KTH Electrical Engineering

Resource-Constrained Multi-Agent Control Systems: Dynamic Event-triggering, Input Saturation, and Connectivity Preservation

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Abstract

A multi-agent system consists of multiple agents cooperating to achieve a common objective through local interactions. An important problem is how to reduce the amount of information exchanged, since agents in practice only have limited energy and communication resources. In this thesis, we propose dynamic event-triggered control strategies to solve consensus and formation problems for multi-agent systems under such resource constraints.

In the first part, we propose dynamic event-triggered control strategies to solve the average consensus problem for first-order continuous-time multi-agent systems. It is proven that the state of each agent converges exponentially to the average of all agents' initial states under the proposed triggering laws if and only if the underlying undirected graph is connected. In the second part, we study the consensus problem with input saturation over directed graphs. It is shown that the underlying directed graph having a directed spanning tree is a necessary and sufficient condition for achieving consensus. Moreover, in order to reduce the overall need of communication and system updates, we propose an event-triggered control strategy to solve this problem. It is shown that consensus is achieved, again, if and only if the underlying directed graph has a directed spanning tree. In the third part, dynamic event-triggered formation control with connectivity preservation is investigated. Single and double integrator dynamics are considered. All agents are shown to converge to the formation exponentially with connectivity preservation. The effectiveness of the theoretical results in the thesis is verified by several numerical examples.

Sammanfattning

Ett fleragentsystem består av en uppsättning agenter som samarbetar för att uppnå ett gemensamt mål via lokala interaktioner. Ett viktigt problem är att reducera mängden information som behöver utbytas, eftersom agenterna vid praktiska tillämpningar har begränsade energi- och kommunikationsresurser. I den här avhandlingen föreslås dynamiska reglerstrategier för att lösa konsensus- och formationsproblem för fleragentsystem under sådana bivillkor.

I den första delen föreslås dynamiska reglerstrategier för att lösa konsensusproblemet för första ordningens fleragentsystem i kontinuerlig tid. Vi bevisar att tillståndet hos varje agent konvergerar exponentiellt mot medelvärdet av alla agenters initialtillstånd om, och endast om, den underliggande grafen är sammanhängande. I den andra delen studeras konsensusproblemet med insignalsbegränsningar över riktade grafer. Vi visar att ett nödvändigt och tillräckligt villkor för att uppnå konsensus är att den underliggande riktade grafen spänns upp av ett träd. För att reducera behovet av att kommunicera och överföra systemuppdateringar föreslås en reglerstrategi baserad på händelsestyrd reglering. Vi bevisar återigen att konsensus uppnås om, och endast om, den underliggande riktade grafen spänns upp av ett träd. I den tredje delen behandlas dynamisk händelsestyrd formationsreglering med bivillkor på konnektiviteten mellan agenterna. Enkel- och dubbelintegratordynamik behandlas. Vi bevisar att alla agenter konvergerar till formationen exponentiellt, och att agenterna förblir sammankoppade. De teoretiska resultaten som härleds i avhandlingen verifieras med hjälp av numeriska exempel.

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Notations

Graph theory

$\mathcal{G}$	undirected graph or directed graph
$\mathcal{V}$	vertex set
$\mathcal{E}$	edge set; when necessary, also denoted by $\mathcal{E}(\mathcal{G})$
A	adjacency matrix
n	number of agents
$\mathcal{I}$	index set $\{1, \ldots, n\}$
v_i	i-th vertex
$\mathcal{N}_i$	in-neighbors of agent i in a directed graph or neighbors of agent i in an undirected graph
L	(weighted) Laplacian matrix
$B(\mathcal{G})$	incidence matrix of $\mathcal{G}$
SCC	strongly connected component of a directed graph
SCC_m	m-th strongly connected component of a directed graph
CC	connected component of an undirected graph
CC_m	m-th connected component of an undirected graph

Linear algebra

$\mathbb{R}^p$	real Euclidean space of dimension p or the p-dimensional vector space
$\mathbb{R}^{n\times m}$	n-by-m real matrix space
$\lVert \cdot \rVert$	Euclidean norm for vectors or the induced 2-norm for matrices
$\mathbf{1}_n$	column one vector of dimension n
I_n	n dimensional identity matrix
$\rho(M)$	spectral radius of matrix M
$\rho_2(M)$	minimum positive eigenvalue of matrix M; M has positive eigenvalues
$A^\top$	transpose of matrix A
$\text{rank}(A)$	rank of matrix A
$\det(M)$	determinant of square matrix M

$M > N$	$M - N$ is positive definite
$M \geq N$	$M - N$ is positive semi-definite
$A \otimes B$	Kronecker product of matrices A and B
$\mathrm{Diag}(x)$	diagonal matrix with the vector x on its diagonal
$c_l(x)$	l-th component of vector x
$x \perp y$	vector x is orthogonal to y, i.e., $x^\top y = 0$
$\emptyset$	empty set
$\lvert S \rvert$	cardinality of set S

Other

$a := b$	denote b as a
$a \Leftrightarrow b$	a and b are equivalent
$a \Rightarrow b$	a implies b

Chapter 1

Introduction

A multi-agent system is composed of multiple agents cooperating to achieve a common objective through local interactions. Interactions are local in the sense that an agent can only interact with a subset of agents. Multi-agent systems have been extensively studied in various disciplines over the past decades and they have broad applications in many areas, for instance, surveillance [1]; monitoring [2]; distributed data mining [3]; learning [4]; software engineering [5]; power grid [6]; transportation [7]; and logistics [8]. A specific class of applications is the cooperation of a group of manned or unmanned vehicles, such as satellite formation flying, heavy-duty vehicle platooning, and autonomous surface vehicle tracking. This kind of applications concerns the problem of vehicle formation control, i.e., making the vehicles move to a desired geometric shape. A vehicle can be represented by an agent and the interactions among vehicles can then be described as the underlying graph of a multi-agent system. In order to describe these applications more precisely, the multi-agent system model needs to take into account realistic constraints, such as energy, communication, sensing, and control constraints. So it is important to mathematically model resource-constrained multi-agent systems and to properly design their control such that a common objective is achieved while resource constraints are satisfied.

The rest of this chapter is organized as follows. Section 1.1 shows some applications that have motivated the work presented in this thesis. Section 1.2 briefly discusses resources constraints in multi-agent control systems. Section 1.3 mathematically models resource-constrained multi-agent control systems. Section 1.4 presents the problems studied in the thesis. Section 1.5 reviews some related literature. Section 1.6 gives the outline of the thesis and discusses the contributions of the author.

1.1 Motivating examples

In this section, we will briefly introduce multi-agent control systems and present some motivating examples.

A multi-agent control system consists of n agents and each agent has its own dynamics and control input. The interactions among agents is described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of vertices (or nodes) $\mathcal{V} = \{v_1, \dots, v_n\}$ and the set of edges (links)

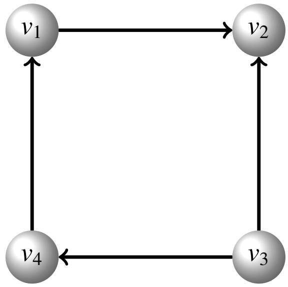


Figure 1.1: A multi-agent system with four agents.

$\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Figure 1.1 shows an example with four agents. Let $\mathcal{I} = \{1, \ldots, n\}$ denote the index set and $x_i \in \mathbb{R}^p$ the state of agent i. The state of an agent might represent physical variables such as attitude, position, temperature, or voltage. The dynamics of each agent is modelled as

$$\dot{x}_i = f_i(x_i, u_i), \; i \in \mathcal{I}, \tag{1.1}$$

where $f_i(\cdot)$ is a function and u_i the control input. The agents are supposed to have a common objective. In order to achieve it, every agent shares state information with its neighbors and determines the proper control input

$$u_i = g_i(x_i, \{x_j\}_{j \in \mathcal{N}_i}), \; i \in \mathcal{I}, \tag{1.2}$$

where $g_i(\cdot)$ is a control law and $\mathcal{N}_i = \{j \in \mathcal{I} : (v_j, v_i) \in \mathcal{E}\}$ is the neighbors of agent i.

In this following, we will introduce three examples of multi-agent control systems.

Satellite formation flying

Multiple satellites may work together to accomplish the objective of one larger, usually more expensive, satellite. This is known as satellite formation flying. It reduces cost and adds flexibility to space programs [9]. More specifically, the benefits of satellite formation flying include simpler designs, faster build times, cheaper replacement creating higher redundancy, unprecedented high resolution, and the ability to view research targets from multiple angles or at multiple times. Figure 1.2 shows the PRISMA formation flying mission [10]. PRISMA was a Swedish-led technology mission to demonstrate formation flying and rendezvous technologies. The mission consisted of two spacecraft, one advanced and highly maneuverable one, called MAIN, and a smaller one without a maneuvering capability, called TARGET. The latter one simply followed the trajectory into which it was injected by the launch system. MAIN had full translational capability, and performed a series of maneuvers around TARGET, on both close and long range approach, using the different sensors provided [11].

The satellite formation control problem is a resource-constrained two-agent system. There are several constraints in this system, but here we only discuss two of them.

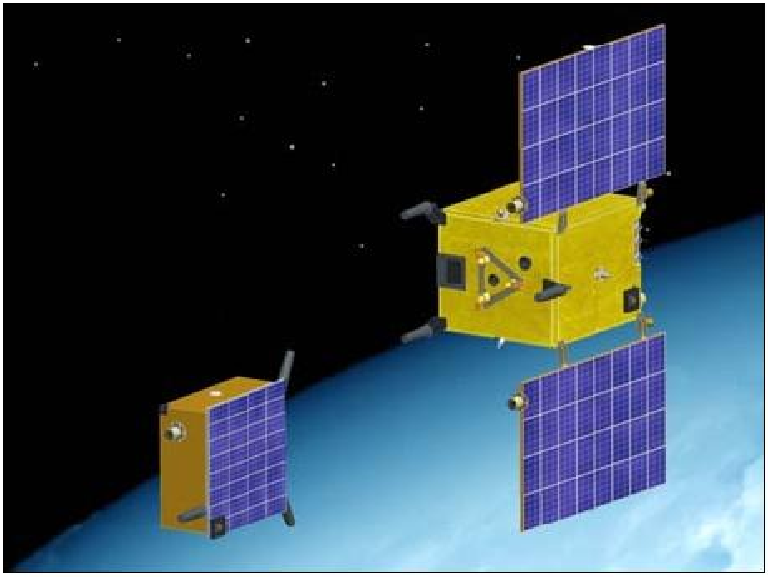


Figure 1.2: Artist's view of the PRISMA formation flying mission [10].

The first one is energy constraints. MAIN has six thrusters arranged to provide torque-free translational capability in all directions. Thus, the control energy is limited and the control input of MAIN should be optimized such that the energy consumed to perform the maneuvers is saved. The second constraint is communication constraint. Although there are two deployable solar panels to power MAIN and there is one body-mounted solar panel to power TARGET, the energy, for instance to be used for communication is limited. Thus, the communication between the ground, MAIN, and TRAGET should be reduced to save power energy. One way to partially overcome these two constraints is by using event-triggered control strategies, as discussed in this thesis.

