \sum_{i=1}^N \frac{\delta_i}{c_i}(\ln \kappa-\ln \xi_i)+2\sigma\|L\|^2\left(\left\|\mathbf{e}_{\mathbf{y}}\right\|^2+\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& \|\mathbf{e}_{\mathbf{x}}\|^2+\left(1-2\sigma\|L\|^2\right)\left\|\mathbf{e}_{\mathbf{y}}\right\|^2 \\
\leq & \sum_{i=1}^N \frac{\delta_i}{c_i}(\ln \kappa-\ln \xi_i)+2\sigma\|L\|^2\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2.
\end{aligned}
$$

If $\sigma \leq \frac{N-1}{2N\|L\|^2}$, the combination of the second terms of (28) and (29) satisfies

$$
\begin{aligned}
& \nu N\left\|P\left(L \otimes I_N+B_0\right)\right\|^2\left(\|\mathbf{e}_{\mathbf{x}}\|^2+\frac{1}{N}\left\|\mathbf{e}_{\mathbf{y}}\right\|^2\right) \\
\leq & \nu N\left\|P\left(L \otimes I_N+B_0\right)\right\|^2\left[\|\mathbf{e}_{\mathbf{x}}\|^2+\left(1-2\sigma\|L\|^2\right)\left\|\mathbf{e}_{\mathbf{y}}\right\|^2\right] \\
\leq & \nu N\left\|P\left(L \otimes I_N+B_0\right)\right\|^2\left[\sum_{i=1}^N \frac{\delta_i}{c_i}(\ln \kappa-\ln \xi_i)\right. \\
& \left.+2\sigma\|L\|^2\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2\right]. \qquad (30)
\end{aligned}
$$

Letting $\nu = \frac{\sqrt{2}}{\sqrt{N-1}\|P(L\otimes I_N+B_0)\|}$, and combining (27)–(30) into (26), one obtains

$$
\begin{aligned}
\dot{V}_2 \leq & -\omega_2\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2+\phi_2\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\| \\
& +C_5\left[\sum_{i=1}^N \frac{\delta_i(t)}{c_i}(\ln \kappa-\ln \xi_i(t))\right], \qquad (31)
\end{aligned}
$$

where $\omega_2 = \beta\lambda - 2\alpha C_2 C_3 - C_4$.

Combining (25) and (31), we have

$$
\begin{aligned}
& \phi_2\dot{V}_1+\phi_1\dot{V}_2 \\
\leq & -\phi_2\omega_1\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|^2+2\phi_1\phi_2\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|-\phi_1\omega_2\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2 \\
& +\phi_1 C_5\left[\sum_{i=1}^N \frac{\delta_i(t)}{c_i}(\ln \kappa-\ln \xi_i(t))\right] \\
= & -\Theta^*(\phi_2\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|^2+\phi_1\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2)-\left(\omega_1-\Theta^*\right)\phi_2\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|^2 \\
& -\left(\omega_2-\Theta^*\right)\phi_1\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\|^2+2\phi_1\phi_2\|\boldsymbol{\varepsilon}_{\mathbf{x}}\|\left\|\boldsymbol{\varepsilon}_{\mathbf{y}}\right\| \\
& +\phi_1 C_5\left[\sum_{i=1}^N \frac{\delta_i(t)}{c_i}(\ln \kappa-\ln \xi_i(t))\right] \\
\leq & -\Theta^*\left(\phi_2 V_1+\frac{\phi_1}{\lambda_M(P)}V_2\right) \\
& +\phi_1 C_5\left[\sum_{i=1}^N \frac{\delta_i(t)}{c_i}(\ln \kappa-\ln \xi_i(t))\right], \qquad (32)
\end{aligned}
$$

where $\Theta^* = \left(\omega_1+\omega_2-\sqrt{(\omega_1-\omega_2)^2+4\phi_1\phi_2}\right)/2$, and $\lambda_M(P)$ is the maximum eigenvalue of P. It follows from (17) and (18) that $\Theta^* > 0$, and $(\omega_1-\Theta^*)(\omega_2-\Theta^*) = \phi_1\phi_2$. Thus,