Heavy-duty vehicle platooning

The formation of a group of heavy-duty vehicles at close intervehicular distances, similar to cyclists in a race, reduces fuel consumption thanks to reduced air resistance. This is a platoon. A vehicle platoon with three vehicles is shown in Figure 1.3. In [12], the authors present an architecture for heavy-duty vehicle platooning to improve the efficiency of the current freight transportation system and experimental results show a significant decrease in fuel and energy consumption.

Vehicle platooning is a formation control problem with input saturation. The desired formation is a line graph. The input saturation follows from that the vehicles have limitations such as maximum acceleration and deceleration. Moreover, continuous communication among vehicles is impossible. One way to model such a system is using event-

Figure 1.3: A platoon of heavy-duty vehicles. *Source:* https://www.scania.com.

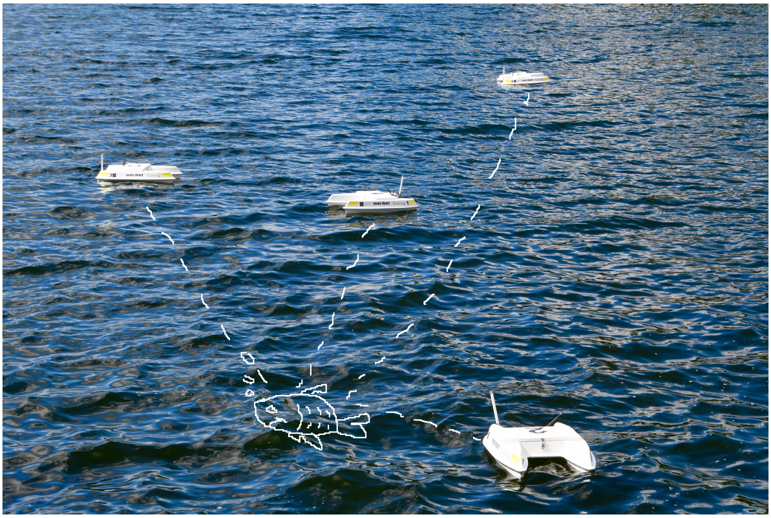


Figure 1.4: Experiments done in Brunnsviken northwest of KTH campus.

triggered multi-agent systems with input saturation.

Autonomous surface vehicle tracking

Autonomous surface vehicles are robotic vehicles that sit on the sea surface and are used for target tracking, environmental sampling, hydrographic or oceanographic surveys, water surface cleaning, etc. One specific example of autonomous surface vehicle tracking is collaborative tracking of fish [13], see Figure 1.4. The autonomous surface vehicles measure the location of the underwater target (the fish) by using sonar. The vehicles create a formation around the target.

Fish tracking is a formation control problem of a resource-constrained multi-agent system. There are several constraints in this system. The first one is that each vehicle is with limited energy since it is battery-powered. Motion and communication consume energy, so it is important to design a proper control law. The second constraint is that the transceiver in each vehicle is simple and have limited communication range. However, the relative distance between any two vehicles may change during operation, in such a way so that the connectivity of the underlying interaction graph cannot be guaranteed. One way to handle these constraints is to consider event-triggered formation control with connectivity preservation using relative positions.

1.2 Resource constraints

A multi-agent control system should be capable of solving missions that are difficult or impossible for an individual agent. Each agent cooperates with other agents and coordinates its action according to the information it gathers. The agent should be able to compute, communicate, sense, and control. Resource constraints are essential for the control design of multi-agent systems as a constrained system can have completely different behavior compared to the unconstrained one. In this section, we will summarize the essential constraints: energy, communication, sensing, and control constraints. For many multi-agent control systems computation constraints are not critical given the development of embedded hardware and software, so we will not discuss them further.

1.2.1 Energy constraints

Multi-agent systems are almost always energy constrained, because agents are usually powered by batteries and battery sizes are limited. For example, for the above autonomous surface vehicle, the size constraint of the vehicles limits the size of its battery and therefore the amount of energy available. For most systems, motion and then communication consume most energy. The motion of an agent is determined by the control input while the communication among agents is realized by wireless transceivers. Thus control and communication resources are limited, and their use should be reduced as much as possible.

Figure 1.5 shows the typical power consumption of a wireless sensor, which is representative for a sensing and communication device on an agent. From this figure, we see that broadcasting, receiving, and listening consume most energy. Thus, in order to save energy, the use of broadcasting, receiving, and listening should be avoided as much as possible. Note that the CPU computations use less energy. Energy used for mobility is of course not included in the figure.

1.2.2 Communication constraints

Communication has a great impact on the performance of multi-agent control systems. Individual agents coordinate their actions through message exchanges. One common way in which agents exchange information with other agents is through a digital communication network. Specifically, an agent broadcasts its message at a given instant to other agents, who need to be listening. There are several communication constraints. The first one is that

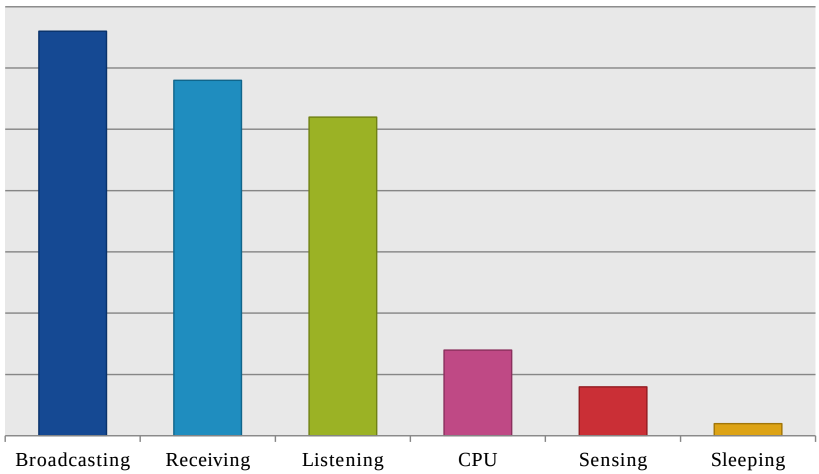


Figure 1.5: Typical power consumption of a wireless sensor [14].

communication equipments have limitations in terms of range, resolution, accuracy, and sensitivity. In this thesis, we consider communication range limitations. The upper bound on the range can often be considered to be fixed, and if the distance between the agents exceed the bound, information cannot be shared among the agents. Hence, only agents within a limited range of each other can exchange information directly. However, agents are often mobile, so the relative distance between any two agents changes over time. Thus, the interactions among agents change and the connectivity of the underlying graph cannot be guaranteed in general.

Another constraint is on the communication channel capacity. Communication is done using radio over a shared channel. The performance of the radio channel is closely related to quantization errors, time delays, bandwidth constraints, data rate constraints, data packet dropouts, and noise. Multiple users of the same channel may cause interference. Frequent use of the communication channel can thus result in time delays or dropouts.

1.2.3 Sensing constraints

Sensing constraints affect the performance of multi-agent control systems. An alternative of using communication to exchange information is active sensing. For instance, autonomous surface vehicles can have sensors to measure the relative distances to other vehicles even if they cannot sense their absolute positions. Sensing can be done by sensors. Similar to communication constraints, there are sensing constraints: sensing cannot be done continuously; energy assigned for sensing is limited; every sensor has limitations in terms of range, resolution, accuracy, sensitivity, etc.

1.2.4 Control constraints

Control and actuator constraints have a vital impact on the closed-loop performance of multi-agent systems. In additional to energy constraints, there are mainly two other control constraints. The first one is that actuators cannot be updated continuously. It is sometimes beneficial for the actuator if the updating frequency is decreased as small as possible. The other constraint is that, in almost all physical applications, actuators have bounded input and output. For instance, vehicles have limitations on mobility, i.e., maximal allowable speed and acceleration. Thus the control input cannot be arbitrary.

1.3 Mathematical modelling

In this section, we will illustrate how to mathematically model multi-agent systems under above resource constraints.

1.3.1 Multi-agent systems under communication constraints

This thesis focus on a special class of multi-agent control systems (1.1)–(1.2). Consider the systems given by integral dynamics

$$\dot{x}_i(t) = u_i(t),\ i \in \mathcal{I},\ t \geq t_0, \tag{1.3}$$

$$u_i(t) = -\sum_{j\in\mathcal{I}} L_{ij} x_j(t), \tag{1.4}$$

where t_0 is a common initial time and L_{ij} is the element of the Laplacian matrix of the underlying graph $\mathcal{G}$. Such a system with two agents is illustrated by Figure 1.6. Each agent has a sensor component to measure and broadcast its state information, and to listen to and receive its neighbor's state information. Each agent also has a control component to generate the control input based on the information it receives from the sensor.

To implement the control (1.4), continuous-time state information from neighbors is needed. In other words, each agent i has to continuously broadcast its own state $x_i(t)$, and continuously listen to and receive its neighbors' states $x_j(t),\ j \in \mathcal{N}_i$. Moreover, each agent i has to continuously update its control input $u_i(t) = \sum_{j\in\mathcal{I}} L_{ij} x_j(t)$. It is impractical to require continuous communication and updating of control input in most real applications.

Reducing the frequency of information exchange among agents is essential to avoid continuous communication and control. In order to realize this, we introduce a model where each agent $i \in \mathcal{I}$ prefers to only broadcast its state at discrete time instants $\{t^i_1, t^i_2, \dots\}$. In this case, the state information received by agent i is $\{x_j(t^j_k),\ j \in \mathcal{N}_i\}_{k=1}^{\infty}$. In other words, at any time instant t, agent i knows $x_j(t^j_{k_j(t)}),\ j \in \mathcal{N}_i$, where $t^j_{k_j(t)} = \max\{t^j_k :\ t^j_k \leq t\}$ is the latest broadcasting time of agent j. Then, the control input is

$$u_i(t) = -\sum_{j\in\mathcal{I}} L_{ij} x_j(t^j_{k_j(t)}). \tag{1.5}$$

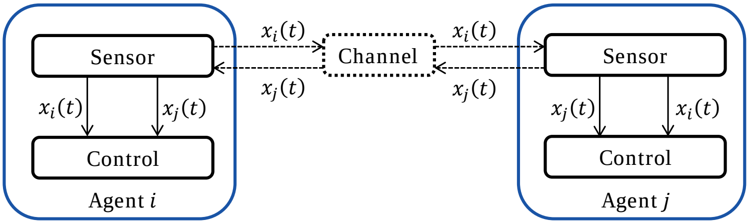


Figure 1.6: Illustration of how agents communicate when the control input has the form (1.4).

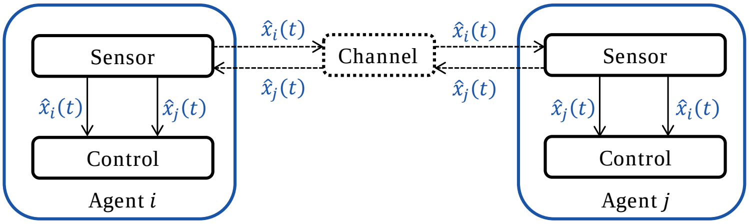


Figure 1.7: Illustration of how agents communicate when the control input has the form (1.5).

For simplicity, let $\hat{x}_i(t) = x_i(t^i_{k_i(t)})$. Figure 1.7 shows that agent i broadcasts its state $x_i(t^i_k)$ at time instants $\{t^i_k\}_{k=1}^{\infty}$ and receives its neighbors' states $\{x_j(t^j_k),\ j \in \mathcal{N}_i\}$ at time instants $\{t^j_k,\ j \in \mathcal{N}_i\}_{k=1}^{\infty}$. By comparing Figures 1.6 and 1.7, we note that in the later every agent does not need to continuously broadcast or receive state information. An essential question is now how to determine the communication instances $\{t^i_k,\ i \in \mathcal{I}\}_{k=1}^{\infty}$ such that desired properties are maintained. In the literature, researchers often consider three kinds of approaches: time-triggered, event-triggered, and self-triggered communication. We discuss each one of them next.

Time-triggered communication

The traditional way for agents to share information is to communicate equidistantly (periodically), i.e.,

$$t^i_1 = t_0,\ t^i_{k+1} = t^i_k + T,\ i \in \mathcal{I},$$

where $T > 0$ is the sampling period. This is called time-triggered approach or periodic sampling. Note that the triggering sequence is equal for each agent. A nice feature of this approach is that analysis and design becomes rather straightforward and the vast literature on sample-data control can be used [15]. Drawbacks are that agents need to take action in a synchronous manner (which is often hard to implement) and it is not energy-efficient to communicate even if the state has not changed at all, for instance.

Event-triggered communication

To make the sampling period T adaptive, we can let communication occur only when a predefined condition is satisfied. This is called event-triggered. Triggering times $\{t_1^i, t_2^i, \dots\}$ is in this case different for different agents. We call $\{t_{k+1}^i - t_k^i\}_{k=1}^{\infty}$ the inter-event times of agent i. The advantages of event-triggered approaches are that they can be implemented in a distributed way and can even give better performance than periodic sampling. However, the design and analysis is less developed.

One common form of event-triggered communication is to use a triggering law defined by

$$t_1^i = t_0,\ t_{k+1}^i = \min\Big\{t:\ F_i\Big(x_i(t), \hat{x}_i(t), \big\{x_j(t), \hat{x}_j(t)\big\}_{j\in N_i}\Big) \geq 0,\ t \geq t_k^i\Big\},\ i \in \mathcal{I}, \tag{1.6}$$

where $F_i(\cdot)$ is a function to be designed. We call (1.6) a static triggering law since it does not involve any extra dynamic variables. There are two well known ways to define the function $F_i(\cdot)$. The first one was introduced in [16]:

$$F_i(\cdot) = (\hat{x}_i(t) - x_i(t))^2 - \frac{\sigma_i a(1 - a|\mathcal{N}_i|)}{|\mathcal{N}_i|}\Big(\sum_{j=1}^{n}(x_j(t) - x_i(t))\Big)^2, \tag{1.7}$$

and the second one in [17]:

$$F_i(\cdot) = (\hat{x}_i(t) - x_i(t))^2 - \frac{\sigma_i a(1 - a|\mathcal{N}_i|)}{|\mathcal{N}_i|}\Big(\sum_{j=1}^{n}(\hat{x}_j(t) - \hat{x}_i(t))\Big)^2, \tag{1.8}$$

where $0 < \sigma_i < 1$ and $0 < a < \frac{1}{|\mathcal{N}_i|}$ are design parameters. It is straightforward to see that the function $F_i(\cdot)$ in (1.7) or (1.8) does not involve any extra dynamic variables but the agent state variables $x_i(t), \hat{x}_i(t)$ and $x_j(t)$, $j \in \mathcal{N}_i$.

Another common form of event-triggered communication is

$$t_1^i = t_0,\ t_{k+1}^i = \min\Big\{t:\ F_i\Big(x_i(t), \hat{x}_i(t), \big\{x_j(t), \hat{x}_j(t)\big\}_{j\in N_i}\Big) \geq \eta_i(t),\ t \geq t_k^i\Big\},\ i \in \mathcal{I}, \tag{1.9}$$

where $\eta_i(t)$ is an internal dynamic variable to be defined. We call (1.9) a dynamic triggering law since it involves an extra dynamic variable. One well known dynamic triggering law introduced in [18] is

$$t_1^i = t_0,\ t_{k+1}^i = \min\Big\{t:\ |\hat{x}_i(t) - x_i(t)| \geq c_0 + c_1 e^{-\alpha t},\ t \geq t_k^i\Big\},\ i \in \mathcal{I}, \tag{1.10}$$

where constants $c_0 \geq 0, c_1 \geq 0,\ c_0 + c_1 > 0$, and $0 < \alpha < \rho_2(L)$. Here $\rho_2(L)$ is the minimum positive eigenvalue of the Laplacian matrix L of the undirected underlying graph $\mathcal{G}$.

A key challenge in event-triggered multi-agent control systems is to exclude Zeno behavior when designing the triggering laws. Zeno is the behavior that there are infinite number of triggers in a finite time interval [19], i.e., that for some i

$$\lim_{k\to+\infty} t_k^i < \infty. \tag{1.11}$$

Self-triggered communication

For event-triggered communication, each agent needs to continuously monitor the triggering laws. However, an agent i could instead at its current triggering time t_k^i predict its next triggering time t_{k+1}^i and broadcast it to its neighbors. In this case, agent i only needs to listen and receive information at $\{t_k^j\}_{k=1}^{\infty}$, $j \in \mathcal{N}_i$ since it knows when these time instances will happen in advance. Each agent broadcasts at its own triggering times, and listen to incoming information from its neighbors at their triggering times. This is called self-triggered. It should be highlighted that it is at the current triggering time instant that next triggering time is determined.

One common form of self-triggered communication is to use a triggering law defined by

$$t_1^i = t_0,\ t_{k+1}^i = \min\Big\{t:\ G_i\Big(t, x_i(t_k^i), t_k^i, \Big\{t_{k_j(t_k^i)}^j, t_{k_j(t_k^i)+1}^j, x_j(t_{k_j(t_k^i)}^j)\Big\}_{j\in N_i}\Big) = 0,\ t \geq t_k^i\Big\},\ i \in \mathcal{I}, \tag{1.12}$$

where $G_i(\cdot)$ is a function to be designed. It can often be chosen related to the function $F_i(\cdot)$ in the event-triggered communication.

1.3.2 Multi-agent systems under sensing constraints

When computing control input (1.4) or (1.5), absolute state information seems to be needed. In some real applications, accurate absolute state information is not available, but relative state information is . We modify the control input (1.4) and (1.5) as

$$u_i(t) = -\sum_{j\in\mathcal{I}} L_{ij}(x_j(t) - x_i(t)), \tag{1.13}$$

and

$$u_i(t) = -\sum_{j\in\mathcal{I}} L_{ij}(x_j(t_k^i) - x_i(t_k^i)),\ t \in [t_k^i, t_{k+1}^i), \tag{1.14}$$

respectively. Figure 1.8 (a) shows that each agent continuously sense the relative state information between itself and its neighbors and use such information to generate its control input; Figure 1.8 (b) shows a similar process except that each agent only senses the relative state information at discrete time instants $\{t_1^i, t_2^i, \dots\}$.

Now, let us compare (1.4) and (1.13). From the property of Laplacian matrices, we know that $\sum_{j\in\mathcal{I}} L_{ij} = 0$. Thus $-\sum_{j\in\mathcal{I}} L_{ij}x_j(t) = -\sum_{j\in\mathcal{I}} L_{ij}(x_j(t) - x_i(t))$. This is to say that the control input (1.4) is same as the control input (1.13). The only difference between (1.4) and (1.13) is that the former seems to use absolute information and the later uses relative information.

Similarly, let us compare (1.5) and (1.14). The control input (1.14) is constant during each interval $[t_k^i, t_{k+1}^i)$. In other words, the control input (1.14) of each agent is not affected by its neighbors during $[t_k^i, t_{k+1}^i)$. On the contrary, the control input (1.5) is not necessarily

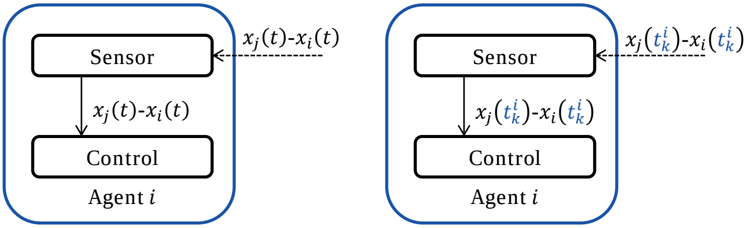


Figure 1.8: Illustration of how one agent gather information. (a) The case that the control input has the form (1.13). (b) The case that the control input has the form (1.14).

a constant during $[t_k^i, t_{k+1}^i)$ since $x_j(t^j_{k_j(t)})$ normally is not a constant for all $t \in [t_k^i, t_{k+1}^i)$. In other words, the control input (1.5) of each agent is affected by its neighbors during each interval $[t_k^i, t_{k+1}^i)$. Another difference between (1.14) and (1.5) is that the (weighted) summation of the control input (1.5) is zero, which does not present in (1.14).

1.3.3 Multi-agent systems under control constraints

The considered control constraints correspond to the following multi-agent system with input saturation

$$\dot{x}_i(t) = \mathrm{sat}_h(u_i(t)),\ i \in \mathcal{I},\ t \geq t_0, \tag{1.15}$$

where $\mathrm{sat}_h(\cdot)$ is the saturation function defined (with slight abuse of notation) as

$$\mathrm{sat}_h(s) = [\mathrm{sat}_h(s_1), \dots, \mathrm{sat}_h(s_p)]^\top, \tag{1.16}$$

where $s = [s_1, \dots, s_p]^\top \in \mathbb{R}^p$ with $p > 0$ and

$$\mathrm{sat}_h(s_i) = \begin{cases} h, & \text{if } s_i \geq h \\ s_i, & \text{if } |s_i| < h \\ -h, & \text{if } s_i \leq -h, \end{cases}$$

with h a positive constant referred to as the saturation level. A multi-agent system described by (1.15) and (1.4) is illustrated in Figure 1.9.

1.4 Problem formulation

In this thesis, we solve consensus and formation problems for resource-constrained multi-agent control systems using dynamic event-triggered control strategies. Each problem is specified below

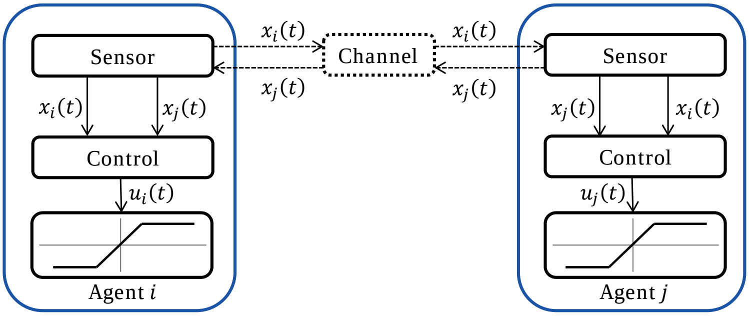


Figure 1.9: Illustration of how one agent communicates with another agent when the control input has the form (1.4). And the control signal is saturated before it is transmitted to the actuator.

Dynamic event-triggered control of multi-agent systems

We first consider multi-agent system (1.3) with event-triggered control input (1.5) over undirected graphs. The problem to solve is to (distributively) determine the triggering times such that average consensus is reached, while continuous exchange of information, continuous update of actuators, and Zeno behavior are avoided.

Multi-agent systems with input saturations

The second problem to study is on the multi-agent system with input saturation (1.15) over directed graphs. Here we consider the problem to find sufficient and necessary connectivity conditions to guarantee that consensus is reached. Again, under the assumption that there are no continuous communication or system updates.