$$
\begin{aligned}
\dot{V}(t) \leq & -\Theta^*\left(\phi_2 V_1+\frac{\phi_1}{\lambda_M(P)}V_2\right)-\eta\zeta\sum_{i=1}^N \delta_i(t) \\
& +\phi_1 C_5\left[\sum_{i=1}^N \frac{\delta_i(t)}{c_i}(\ln \kappa-\ln \xi_i(t))\right] \\
\leq & -\Theta^*\left(\phi_2 V_1+\frac{\phi_1}{\lambda_M(P)}V_2\right)-\eta\zeta\sum_{i=1}^N \delta_i(t) \\
& +\frac{\phi_1 C_5}{\min_i\{c_i\}}(\ln \kappa-\ln a)\sum_{i=1}^N \delta_i(t) \\
= & -\Theta^*(\phi_2 V_1+\frac{\phi_1}{\lambda_M(P)}V_2)-\frac{\eta\zeta}{2}\sum_{i=1}^N \delta_i(t) \\
\leq & -k_v V(t), \qquad (33)
\end{aligned}
$$

where $k_v = \min\left\{\Theta^*, \frac{\Theta^*}{\lambda_M(P)}, \frac{\eta}{2}\right\}$. By LaSalle's invariance principle (Khalil, 2002), we conclude that $\boldsymbol{\varepsilon}_{\mathbf{x}}$ and $\boldsymbol{\varepsilon}_{\mathbf{y}}$ converge to zero exponentially.

Remark 11. Based on (33), $k_v > 0$ represents the actual lower bound on the convergence rate, which relies on Θ^*, $\lambda_M(P)$, and η. The variable $\delta_i(t)$ plays an important role in the convergence analysis. If $\xi_i(t)$ is a strictly positive constant, the above analysis remains valid since the stochastic event trigger is an extension of its deterministic counterpart.

Remark 12. The proposed algorithm achieves the exact NE with exponential convergence, which outperforms the convergence guarantees of event-triggered algorithms in Yu et al. (2022) and Xu et al. (2022), which only converge to a neighborhood of the NE. Moreover, Tsang et al. (2020) showed that their stochastic event-triggered optimization algorithm converges to the proximity of the optimal point with arbitrary accuracy. Hence, our algorithm and analysis provide a stronger convergence guarantee.

4.3. Analysis on Zeno behavior

Theorem 13. *For a multi-agent system, the distributed NE seeking algorithm composed of (4) and (6) under the stochastic event-triggering law (13) does not exhibit Zeno behavior.*

The proof of Theorem 13 is provided in Huo et al. (2023).

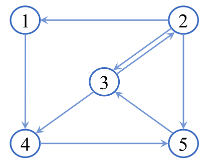

Fig. 1. Communication topology for BSNs.

Remark 14. Excluding Zeno behavior validates the well-posedness of the proposed stochastic event-triggered algorithm.

5. Numerical simulations

We consider the spectrum access game in energy harvesting BSNs introduced in Section 1. Following the formulation in Niyato and Hossain (2007), we model the spectrum access problem as an oligopoly market where $N = 5$ BSNs compete to lease the spectrum size $x_i \in [0, 16]$ from the primary base station. The objective is to minimize the cost associated with leasing the spectrum, which is determined by a pricing function $p_i(\mathbf{x}) = m_i^c + q_i \left(\sum_{j=1}^N x_j\right)^\tau$, where $m_i^c, q_i \geq 0$, for $i \in \{1, 2, \ldots, 5\}$, and $\tau \geq 1$. The BSNs can improve transmission performance using adaptive modulation, earning revenue r_i per unit of the achievable transmission rate. If each BSN utilizes uncoded quadrature amplitude modulation with a square constellation, its spectral efficiency is calculated as:

$$u_i = \log_2 \left(1 + \frac{1.5 s_i}{\ln\left(\frac{0.2}{\mathbf{BER}_i^{\text{tar}}}\right)}\right), \tag{34}$$

where s_i is the received signal-to-noise ratio, indicating the quality of the signal received by the health center, and $\mathbf{BER}_i^{\text{tar}}$ is the target bit error rate level in the single-input single-output Gaussian noise channel. The revenue earned by BSN i is given by $r_i u_i x_i$, and the cost incurred by BSN i is expressed as $f_i(\mathbf{x}) = x_i p_i(\mathbf{x}) - r_i u_i x_i$.