Event-triggered formation control connectivity preservation

The third problem to investigate is on both first-order multi-agent system (1.3) and second-order

$$\begin{cases} \dot{x}_i(t) = r_i(t), \\ \dot{r}_i(t) = u_i(t), \ i \in \mathcal{I}, \ t \geq t_0. \end{cases}$$

We assume all agents have the communication radius $\Delta > 0$. The problem to solve is to propose distributed event-triggered control together with triggering laws such that a desired formation is achieved while connectivity is preserved.

Table 1.1: Formation control principles [20].

	Position-based	Displacement-based	Distance-based
Sensors	Positions	Relative positions	Relative positions
Controls	Positions	Relative positions	Inter-agent distances
Coordinates	Global coordinate system	Orientation aligned local coordinate systems	Local coordinate systems
Interactions	Usually not required	Existence of a spanning tree	Rigidity or persistence

1.5 Related work

In this section, we review the literature about formation control, consensus control, connectivity preservation, saturation constraints, and event- and self-triggered control of multi-agent systems.

1.5.1 Formation control

Generally speaking, formation control for a multi-agent system is about making the agents move to a desired geometric shape. In the survey paper [20], the authors categorize the existing results on formation control into position-, displacement-, and distance-based control according to types of sensed and controlled variables. In position-based control, agents sense their own positions with respect to a global coordinate system. They actively control their own positions to achieve the desired formation, which is prescribed by desired positions with respect to the global coordinate system. This kind of work is found in [21–24]. In displacement-based control, agents actively control displacements of their neighboring agents to achieve the desired formation, which is specified by the desired displacements with respect to a global coordinate system under the assumption that each agent is able to sense relative positions to its neighboring agents with respect to the global coordinate system. This implies that the agents need to know the orientation of the global coordinate system. However, the agents require neither knowledge on the global coordinate system itself nor their positions with respect to the coordinate system. This kind of work is found in [25–29]. In distance-based control, inter-agent distances are actively controlled to achieve the desired formation, which is given by the desired inter-agent distances. Individual agents are assumed to be able to sense relative positions to their neighboring agents with respect to their own local coordinate systems. The orientations of local coordinate systems are not necessarily aligned with each other. This kind of work is found in [30–33].

In [20], the authors summarize these formation control principles as in Table 1.1.

1.5.2 Consensus

The consensus problem has a long history in computer science, particular in distributed computing [34]. For multi-agent control systems, consensus means that the group of agents reach an agreement upon a certain quantity of interest that may depend on the initial states of all agents. In the study of complex networks, the synchronization has sometime a similar meaning as consensus.

There is a huge amount of research work on consensus or synchronization in the past decades. Here we only recall some of them. In [35–39], the authors introduce a theoretical framework for analysis of consensus for first-order linear multi-agent systems with an emphasis on the role of directed information flow, robustness to changes in network topology due to link/node failures, time-delays, and performance guarantees. One fundamental result is that the performance of the consensus protocol is determined by the algebraic connectivity. Consensus is achieved if and only if the underlying fixed undirected graph is connected or directed graph has a directed spanning tree [35–37]. In [40], the authors study general linear multi-agent systems with directed communication graphs. Similar work can be found in earlier papers [41, 42], in which the authors present a framework for analyzing synchronization of linearly coupled ordinary differential equations. In [43], the authors use a high-gain methodology to construct linear decentralized consensus controllers for general linear multi-agent systems with time-invariant and time-varying topologies. In [44], the authors consider consensus for first-order multi-agent systems with stochastically switching topologies modeled as a stochastic process. In [45], the authors study asynchronous consensus problems for continuous-time multi-agent systems with discontinuous information transmission. In [46], the authors investigate the joint effect of agent dynamics, network topologies and communication data rate on the consensus problem. In [47], the authors consider nonlinear consensus protocols.

1.5.3 Connectivity preservation

In the study of distributed coordination, such as consensus and formation control, one vital assumption is that the associated communication graph is connected or has a directed spanning tree, at least in some average sense. However, in realistic applications, it is difficult to guarantee this assumption. For example, in mobile robot networks with limited communication range, connectivity of the initial deployment of the robots do not guarantee connectivity in the future.

Motivated by this, many researchers have studied connectivity preservation for multi-agent systems. In particular, the control should ensure that the associated communication graph remains connected during the evolution of the system. For instance, in [48], the authors present a geometric analysis of wireless connectivity in vehicle networks. In [49], the authors present a decentralized control strategy that drives a system of multiple nonholonomic kinematic unicycles to agreement and maintains at the same time the connectivity properties of the initially formed communication graph. In [25], the authors design nonlinear control input based on an edge-tension function to solve the formation control problem while ensuring connectedness. In [50], the authors propose a centralized feedback control framework based on artificial potential fields to maintain graph connectivity. In [51], the authors introduce a general class of distributed potential functions guaranteeing connectivity for single-integrator agents. In [52], based on the navigation function formalism, the authors develop a decentralized controller to enable a group of agents to achieve a desired global configuration while maintaining global network connectivity. In [53], the authors provide a decentralized robust control approach, which guarantees that connectivity is maintained when certain bounded input terms are added to the control law.

1.5.4 Saturation constraints

Physical systems are subject to saturation constraints, for examples vehicles have limitations on maximal allowable speed and acceleration. A common saturation constraint is on the actuator, i.e., input saturation. Another example saturation is due to limited sensor capacity. Saturations lead to nonlinearities in the closed-loop dynamics. Thus the behavior of each agent is affected and special attention needs to be taken in order to understand the influence on the system . For example, [54] studies global consensus for discrete-time multi-agent systems with input saturation constraints; [55] considers a leader-following consensus problem for continuous-time multi-agent systems subject to input saturations; both [56] and [57] investigate necessary and sufficient initial conditions for achieving consensus in the presence of output (sensor) saturations; [58] shows that the distributed consensus protocol asymptotically leads to consensus, for multi-agent systems with input saturations and directed topologies and [59] achieves the same result under a more general problem settings.

1.5.5 Event-triggered control of multi-agent systems

Continuous communication cannot usually be implemented in multi-agent systems, since the interactions among agents are typically realized over a digital communication channel with limited capacity. Moreover, in order to simplify and reduce communication, the information exchange should be kept as small as possible. In order to realize this, event-triggered control is introduced in [60–63]. Instead of using the continuous state, the event-triggered control is piecewise constant between any two consecutive triggering times. Many researchers study event-triggered control for multi-agent systems recently [16–18, 64–75]

1.5.6 Self-triggered control of multi-agent systems

To overcome some drawbacks of event-triggered control, for example, continuous monitoring of the triggering laws, self-triggered are proposed for single-agent systems [76–78]. Many researchers investigate self-triggered control for multi-agent systems [16, 65, 66, 70, 72, 73]. For self-triggered single-agent systems the next triggering time is determined at the previous triggering instance. However, self-triggered approaches for multi-agent systems mentioned above are not in accordance with this. Although continuous broadcasting, receiving, and sensing are avoided, continuous listening is still needed since the triggering times are determined during runtime and not known in advance. To overcome this disadvantage, some researchers introduce local clock variables to design the self-triggered policy [79], others combine event-triggered control with periodic sampling [80–83], and some present cloud-supported control algorithm [84–86].

1.6 Thesis outline and contributions

The rest of the thesis is organized as follows.

Chapter 2: Algebraic graph theory

In Chapter 2, we review the key definitions and results from algebraic graph theory that will be used in this thesis.

Chapter 3: Dynamic event-triggered control of multi-agent systems

In Chapter 3, we propose dynamic event-triggered approaches to solve the average consensus problem for first-order continuous-time multi-agent systems over undirected graphs. More specifically, two distributed dynamic triggering laws and one self-triggered algorithm are proposed to determine the triggering times. Compared with existing triggering laws, the proposed triggering laws involve internal dynamic variables which play an essential role to guarantee that the triggering time sequence does not exhibit Zeno behavior. Moreover, our dynamic triggering laws include some existing triggering laws as special cases. More importantly, continuous listening is avoided in our proposed self-triggered algorithm. The main idea is that each agent predicts its next triggering time and broadcasts it to its neighbors at the current triggering time. Thus each agent only needs to sense and broadcast at its triggering times, and to listen to and receive incoming information from its neighbors at their triggering times. It is proven that the proposed laws make the state of each agent converge exponentially to the average of the agents' initial states if and only if the underlying graph is connected.

The covered material is based on the following contributions.

- X. Yi, K. Liu, D. V. Dimarogonas and K. H. Johansson, "Distributed dynamic event-triggered control for multi-agent systems," *in IEEE Conference on Decision and Control*, 2017.
- X. Yi, K. Liu, D. V. Dimarogonas and K. H. Johansson, "Dynamic event-triggered and self-triggered control for multi-agent systems," *in Preparation*.

Chapter 4: Multi-agent systems with input saturation

In Chapter 4, we consider the consensus problem for multi-agent systems with input saturation over directed graphs. It is shown that the underlying directed graph having a directed spanning tree is a necessary and sufficient condition for consensus; thus, this condition for consensus without input saturation extends to the case with saturation constraints. Moreover, in order to reduce the overall need of communication and system updates, we then consider event-triggered control and propose a dynamic triggering law. Furthermore, in order to avoid continuous listening, we also propose a self-triggered algorithm. It is shown that Zeno behavior is excluded for these systems and that consensus is achieved, again, if and only if the underlying directed graph has a directed spanning tree.

The covered material is based on the following contribution.

- X. Yi, T. Yang, J. Wu, and K. H. Johansson, "Distributed event-triggered control for global consensus of multi-agent systems with input saturation," *Submitted to Automatica*.

Chapter 5: Event-triggered formation control with connectivity preservation

In Chapter 5, event-triggered and self-triggered control algorithms are proposed to establish pre-specified formations under connectivity preservation. Each agent only needs to update its control input by sensing the relative state to its neighbors and to broadcast its triggering information at its own triggering times. The agents listen to and receive neighbors' triggering information at their triggering times. Two types of system dynamics, single integrators and double integrators, are considered. It is shown that all agents converge to the pre-specified formation exponentially with connectivity preservation and exclusion of Zeno behavior.

The covered material is based on the following contributions.

- X. Yi, J. Wei, D. V. Dimarogonas, and K. H. Johansson, "Formation control for multi-agent systems with connectivity preservation and event-triggered controllers," *in IFAC World Congress*, 2017.

- X. Yi, J. Wei, D. V. Dimarogonas, and K. H. Johansson, "Event-triggered formation control of multi-agent systems with connectivity preservation," *in Preparation*.

Chapter 6: Conclusions and future research

In Chapter 6, we present a summary of the results, and discuss directions for future research.

Contributions not covered in the thesis

The following publications by the author are not covered in the thesis, but contain related material:

- X. Yi, T. Yang, J. Wu, and K. H. Johansson, "Event-triggered control for multi-agent systems with output saturation," *in Chinese Control Conference*, 2017.

- J. Wei, X. Yi, H. Sandberg, and K. H. Johansson, "Nonlinear consensus protocols with applications to quantized systems," *in IFAC World Congress*, 2017.

- J. Wei, X. Yi, H. Sandberg, and K. H. Johansson, “Nonlinear consensus protocols with applications to quantized communication and actuation,” *Submitted to IEEE Transactions on Control of Network Systems*.
- X. Yi, J. Wei, and K. H. Johansson, “Self-triggered control for multi-agent systems with quantized communication or sensing,” *in IEEE Conference on Decision and Control*, 2016.