We simulate communication between BSNs using the directed graph presented in Fig. 1. We set $a_{ij} = 1$ if $a_{ij} > 0$, $\tau = 1$, $\mathbf{BER}_i^{\text{tar}} = 10^{-4}$, and $r_i = 20$ for $i = \{1, 2, \ldots, 5\}$. The constants $m_1^c = 5.7$, $m_2^c = 10.7$, $m_3^c = 10.3$, $m_4^c = 9.7$, $m_5^c = 15$, $q_1 = 1.1$, $q_2 = 1.2$, $q_3 = 1.3$, $q_4 = 1.4$, $q_5 = 1.5$, and $s_1 = 12\,\text{dB}$, $s_2 = 14\,\text{dB}$, $s_3 = 15\,\text{dB}$, $s_4 = 16\,\text{dB}$, $s_5 = 18\,\text{dB}$. By centralized calculation, we determine that the NE for this system is $\mathbf{x}^* = [2.000, 3.987, 6.011, 8.018, 9.990]^T$. The initial actions are $\mathbf{x}(0) = [14, 12, 10, 4, 2]^T$, and the initial estimates are $\mathbf{y}_1(0) = [0, 1.5, 2.5, 3.5, 4.5]^T$, $\mathbf{y}_2(0) = [2.5, 3.5, 4.5, 5.5, 6.5]^T$, $\mathbf{y}_3(0) = [4.5, 5.5, 6.5, 7.5, 8.5]^T$, $\mathbf{y}_4(0) = [6.5, 7.5, 8.5, 9.5, 10.5]^T$, and $\mathbf{y}_5(0) = [8.5, 9.5, 10.5, 11.5, 12.5]^T$. We set step sizes as $\alpha = 0.14$, and $\beta = 1.5$. For the stochastic event-triggering law (13), we choose $\eta = 10$, $\kappa = 1.075$, $\sigma_i = 0.8/d_i^{in}$, $c_i = 1$, and $a = 0.05$.

Fig. 2 shows the evolution of BSN's action and estimates, with gray lines representing the estimates and black dashed lines representing $\mathbf{x}^*$, which indicates that all players' actions and estimates converge to the NE as expected. Fig. 3 displays the triggering times for each BSN, demonstrating that continuous communication is successfully avoided under (13).

We evaluate the effectiveness of our proposed stochastic event-triggering law by comparing it with the static law used in a previous study (Yu et al., 2022) and the dynamic law proposed by Zhang et al. (2021). Due to the randomness of (13), we conducted the simulation 100 times to obtain the empirical mean. Fig. 4 shows that (13) achieves the lowest average communication rate and significantly reduces the peak value of $\Gamma(t)$ compared to the other two laws, which implies that (13) requires much less bandwidth. We also summarized some involved metrics for triggering times and communication intervals in Table 1. Our proposed algorithm has fewer triggering counts and longer communication intervals compared to the other two laws.

The stochastic event-triggered algorithm can reduce communication costs by relaxing the triggering conditions. However, it is difficult to mathematically characterize communication rates

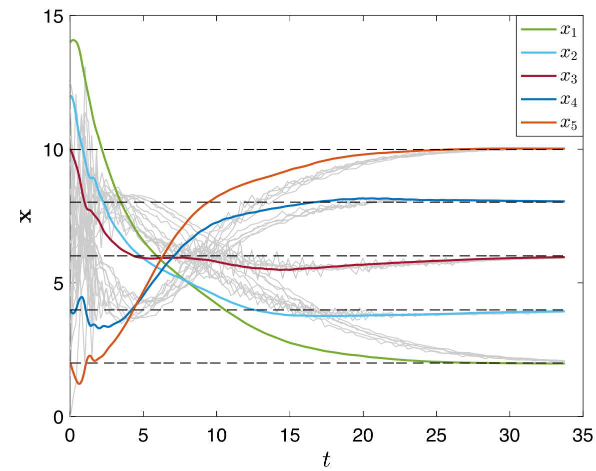

Fig. 2. The evolution of BSNs' actions and estimates.