Contribution by the Author

The order of authors reflects their contribution to each paper. The first author has the most important contribution. In all the listed publications, all the authors were actively involved in formulating the problems, developing the solutions, evaluating the results, and writing the paper.

Chapter 2

Algebraic graph theory

In this chapter, the key definitions and results from algebraic graph theory are reviewed. See [87] for more detailed definitions.

2.1 Directed graphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ denote a weighted directed graph (digraph) associated with a multi-agent system, where the set of vertices (or nodes) $\mathcal{V} = \{v_1, \ldots, v_n\}$, the set of edges[1] (links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $A = (a_{ij})$ with nonnegative elements a_{ij}. An edge of $\mathcal{G}$ is denoted by $(v_i, v_j) \in \mathcal{E}$ if there is a directed link from agent i to agent j with weight $a_{ji} > 0$, i.e., agent i can send information to agent j. The adjacency elements associated with the edges of the graph are positive, i.e., $(v_i, v_j) \in \mathcal{E}$ if and only if $a_{ji} > 0$. It is assumed that $a_{ii} = 0,\ \forall i \in \mathcal{I}$. The in-degree of agent i is defined as $\deg_i^{\text{in}} = \sum_{j=1}^{n} a_{ij}$. The degree matrix of $\mathcal{G}$ is defined as $\text{Deg} = \text{Diag}([\deg_1^{\text{in}}, \ldots, \deg_n^{\text{in}}])$. The (weighted) Laplacian matrix associated with $\mathcal{G}$ is defined as $L = \text{Deg} - A$. Let $\mathcal{N}_i = \{j \in \mathcal{I} \mid a_{ij} > 0\}$ denotes the in-neighbors of agent i. A directed path from v_i to v_j is a directed subgraph of $\mathcal{G}$ with distinct vertices $v_i, v_{i_1}, \ldots, v_{i_k}, v_j$ and edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{k-1}}, v_{i_k}), (v_{i_k}, v_j)$.

Definition 2.1. *A digraph $\mathcal{G}$ is strongly connected if for any two distinct vertices v_i and v_j, there exits a directed path from v_i to v_j.*

$\mathcal{G}$ is strongly connected is equivalent to L is irreducible. Strong connectivity requires that any vertex is accessible to all other vertices, while the following weaker connectivity condition only requires that one vertex can access all other vertices.

Definition 2.2. *A digraph $\mathcal{G}$ has a directed spanning tree if there exists one vertex such that there exits a directed path from this vertex to any other vertex.*

By Perron-Frobenius Theorem [88], we have the following result (see [42] or [89] for a proof).

[1] We sometimes use $\mathcal{E}(\mathcal{G})$ to highlight that this is the edge set of $\mathcal{G}$.

Lemma 2.1. *If L is the Laplacian matrix associated with a digraph $\mathcal{G}$ that has a directed spanning tree, then rank$(L) = n - 1$, and zero is an algebraically simple eigenvalue of L, and there is a nonnegative vector $\xi = [\xi_1, \ldots, \xi_n]^\top$ such that $\xi^\top L = 0$ and $\sum_{i=1}^n \xi_i = 1$. Moreover, if $\mathcal{G}$ is strongly connected, then $\xi_i > 0$, $\forall i \in \mathcal{I}$.*

The following result from [72] is also useful for our analysis.

Lemma 2.2. *Suppose that L is the Laplacian matrix associated with a digraph $\mathcal{G}$ that is strongly connected and ξ is the vector defined in Lemma 2.1. Let $\Xi = Diag(\xi)$, $U = \Xi - \xi\xi^\top$, and $R = \frac{1}{2}(\Xi L + L^\top \Xi)$. Then $R = \frac{1}{2}(UL + L^\top U)$ and*

$$U \geq \frac{\rho_2(U)}{\rho(L^\top L)} L^\top L \geq 0 \text{ and } R \geq \frac{\rho_2(R)}{\rho(U)} U \geq 0. \tag{2.1}$$

By proper row and column permutations, any Laplacian matrix L can be written in Perron-Frobenius form (see Definition 2.3 in [90]):

$$L = \begin{bmatrix} L^{1,1} & L^{1,2} & \cdots & L^{1,M} \\ 0 & L^{2,2} & \cdots & L^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^{M,M} \end{bmatrix}, \tag{2.2}$$

where $L^{m,m}$ is a n_m-by-n_m matrix and is associated with the m-th strongly connected component (SCC) of $\mathcal{G}$, denoted by SCC_m, $m = 1, \ldots, M$. Hence, a digraph $\mathcal{G}$ is strongly connected if and only if $M = 1$. In the following, without loss of generality, we assume that L has the form (2.2).

SCC_m is called closed if and only if there are no edges from vertices outside SCC_m to vertices inside SCC_m, i.e., $L^{m,q} = 0$, $\forall q > m$. The following result, which follows from Lemma 1 in [91], gives an equivalent description of a digraph that has a directed spanning tree.

Lemma 2.3. *The digraph $\mathcal{G}$ contains a directed spanning tree if and only if for each $m = 1, \ldots, M - 1$, SCC_m is not closed.*

Let us illustrate this construction with an example.

Example 2.1. *Figure 2.1 shows a digraph of 7 vertices having multiple directed spanning trees. For example, one of the directed spanning tree is described by edges $(v_7, v_5), (v_5, v_6), (v_6, v_3), (v_3, v_4),$ The graph can be divided into two strongly connected components, as indicated in the*

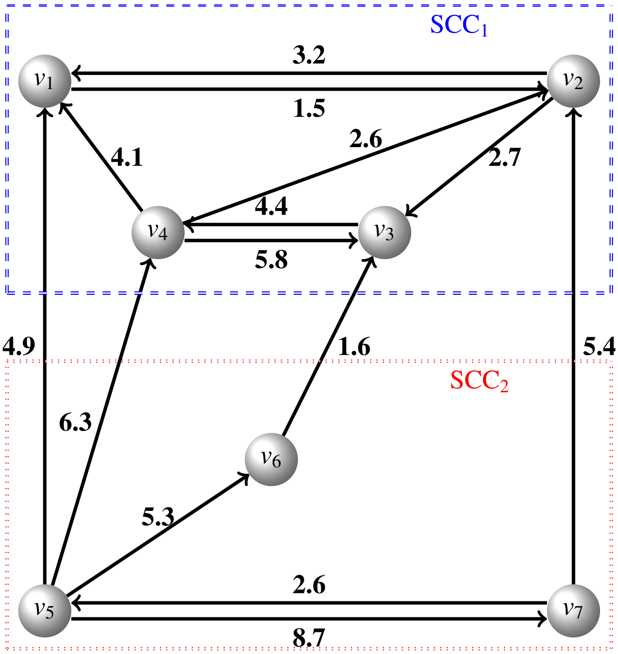


Figure 2.1: An example of a digraph which contains directed spanning trees. The subgraph in the dashed lines is the first strongly connected component, and the subgraph in the dotted lines is the second strongly connected component

figure. The corresponding Laplacian matrix

$$L = \left[\begin{array}{cccc|ccc} 12.2 & -3.2 & 0 & -4.1 & -4.9 & 0 & 0 \\ -1.5 & 9.5 & 0 & -2.6 & 0 & 0 & -5.4 \\ 0 & -2.7 & 10.1 & -5.8 & 0 & -1.6 & 0 \\ 0 & 0 & -4.4 & 10.7 & -6.3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2.6 & 0 & -2.6 \\ 0 & 0 & 0 & 0 & -5.3 & 5.3 & 0 \\ 0 & 0 & 0 & 0 & -8.7 & -7 & 15.7 \end{array}\right],$$

has the form (2.2).

For SCC$_m$ with $m < M$, define an auxiliary matrix $\tilde{L}^{m,m} = [\tilde{L}^{m,m}_{ij}]^{n_m}_{i,j=1}$ as

$$\tilde{L}^{m,m}_{ij} = \begin{cases} L^{m,m}_{ij} & i \neq j, \\ -\sum_{r=1,r\neq i}^{n_m} L^{m,m}_{ir} & i = j. \end{cases}$$

Example 2.2. *In Example 2.1,*

$$\tilde{L}^{1,1} = \begin{bmatrix} 7.3 & -3.2 & 0 & -4.1 \\ -1.5 & 4.1 & 0 & -2.6 \\ 0 & -2.7 & 8.5 & -5.8 \\ 0 & 0 & -4.4 & 4.4 \end{bmatrix}.$$

Similar to Lemma 2.2, we have the following lemma.

Lemma 2.4. *Let $\xi^m = [\xi_1^m, \ldots, \xi_{n_m}^m]^\top$ be the positive left eigenvector of the irreducible $\tilde{L}^{m,m}$ corresponding to the eigenvalue zero and the sum of its components is* 1. *Denote $\Xi^m = Diag(\xi^m)$, $Q^m = \frac{1}{2}[\Xi^m L^{m,m} + (\Xi^m L^{m,m})^\top]$, $m = 1, \ldots, M$, and $U^M = \Xi^M - \xi^M(\xi^M)^\top$. Then*

$$Q^m > 0,\ m = 1, \ldots, M-1,\ Q^M \geq 0,\ U^M \geq 0,\ \textit{and}\ Q^M \geq \frac{\rho_2(Q^M)}{\rho(U^M)} U^M. \tag{2.3}$$

Proof. For the proof of $Q^m > 0$ for all $m < M$, see Lemma 3.1 in [92].
$Q^M \geq 0$ is straightforward since we can regard Q^M as the Laplacian matrix of a connected undirected graph.
$U^M \geq 0$ is also straightforward since we can regard U^M as the Laplacian matrix of a complete graph.
The idea of the proof of $Q^M \geq \frac{\rho_2(Q^M)}{\rho(U^M)} U^M$ follows a similar trend as the proof of (2.1), and it can be found in [72]. We thus omit the proof here. □

Let n_e denotes the number of edges in $\mathcal{G}$, i.e., $n_e = |\mathcal{E}(\mathcal{G})|$ and label the edges in $\mathcal{G}$ as $e_1, \ldots, e_{n_e}$. Define $W = \text{Diag}([\omega(e_1), \cdots, \omega(e_{n_e})])$, where $\omega(e_k) = a_{ij}$ with e_k being the label of edge (v_i, v_j).

Definition 2.3. *The n-by-n_e incidence matrix $B(\mathcal{G}) = (B_{ij})$ is defined as*

$$\text{B}_{ij} = \begin{cases} -1 & \textit{if vertex } v_i \textit{ is the tail of edge } e_j, \\ 1 & \textit{if vertex } v_i \textit{ is the head of edge } e_j, \\ 0 & \textit{otherwise.} \end{cases}$$

2.2 Undirected graphs

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is undirected if $A = A^\top$. In an undirected graph, a path of length k between vertex v_i and vertex v_j is a subgraph with distinct vertices $v_{i_0} = v_i, \ldots, v_{i_k} = v_j \in \mathcal{V}$ and edges $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}$, $j = 0, \ldots, k-1$.