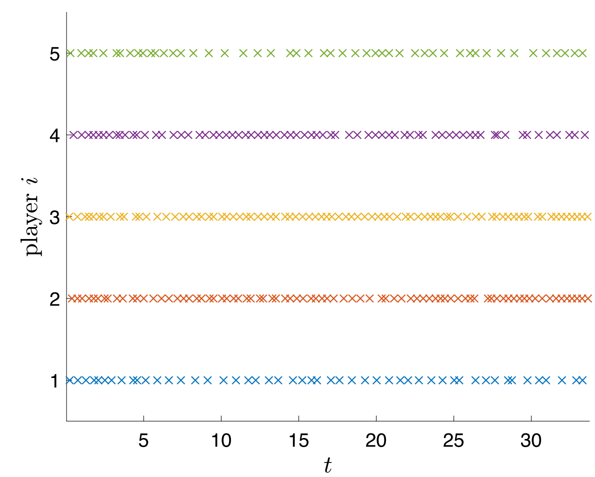

Fig. 3. The triggering times for each BSN.

Table 1
Triggering counts and statistics of communication intervals for players.

	Player	1	2	3	4	5
Trigger count	Static	430	420	392	215	407
	Dynamic	220	185	172	92	234
	Stochastic	**50**	**81**	**82**	**70**	**49**
Max interval	Static	0.100	0.150	0.175	0.225	0.150
	Dynamic	0.325	0.200	0.275	0.550	0.225
	Stochastic	**1.050**	**0.875**	**0.775**	**1.125**	**1.250**
Mean interval	Static	0.078	0.080	0.086	0.157	0.083
	Dynamic	0.154	0.182	0.197	0.367	0.144
	Stochastic	**0.675**	**0.416**	**0.413**	**0.478**	**0.688**
Min interval	Static	0.050	0.050	0.050	0.075	0.050
	Dynamic	0.125	0.125	0.125	0.200	0.075
	Stochastic	**0.250**	**0.200**	**0.200**	**0.200**	**0.225**

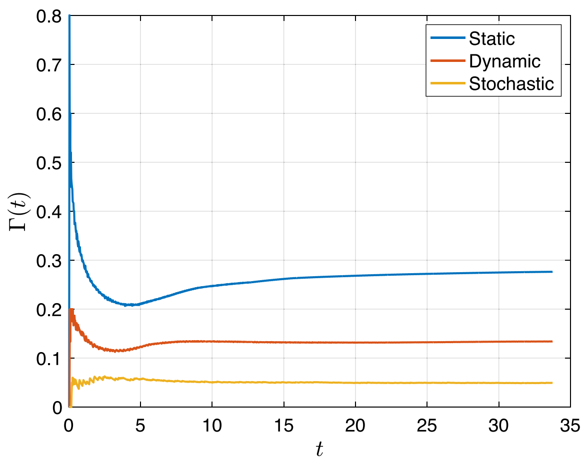

Fig. 4. Average communication rates for the static, dynamic, and stochastic event-triggering laws.

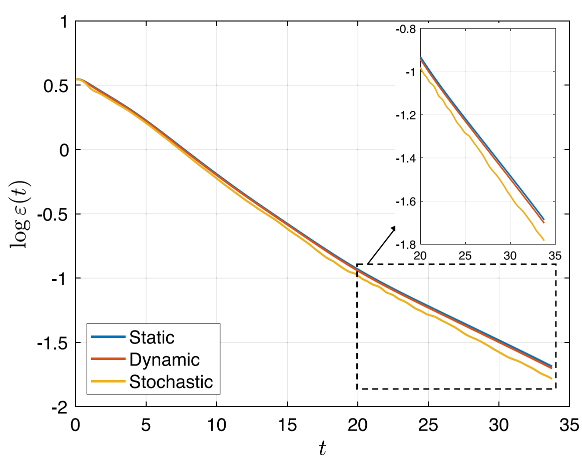

Fig. 5. Convergence performance for the static, dynamic, and stochastic event-triggering laws in semi-log scale.