Definition 2.4. *An undirected graph is connected if there exists at least one path between any two vertices. And an undirected graph is complete if any two distinct vertices are connected by an edge.*

Similar to the definition of SCC in digraphs, by proper row and column permutations, we can rewrite any Laplacian matrix L associated with undirected graphs in the following form

$$L = \begin{bmatrix} L^{1,1} & 0 & \cdots & 0 \\ 0 & L^{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^{M,M} \end{bmatrix}, \tag{2.4}$$

where $L^{m,m}$ is a n_m-by-n_m matrix and is associated with the m-th connected component (CC) of $\mathcal{G}$, denoted by CC_m, $m = 1, \ldots, M$. Hence, a disconnected graph has more than one CC and $L^{m,m}$ is the Laplacian matrix of CC_m.

Obviously, there is a one-to-one correspondence between a graph and its adjacency matrix or its Laplacian matrix. If we let $K_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$, then we can treat K_n as the Laplacian matrix of a complete graph with n vertices and edge weight $\frac{1}{n}$.

For a connected graph we have the following well known results.

Lemma 2.5. *([93]) If an undirected graph $\mathcal{G}$ is connected, then its Laplacian matrix L is positive semi-definite, i.e., $z^\top Lz \geq 0$ for any $z \in \mathbb{R}^n$. Moreover, $z^\top Lz = 0$ if and only if $z = a\mathbf{1}_n$ for some $a \in \mathbb{R}$.*

For undirected graphs, the incidence matrix can be defined after arbitrarily assigning a direction to each edge. The following results are also useful for our analysis.

Lemma 2.6. *([87]) For any undirected graph $\mathcal{G}$, $B(\mathcal{G})B(\mathcal{G})^\top$ is independent of the labels and orientations given to $\mathcal{G}$, and $B(\mathcal{G})WB(\mathcal{G})^\top = L$.*

Example 2.3. *Figure 2.2 (a) shows an undirected graph $\mathcal{G}$ and Figure 2.2 (b) shows an example of assigning a direction to each edge of $\mathcal{G}$. Then*

$$L = \begin{bmatrix} 3.4 & -3.4 & 0 & 0 \\ -3.4 & 9.8 & -2.1 & -4.3 \\ 0 & -2.1 & 3.2 & -1.1 \\ 0 & -4.3 & -1.1 & 5.4 \end{bmatrix}, \quad B(\mathcal{G}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix},$$

$$W = \begin{bmatrix} 3.4 & 0 & 0 & 0 & 0 \\ 0 & 2.1 & 0 & 0 & 0 \\ 0 & 0 & 1.1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4.3 \end{bmatrix}.$$

And one can easily verify that $B(\mathcal{G})WB(\mathcal{G})^\top = L$.

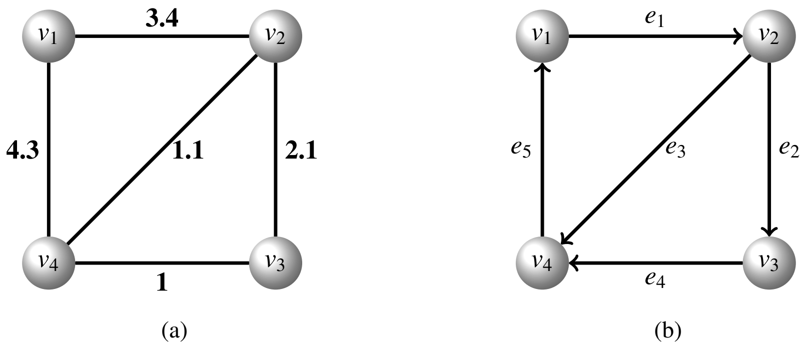


Figure 2.2: (a) An example of an undirected graph $\mathcal{G}$. (b) An example of assigning a direction to each edge of $\mathcal{G}$.

Lemma 2.7. *Assume $\mathcal{G}$ is undirected and connected, then $K_nL = L$, $\rho_2(B(\mathcal{G})B(\mathcal{G})^\top) > 0$, $\rho(K_n) = 1$ and*

$$0 \leq \rho_2(B(\mathcal{G})B(\mathcal{G})^\top)K_n \leq B(\mathcal{G})B(\mathcal{G})^\top,\ 0 \leq \rho_2(L)K_n \leq L. \tag{2.5}$$

Proof. $K_nL = L$ is straightforward. $\rho_2(B(\mathcal{G})B(\mathcal{G})^\top) > 0$ is also straightforward since $\mathcal{G}$ is connected.
From Geršgorin Disc Theorem [88], we know that $\rho(K_n) \leq 1$. From $\det(K_n - I_n) = 0$, we know that 1 is an eigenvalue of K_n. Thus $\rho(K_n) = 1$.
The proof of the rest results is similar to the proof of Lemma 2.4. We thus omit the proof here. □

Chapter 3

Dynamic event-triggered control of multi-agent systems

In this chapter, we consider the average consensus problem for the first-order continuous-time multi-agent systems over undirected graphs. In order to avoid continuous communication between agents and system updates, we use event-triggered control input. We propose two distributed dynamic triggering laws and one self-triggered algorithm to design the triggering times. The idea behind these approaches will also play an important role in the following chapters.

In [94], by introducing an internal dynamic variable, a new class of event-triggered mechanisms is presented and it is extended to discrete-time setting in [95]. The idea of using internal dynamic variables in event-triggered and self-triggered control can also be found in [79, 83, 96–98]. In this chapter, we modify the dynamic event triggering mechanism in [94] and extend it to multi-agent systems in a distributed manner. We propose two dynamic triggering laws which are distributed in the sense that they do not require any a priori knowledge of global network parameters, and we prove that our proposed dynamic triggering laws yield consensus exponentially fast, and we show that they are free from Zeno behavior. We show also that the triggering laws in [16–18] are special cases of our dynamic triggering laws. The main disadvantage of our dynamic triggering laws is that continuous sensing and listening are still needed. To overcome this, we then propose one self-triggered algorithm. The idea of avoiding continuous sensing in our presented self-triggered algorithm is by simple calculation since the control input is piece-wise constant. The main idea of avoiding continuous listening is that each agent predicts (determines) its next triggering time and broadcasts it to its neighbors at the current triggering time. Thus each agent knows its neighbors' next triggering time in advance. As a result, each agent only needs to sense its state information and broadcast its triggering information at its triggering times, and to listen to and receive incoming information from its neighbors at their triggering times. This is to say that, in terms of avoiding continuous listening, our self-triggered algorithm improves the self-triggered algorithms in [16, 65, 66, 70, 72] and other papers using a similar approach. Although continuous sensing, broadcasting, listening, and receiving are also avoided in [80–83] by combining event-triggered control with periodic sampling, periodic sensing and listening are still needed. Moreover, it is not clear how to show that the average inter-event time is strictly larger than the required sampling period in theory. The presented self-triggered algorithm is reminiscent of the event-triggered cloud access in [84–86]. The main difference between this paper and the cloud access mentioned above is that we do not need the cloud to store data. Moreover, we use different technical methods.

The rest of this chapter is organized as follows. Section 3.1 reviews the average consensus problem for the first-order continuous-time multi-agent systems with event-triggered control input. Section 3.2 presents two distributed dynamic triggering laws to determine triggering times such that the average consensus is achieved exponentially. A self-triggered algorithm to solve the aforementioned problem is presented in Section 3.3. Simulations are given in Section 3.4. Finally, the chapter is concluded in Section 3.5.

3.1 Problem formulation

We consider a set of n agents modelled as single integrators

$$\dot{x}_i(t) = u_i(t),\ i \in \mathcal{I},\ t \geq 0, \tag{3.1}$$

where $x_i(t) \in \mathbb{R}$ is the state and $u_i(t) \in \mathbb{R}$ is the control input.

Remark 3.1. *For the ease of presentation, we study the case where all the agents have scalar states, i.e., $x_i \in \mathbb{R}$. However, the analysis in this chapter is also valid for the cases where the agents have vector-valued states, i.e., $x_i \in \mathbb{R}^p$.*

Definition 3.1. *The average consensus for the multi-agent system (3.1) is achieved if $\lim_{t\to\infty} x_i(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(0),\ \forall i \in \mathcal{I}$.*

The classic distributed consensus protocol is given by [38, 39],

$$u_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t),$$

where L_{ij} is the element of the Laplacian matrix L. In this chapter, we assume that the underlying graph $\mathcal{G}$ is undirected.

To implement the above consensus protocol, a continuous exchange of information among agents and a continuous update of actuators are needed. However, it is often impractical to require continuous communication and update in real applications.

Inspired by the idea of event-triggered control for multi-agent systems [16], we use the following event-triggered control input

$$u_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t^j_{k_j(t)}). \tag{3.2}$$

Note that the event-triggered control input (3.2) only updates at the triggering times and it remains constant between any two consecutive triggering times.

Our goal in this chapter is to solve the following problem.

Problem 3.1. *Propose methods to determine the triggering times such that average consensus is reached, while continuous exchange of information, continuous update of actuators, and Zeno behavior are avoided.*

For simplicity, let $x(t) = [x_1(t), \dots, x_n(t)]^\top$, $\hat{x}_i(t) = x_i(t^i_{k_i(t)})$, $\hat{x}(t) = [\hat{x}_1(t), \dots, \hat{x}_n(t)]^\top$, $e_i(t) = \hat{x}_i(t) - x_i(t)$, and $e(t) = [e_1(t), \cdots, e_n(t)]^\top = \hat{x}(t) - x(t)$. Then we can rewrite the multi-agent system with agent dynamics as in (3.1) and event-triggered control input as in (3.2) in the following stack vector form:

$$\dot{x}(t) = -L\hat{x}(t) = -L(x(t) + e(t)).$$

3.2 Dynamic triggering laws

In this section, we propose two distributed dynamic triggering laws to design the triggering times such that the average consensus can be achieved.

3.2.1 Continuous approach

We first show that the average state in (3.1) is constant.

Lemma 3.1. *Consider the multi-agent system (3.1)–(3.2), and assume that the underlying graph $\mathcal{G}$ is undirected. The average of all agents' states $\bar{x}(t) = \frac{1}{n}\sum_{i=1}^n x_i(t)$ is constant, i.e., $\bar{x}(t) = \bar{x}(0),\ \forall t \geq 0$.*

Proof. It follows from (3.1) and (3.2) that the time derivative of the average value is given by

$$\dot{\bar{x}}(t) = \frac{1}{n}\sum_{i=1}^n \dot{x}_i(t) = -\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n L_{ij}x_j(t^j_{k_j(t)}) = -\frac{1}{n}\sum_{j=1}^n x_j(t^j_{k_j(t)})\sum_{i=1}^n L_{ij} = 0.$$

Thus $\bar{x}(t)$ is constant. □

Now, consider a Lyapunov candidate as follows

$$\begin{aligned} V(x(t)) =& \frac{1}{2}x^\top(t)K_n x(t) = \frac{1}{2}x^\top(t)[I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top]x(t) \\ =& \frac{1}{2}\sum_{i=1}^n x_i^2(t) - \frac{n}{2}\bar{x}^2(0) = \frac{1}{2}\sum_{i=1}^n [x_i(t) - \bar{x}(0)]^2. \end{aligned} \tag{3.3}$$

Then the derivative of $V(x(t))$ along the trajectories of the multi-agent system (3.1)–(3.2) satisfies

$$\dot{V}(x(t)) = \sum_{i=1}^n [x_i(t) - \bar{x}(0)]\dot{x}_i(t) = \sum_{i=1}^n x_i(t)\dot{x}_i(t) - \bar{x}(0)\sum_{i=1}^n \dot{x}_i(t) = \sum_{i=1}^n x_i(t)\dot{x}_i(t)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} (-L_{ij} x_j(t^j_{k_j(t)})) = -\sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} L_{ij}(x_j(t) + e_j(t)) \\
&\overset{*}{=} -\sum_{i=1}^{n} q_i(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_i(t) L_{ij} e_j(t) = -\sum_{i=1}^{n} q_i(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} e_i(t) L_{ij} x_j(t) \\
&= -\sum_{i=1}^{n} q_i(t) - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} e_i(t) L_{ij}(x_j(t) - x_i(t)) \\
&\leq -\sum_{i=1}^{n} q_i(t) - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} L_{ij} e_i^2(t) - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} L_{ij} \frac{1}{4}(x_j(t) - x_i(t))^2 \\
&= -\sum_{i=1}^{n} q_i(t) + \sum_{i=1}^{n} L_{ii} e_i^2(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{4} L_{ij}(x_j(t) - x_i(t))^2 \\
&\overset{*}{=} -\sum_{i=1}^{n} \frac{1}{2} q_i(t) + \sum_{i=1}^{n} L_{ii} e_i^2(t), \qquad (3.4)
\end{aligned}
$$

where

$$
q_i(t) = -\frac{1}{2} \sum_{j=1}^{n} L_{ij}(x_j(t) - x_i(t))^2 \geq 0, \qquad (3.5)
$$

and the equalities denoted by $\overset{*}{=}$ hold since

$$
\sum_{i=1}^{n} q_i(t) = -\sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{n} L_{ij}(x_j(t) - x_i(t))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i(t) L_{ij} x_j(t) = x^\top(t) L x(t),
$$

and the inequality holds since $ab \leq a^2 + \frac{1}{4} b^2$.