under different event-triggering laws as this involves calculating the frequency of $\rho_i > 0$ for deterministic event triggers or $\xi_i(t) > \kappa \exp(-c_i\rho_i(t)/\delta_i(t))$ for stochastic ones. Therefore, most works on distributed algorithms with event-triggered mechanisms provide only convergence analysis, without theoretical estimates on the communication rate (Cao & Başar, 2020; Nowzari et al., 2019; Qian & Wan, 2021; Tsang et al., 2019, 2020; Xia et al., 2022; Yi et al., 2018; Zhang et al., 2021; Zhao et al., 2021). Usually, numerical simulations are provided to illustrate the reduction of communication cost through the proposed event trigger, as we do in our work. Although some works offer lower bounds for the minimum inter-event time by proving the exclusion of Zeno behavior, these bounds are too loose to be compared (Nowzari et al., 2019; Qian & Wan, 2021; Tsang et al., 2020; Zhao et al., 2021). Precisely quantifying the communication rate is therefore a challenging direction for future research.

Fig. 5 shows that the proposed algorithm preserves comparable convergence performance with such low communication costs, even a slightly faster convergence rate. In other words, the proposed algorithm can better balance the communication efficiency and convergence performance.

6. Conclusion and future work

In this paper, we proposed a novel stochastic event-triggered algorithm for distributed constrained NE seeking problems to improve communication efficiency. Specifically, a player transmits its message with a probability that increases as the value of the triggering function increases. We prove the exponential convergence to the exact NE and demonstrate the non-existence of Zeno behavior. Numerical simulations illustrate the practical significance of our proposed algorithm, which offers much lower communication rates and comparable convergence performance.

Potential future work includes a rigorous analysis of a tradeoff between communication rates and convergence rates, as well as a systematic design of the parameters in the algorithm.

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References

Cao, X., & Başar, Tamer (2020). Decentralized online convex optimization with event-triggered communications. *IEEE Transactions on Signal Processing*, *69*, 284–299.

Facchinei, F., & Kanzow, Christian (2007). Generalized Nash equilibrium problems. *4OR*, *5*(3), 173–210.

Facchinei, F., & Pang, Jong-Shi (2003). *Finite-dimensional variational inequalities and complementarity problems*. Springer Science & Business Media.

Frihauf, P., Krstic, Miroslav, & Basar, Tamer (2011). Nash equilibrium seeking in noncooperative games. *IEEE Transactions on Automatic Control*, *57*(5), 1192–1207.

Gadjov, D., & Pavel, Lacra (2018). A passivity-based approach to Nash equilibrium seeking over networks. *IEEE Transactions on Automatic Control*, *64*(3), 1077–1092.

Ghaderi, J., & Srikant, Rayadurgam (2014). Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate. *Automatica*, *50*(12), 3209–3215.

Govindan, S., & Wilson, Robert (2003). A global Newton method to compute Nash equilibria. *Journal of Economic Theory*, *110*(1), 65–86.

Huo, W., Tsang, Kam Fai Elvis, Yan, Yamin, Johansson, Karl Henrik, & Shi, Ling (2023). Distributed Nash equilibrium seeking with stochastic event-triggered mechanism. arXiv preprint arXiv:2304.10338.

Kannan, A., & Shanbhag, Uday V. (2012). Distributed computation of equilibria in monotone Nash games via iterative regularization techniques. *SIAM Journal on Optimization*, *22*(4), 1177–1205.

Khalil, H. K. (2002). *Nonlinear systems* (3rd ed.). Prentice Hall.

Liang, S., Yi, Peng, Hong, Yiguang, & Peng, Kaixiang (2022). Exponentially convergent distributed Nash equilibrium seeking for constrained aggregative games. *Autonomous Intelligent Systems*, *2*(1), 1–8.

Liu, J., & Yi, Peng (2023). Predefined-time distributed Nash equilibrium seeking for noncooperative games with event-triggered communication. *IEEE Transactions on Circuits and Systems II: Express Briefs*.

Niyato, D., & Hossain, Ekram (2007). A game-theoretic approach to competitive spectrum sharing in cognitive radio networks. In *IEEE wireless communications and networking conference* (pp. 16–20).

Nowzari, C., Garcia, Eloy, & Cortés, Jorge (2019). Event-triggered communication and control of networked systems for multi-agent consensus. *Automatica*, *105*, 1–27.

Qian, Y.-Y., & Wan, Yan (2021). Design of distributed adaptive event-triggered consensus control strategies with positive minimum inter-event times. *Automatica*, *133*, Article 109837.

Rahman, S., Rashid, Md Mamunur, & Alam, Md Zahangir (2019). Efficient energy allocation in wireless sensor networks based on non-cooperative game over Gaussian fading channel. In *International conference on advances in electrical engineering* (pp. 201–206).

Salehisadaghiani, F., & Pavel, Lacra (2016). Distributed Nash equilibrium seeking: A gossip-based algorithm. *Automatica*, *72*, 209–216.

Salehisadaghiani, F., Shi, Wei, & Pavel, Lacra (2019). Distributed Nash equilibrium seeking under partial-decision information via the alternating direction method of multipliers. *Automatica*, *103*, 27–35.

Shi, C.-X., & Yang, Guang-Hong (2019). Distributed Nash equilibrium computation in aggregative games: An event-triggered algorithm. *Information Sciences*, *489*, 289–302.

Stankovic, M. S., Johansson, Karl H., & Stipanovic, Dušan M. (2011). Distributed seeking of Nash equilibria with applications to mobile sensor networks. *IEEE Transactions on Automatic Control*, *57*(4), 904–919.
Tsang, K. F. E., Wu, Junfeng, & Shi, Ling (2019). Zeno-free stochastic distributed event-triggered consensus control for multi-agent systems. In *American control conference* (pp. 778–783).
Tsang, K. F. E., Wu, Junfeng, & Shi, Ling (2020). Distributed optimisation with stochastic event-triggered multi-agent control algorithm. In *IEEE conference on decision and control* (pp. 6222–6227).
Wang, Z., Liu, Feng, Ma, Zhiyuan, Chen, Yue, Jia, Mengshuo, Wei, Wei, & Wu, Qiuwei (2021). Distributed generalized Nash equilibrium seeking for energy sharing games in prosumers. *IEEE Transactions on Power Systems*, *36*(5), 3973–3986.
Wu, J., Jia, Qing-Shan, Johansson, Karl Henrik, & Shi, Ling (2012). Event-based sensor data scheduling: Trade-off between communication rate and estimation quality. *IEEE Transactions on Automatic Control*, *58*(4), 1041–1046.
Xia, L., Li, Qing, Song, Ruizhuo, & Ge, Shuzhi Sam (2022). Distributed optimized dynamic event-triggered control for unknown heterogeneous nonlinear MASs with input-constrained. *Neural Networks*, *154*, 1–12.
Xu, W., Wang, Zidong, Hu, Guoqiang, & Kurths, Jürgen (2022). Hybrid Nash equilibrium seeking under partial-decision information: An adaptive dynamic event-triggered approach. *IEEE Transactions on Automatic Control*.
Ye, M., & Hu, Guoqiang (2017). Distributed Nash equilibrium seeking by a consensus based approach. *IEEE Transactions on Automatic Control*, *62*(9), 4811–4818.
Yi, X., Liu, Kun, Dimarogonas, Dimos V., & Johansson, Karl H. (2018). Dynamic event-triggered and self-triggered control for multi-agent systems. *IEEE Transactions on Automatic Control*, *64*(8), 3300–3307.
Yu, R., Tang, Yutao, Yi, Peng, & Li, Li (2022). Distributed Nash equilibrium seeking dynamics with discrete communication. *IEEE Transactions on Neural Networks and Learning Systems*.
Zhang, K., Fang, Xiao, Wang, Dandan, Lv, Yuezu, & Yu, Xinghuo (2021). Distributed Nash equilibrium seeking under event-triggered mechanism. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *68*(11), 3441–3445.
Zhao, H., Meng, Xiangyu, & Wu, Sentang (2021). Distributed edge-based event-triggered coordination control for multi-agent systems. *Automatica*, *132*, Article 109797.

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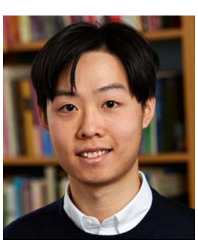

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