Similar to [16] and [65], the following law can be used to determine the triggering times:

$$
t^i_1 = 0,\ t^i_{k+1} = \min\left\{t :\ L_{ii} e_i^2(t) - \frac{\sigma_i}{2} q_i(t) \geq 0,\ t \geq t^i_k\right\},\ k = 1, 2, \ldots \qquad (3.6)
$$

with $\sigma_i \in (0, 1)$. From the way to determine the triggering times by (3.6), we have

$$
L_{ii} e_i^2(t) \leq \frac{\sigma_i}{2} q_i(t),\ \forall t \geq 0. \qquad (3.7)
$$

Then, from (3.4) and (3.7), we have

$$
\begin{aligned}
\dot{V}(x(t)) &\leq -\sum_{i=1}^{n} \frac{1}{2} q_i(t) + \sum_{i=1}^{n} L_{ii} e_i^2(t) \leq -\frac{1}{2}(1 - \sigma_{\max}) \sum_{i=1}^{n} q_i(t) = -\frac{1}{2}(1 - \sigma_{\max}) x^\top(t) L x(t) \\
&\leq -\frac{1}{2}(1 - \sigma_{\max}) \rho_2(L) x^\top(t) K_n x(t) = -(1 - \sigma_{\max}) \rho_2(L) V(x(t)), \qquad (3.8)
\end{aligned}
$$

where $\sigma_{\max} = \max\{\sigma_1, \ldots, \sigma_n\} < 1$ and the last inequality holds due to (2.5). Then

$$V(x(t)) \leq V(x(0))e^{-(1-\sigma_{\max})\rho_2(L)t}. \tag{3.9}$$

This implies that system (3.1)–(3.2) reaches average consensus exponentially if the underlying graph $\mathcal{G}$ is connected.

Remark 3.2. *(3.6) is a static triggering law since it does not involve any extra dynamic variables but the agent state variables $x_i(t), \hat{x}_i(t)$ and $x_j(t), j \in \mathcal{N}_i$. The static triggering law (3.6) is distributed since each agent's control action only depends on its own state information and its neighbors' state information, without any a prior knowledge of any global parameters, such as the eigenvalues of the Laplacian matrix.*

Remark 3.3. *If we consider the same graph as in [16], i.e., $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$, then $L_{ii} = |\mathcal{N}_i|$. Since $a(1-a|\mathcal{N}_i|) \leq \frac{1}{4|\mathcal{N}_i|}$ and $(\sum_{j=1}^n (x_j(t)-x_i(t)))^2 \leq 2|\mathcal{N}_i| \sum_{j=1}^n (x_j(t)-x_i(t))^2$, we have $\frac{\sigma_i a(1-a|\mathcal{N}_i|)}{|\mathcal{N}_i|}(\sum_{j=1}^n (x_j(t) - x_i(t)))^2 \leq \frac{\sigma_i}{2|\mathcal{N}_i|} q_i(t)$. In other words, the distributed triggering law (10) in [16] is a special case of the static triggering law (3.6).*

The main purpose of using event-triggered control is to reduce the overall need of actuation updates and communication between agents, so it is essential to exclude Zeno behavior. However, as stated in [16], Zeno behavior may not be excluded under (3.6). In order to explicitly exclude Zeno behavior, in the following we propose a dynamic triggering law to determine the triggering times.

Inspired by [94], we propose the following internal dynamic variable η_i to agent i:

$$\dot{\eta}_i(t) = -\beta_i \eta_i(t) - \delta_i (L_{ii} e_i^2(t) - \frac{\sigma_i}{2} q_i(t)), \; i \in \mathcal{I}, \tag{3.10}$$

where $\eta_i(0) > 0$, $\beta_i > 0$, $\delta_i \in [0, 1]$, and $\sigma_i \in [0, 1)$ are design parameters and can be arbitrarily chosen in the given intervals. These dynamic variables are correlated in the triggering law, as defined in our first main result.

Theorem 3.1. *Consider the multi-agent system (3.1)–(3.2). Suppose that the underlying graph $\mathcal{G}$ is undirected. Given $\theta_i > \frac{1-\delta_i}{\beta_i}$ and the first triggering time $t_1^i = 0$, agent i determines the triggering times $\{t_k^i\}_{k=2}^{\infty}$ by*

$$t_{k+1}^i = \min\Big\{t : \; \theta_i (L_{ii} e_i^2(t) - \frac{\sigma_i}{2} q_i(t)) \geq \eta_i(t), \; t \geq t_k^i\Big\}, \tag{3.11}$$

with $q_i(t)$ defined in (3.5) and $\eta_i(t)$ defined in (3.10). Then, (i) average consensus is achieved exponentially if and only if $\mathcal{G}$ is connected; and (ii) there is no Zeno behavior.

Proof. (i) The necessity is straightforward and we only prove sufficiency here. From the way to determine the triggering times by (3.11), we have

$$\theta_i (L_{ii} e_i^2(t) - \frac{\sigma_i}{2} q_i(t)) \leq \eta_i(t), \; \forall t \geq 0. \tag{3.12}$$

From (3.10) and (3.12), we have

$$\dot{\eta}_i(t) \geq -\beta_i \eta_i(t) - \frac{\delta_i}{\theta_i}\eta_i(t),\ \forall t \geq 0.$$

Thus

$$\eta_i(t) \geq \eta_i(0)e^{-(\beta_i+\frac{\delta_i}{\theta_i})t} > 0,\ \forall t \geq 0. \tag{3.13}$$

Consider a Lyapunov candidate as follows

$$W(x(t),\eta(t)) = V(x(t)) + \sum_{i=1}^{n} \eta_i(t), \tag{3.14}$$

where $\eta(t) = [\eta_1(t), \ldots, \eta_n(t)]^\top$. Then the derivative of $W(x(t), \eta(t))$ along the trajectories of the multi-agent system (3.1)–(3.2) and system (3.10) satisfies

$$\begin{aligned}
\dot{W}(x(t),\eta(t)) =& \dot{V}(x(t)) + \sum_{i=1}^{n} \dot{\eta}_i(t) \\
\leq& -\sum_{i=1}^{n} \frac{1}{2} q_i(t) + \sum_{i=1}^{n} L_{ii}e_i^2(t) - \sum_{i=1}^{n} \beta_i \eta_i(t) + \sum_{i=1}^{n} \delta_i(\frac{\sigma_i}{2} q_i(t) - L_{ii}e_i^2(t)) \\
=& -\sum_{i=1}^{n} \frac{1}{2}(1-\sigma_i) q_i(t) - \sum_{i=1}^{n} \beta_i \eta_i(t) + \sum_{i=1}^{n} (\delta_i - 1)(\frac{\sigma_i}{2} q_i(t) - L_{ii}e_i^2(t)) \\
\leq& -\sum_{i=1}^{n} \frac{1}{2}(1-\sigma_i) q_i(t) - \sum_{i=1}^{n} \beta_i \eta_i(t) + \sum_{i=1}^{n} \frac{1-\delta_i}{\theta_i}\eta_i(t) \\
=& -\sum_{i=1}^{n} \frac{1}{2}(1-\sigma_i) q_i(t) - \sum_{i=1}^{n} \Big(\beta_i - \frac{1-\delta_i}{\theta_i}\Big)\eta_i(t) \\
\leq& -(1-\sigma_{\max})\sum_{i=1}^{n} \frac{1}{2} q_i(t) - k_d \sum_{i=1}^{n} \eta_i(t) \\
\leq& -(1-\sigma_{\max})\rho_2(L)V(x(t)) - k_d \sum_{i=1}^{n} \eta_i(t) \\
\leq& -k_W W(x(t),\eta(t)),
\end{aligned}$$

where $k_d = \min_i \Big\{\beta_i - \frac{1-\delta_i}{\theta_i}\Big\} > 0$ and $k_W = \min\Big\{(1-\sigma_{\max})\rho_2(L), k_d\Big\} > 0$. Then

$$V(x(t)) \leq W(x(t),\eta(t)) \leq W(x(0),\eta(0))e^{-k_W t},\ \forall t \geq 0. \tag{3.15}$$

This implies that system (3.1)–(3.2) reaches average consensus exponentially.
(ii) Next, we prove that there is no Zeno behavior by contradiction. Suppose there exists Zeno behavior. Then there exists an agent i, such that $\lim_{k\to+\infty} t_k^i = T_0$ where T_0 is a positive constant.

Whether $\mathcal{G}$ is connected or not, from the proof in (i) we know that all the agents in the same CC reach consensus and there is a result similar to (3.15). Thus, we know that there exists a positive constant $M_0 > 0$ such that $|x_i(t)| \leq M_0$ for all $t \geq 0$ and $i = 1, \dots, n$. Then, we have

$$|u_i(t)| \leq 2M_0 L_{ii}, \ \forall t \geq 0.$$

Let $\varepsilon_0 = \frac{\sqrt{\eta_i(0)}}{4\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\delta_i}{\theta_i})T_0} > 0$. Then from the property of limit, there exists a positive integer $N(\varepsilon_0)$ such that

$$t_k^i \in [T_0 - \varepsilon_0, T_0], \ \forall k \geq N(\varepsilon_0). \tag{3.16}$$

Noting $q_i(t) \geq 0$ and (3.13), we can conclude that one necessary condition to guarantee that the inequality in (3.11) holds is

$$|\hat{x}_i(t) - x_i(t)| \geq \sqrt{\frac{\eta_i(0)}{\theta_i L_{ii}}} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})t}. \tag{3.17}$$

Again noting $|\dot{x}_i(t)| = |u_i(t)| \leq 2M_0 L_{ii}$ and $|\hat{x}_i(t_k^i) - x_i(t_k^i)| = 0$ for any triggering time t_k^i, we can conclude that one necessary condition to guarantee that the above inequality holds is

$$(t - t_k^i)2M_0 L_{ii} \geq \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})t}. \tag{3.18}$$

Now suppose that the $N(\varepsilon_0)$-th triggering time of agent i, $t_{N(\varepsilon_0)}^i$, has been determined. Let $t_{N(\varepsilon_0)+1}^i$ and $\tilde{t}_{N(\varepsilon_0)+1}^i$ denote the next triggering time determined by (3.11) and (3.18), respectively. Then

$$\begin{aligned} t_{N(\varepsilon_0)+1}^i - t_{N(\varepsilon_0)}^i \geq & \tilde{t}_{N(\varepsilon_0)+1}^i - t_{N(\varepsilon_0)}^i = \frac{\sqrt{\eta_i(0)}}{2\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})\tilde{t}_{N(\varepsilon_0)+1}^i} \\ \geq & \frac{\sqrt{\eta_i(0)}}{2\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})t_{N(\varepsilon_0)+1}^i} \geq \frac{\sqrt{\eta_i(0)}}{2\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})T_0} = 2\varepsilon_0, \end{aligned} \tag{3.19}$$

which contradicts to (3.16). Therefore, Zeno behavior is excluded. □

Remark 3.4. *(3.11) is a dynamic triggering law since it involves the extra dynamic variables $\eta_i(t)$. Similar to the static triggering law (3.6), it is also distributed. The static triggering law (3.6) can be seen as a limit case of the dynamic triggering law (3.11) when θ_i grows large. Thus, from the analysis in Remark 3.3, we can conclude that the distributed triggering law (10) in [16] is a special case of the dynamic triggering law (3.11).*

Remark 3.5. *If we choose $\delta_i = 0$ in (3.10) and $\sigma_i = 0$ in (3.11), then $\eta_i(t) = \eta_i(0)e^{-\beta_i t}$ and now the inequality in (3.11) is $|e_i(t)| \geq \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}} e^{-\frac{\beta_i}{2}t}$. The later is the triggering function (7) in [18] with $c_0 = 0$, $c_1 = \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}}$, $\alpha = \frac{\beta_i}{2}$. However, we do not need the constraint $\alpha < \rho_2(L)$ which is necessary in [18].*

If we choose β_i large enough, then $k_W = (1-\sigma_{\max})\rho_2(L)$. Hence, in this case, from (3.9) and (3.15), we know that the trajectories of the multi-agent system (3.1) –(3.2) under static triggering law (3.6) and dynamic triggering law (3.11) have the same guaranteed decay rate given by (3.9).

Remark 3.6. *Intuitively, from (3.13), one can conclude that the larger $\eta_i(0)$ the larger the inter-event time. This is also consistent with the definition of ε_0. However, how do those design parameters $\eta_i(0), \beta_i, \xi_i, \sigma_i, \theta_i$ affect the inter-event times and decay rate in theory is unclear. We leave this as future study.*

3.2.2 Discontinuous approach

In the above static and dynamic triggering laws, continuous updating of the control input is avoided. However, in order to monitor the inequalities (3.6) and (3.11), each agent still needs to continuously monitor its neighbors's states, which means continuous broadcasting and continuous receiving are still needed. In what follows, we will modify the above results to avoid these two requirements.

We upper-bound the derivative of $V(x(t))$ along the trajectories of the multi-agent system (3.1)–(3.2) in a different way. Similar to the derivation process to get (3.4), we have

$$\begin{aligned}
\dot{V}(x(t)) &= \sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} -L_{ij}\hat{x}_j(t) = -\sum_{i=1}^{n}(\hat{x}_i(t) - e_i(t)) \sum_{j=1}^{n} L_{ij}\hat{x}_j(t) \\
&\overset{**}{=} -\sum_{i=1}^{n} \hat{q}_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} e_i(t)L_{ij}\hat{x}_j(t) \\
&= -\sum_{i=1}^{n} \hat{q}_i(t) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} e_i(t)L_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) \\
&\leq -\sum_{i=1}^{n} \hat{q}_i(t) - \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} L_{ij}e_i^2(t) - \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} L_{ij}\frac{1}{4}(\hat{x}_j(t) - \hat{x}_i(t))^2 \\
&= -\sum_{i=1}^{n} \hat{q}_i(t) + \sum_{i=1}^{n} L_{ii}e_i^2(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{4}L_{ij}(\hat{x}_j(t) - \hat{x}_i(t))^2 \\
&\overset{**}{=} -\sum_{i=1}^{n} \frac{1}{2}\hat{q}_i(t) + \sum_{i=1}^{n} L_{ii}e_i^2(t),
\end{aligned} \tag{3.20}$$

where

$$\hat{q}_i(t) = -\frac{1}{2}\sum_{j=1}^{n} L_{ij}(\hat{x}_j(t) - \hat{x}_i(t))^2 \geq 0, \tag{3.21}$$

and the equalities denoted by $\stackrel{**}{=}$ hold since

$$\sum_{i=1}^{n}\hat{q}_i(t) = -\sum_{i=1}^{n}\frac{1}{2}\sum_{j=1}^{n}L_{ij}(\hat{x}_j(t)-\hat{x}_i(t))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\hat{x}_i(t)L_{ij}\hat{x}_j(t) = \hat{x}^\top(t)L\hat{x}(t),$$

and the inequality holds since $ab \le a^2 + \frac{1}{4}b^2$.

Similar to [17] and [65], the following law can be used to determine the triggering times:

$$t_1^i = 0,\ t_{k+1}^i = \min\Big\{t:\ L_{ii}e_i^2(t) - \frac{\sigma_i}{2}\hat{q}_i(t) \ge 0,\ t \ge t_k^i\Big\},\ k = 1,2,\dots \tag{3.22}$$

with $\sigma_i \in (0,1)$. From the way to determine the triggering times by (3.22), we have

$$L_{ii}e_i^2(t) \le \frac{\sigma_i}{2}\hat{q}_i(t),\ \forall t \ge 0. \tag{3.23}$$

Then, from (3.20) and (3.23), we have

$$\begin{aligned}\dot{V}(x(t)) &\le -\sum_{i=1}^{n}\frac{1}{2}\hat{q}_i(t) + \sum_{i=1}^{n}L_{ii}e_i^2(t) \le -\frac{1}{2}(1-\sigma_{\max})\sum_{i=1}^{n}\hat{q}_i(t)\\ &= -\frac{1}{2}(1-\sigma_{\max})\hat{x}^\top(t)L\hat{x}(t).\end{aligned}$$

Furthermore,

$$\begin{aligned}x^\top(t)Lx(t) &= (\hat{x}(t)+e(t))^\top L(\hat{x}(t)+e(t)) \le 2\hat{x}^\top(t)L\hat{x}(t) + 2e^\top(t)Le(t)\\ &\le 2\hat{x}^\top(t)L\hat{x}(t) + 2\|L\|\|e(t)\|^2 \le 2\hat{x}^\top(t)L\hat{x}(t) + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\sum_{i=1}^{n}\hat{q}_i(t)\\ &= \Big(2 + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\Big)\hat{x}^\top(t)L\hat{x}(t),\end{aligned} \tag{3.24}$$

where the first inequality holds since L is positive semi-definite and $a^\top Lb \le 2a^\top La + 2b^\top Lb, \forall a,b \in \mathbb{R}^n$, the second inequality holds since $a^\top La \le \|L\|\|a\|^2, \forall a \in \mathbb{R}^n$, and the last inequality holds due to (3.23). We then obtain

$$\begin{aligned}\dot{V}(x(t)) &\le -\frac{(1-\sigma_{\max})\min_i L_{ii}}{4\min_i L_{ii} + 2\|L\|\sigma_{\max}}x^\top(t)Lx(t) = -\frac{(1-\sigma_{\max})\min_i L_{ii}}{2\min_i L_{ii} + \|L\|\sigma_{\max}}\rho_2(L)x^\top(t)K_n x(t)\\ &= -\frac{(1-\sigma_{\max})\min_i L_{ii}}{2\min_i L_{ii} + \|L\|\sigma_{\max}}\rho_2(L)V(x(t)).\end{aligned}$$

Hence,

$$V(x(t)) \le V(x(0))e^{-\frac{(1-\sigma_{\max})\min_i L_{ii}}{2\min_i L_{ii}+\|L\|\sigma_{\max}}\rho_2(L)t},\ \forall t \ge 0. \tag{3.25}$$

This implies that system (3.1)–(3.2) reaches average consensus exponentially if the underlying graph $\mathcal{G}$ is connected.

Remark 3.7. *Similar to the analysis in Remark 3.2, (3.22) is a static triggering law and it is also distributed. Moreover, similar to the analysis in Remark 3.3, we can conclude that the distributed triggering law (6) in [17] is a special case of the static triggering law (3.22).*

In [82] it is argued that the distributed triggering law (6) in [17] "does not discard the possibility of an infinite number of events happening in a finite time period". Zeno behavior may also not be excluded under the static triggering law (3.22). In the following, in order to explicitly exclude Zeno behavior, we will replace the static triggering law (3.22) by the dynamic one.

Similar to (3.10), we propose an internal dynamic variable χ_i to agent i:

$$\dot{\chi}_i(t) = -\beta_i\chi_i(t) - \delta_i(L_{ii}e_i^2(t) - \frac{\sigma_i}{2}\hat{q}_i(t)),\ i \in \mathcal{I} \tag{3.26}$$

where $\chi_i(0) > 0$, $\beta_i > 0$, $\xi_i \in [0,1]$, and $\sigma_i \in [0,1)$ are design parameters and can be arbitrarily chosen in the given intervals. Our second main result is given in the following theorem.

Theorem 3.2. *Consider the multi-agent system (3.1)–(3.2). Suppose that the underlying graph $\mathcal{G}$ is undirected. Given $\theta_i > \frac{1-\xi_i}{\beta_i}$ and the first triggering time $t_1^i = 0$, agent i determines the triggering times $\{t_k^i\}_{k=2}^{\infty}$ by*

$$t_{k+1}^i = \min\Big\{t:\ \theta_i(L_{ii}e_i^2(t) - \frac{\sigma_i}{2}\hat{q}_i(t)) \geq \chi_i(t),\ t \geq t_k^i\Big\}, \tag{3.27}$$

with $\hat{q}_i(t)$ defined in (3.21) and $\chi_i(t)$ defined in (3.26). Then, (i) average consensus is achieved exponentially if and only if $\mathcal{G}$ is connected; and (ii) there is no Zeno behavior.

Proof. (i) The necessity is straightforward and we only prove sufficiency here. Similar to (3.13), we have

$$\chi_i(t) \geq \chi_i(0)e^{-(\beta_i+\frac{\delta_i}{\theta_i})t} > 0. \tag{3.28}$$

Consider a Lyapunov candidate as follows

$$F(x(t),\chi(t)) = V(x(t)) + \sum_{i=1}^{n}\chi_i(t), \tag{3.29}$$

where $\chi(t) = [\chi_1(t),\dots,\chi_n(t)]^\top$. Then the derivative of $F(x(t),\chi(t))$ along the trajectories of the multi-agent system (3.1)–(3.2) and system (3.26) satisfies

$$\begin{aligned}\dot{F}(x(t),\chi(t)) =& \dot{V}(x(t)) + \sum_{i=1}^{n}\dot{\chi}_i(t)\\ \leq& -\sum_{i=1}^{n}\frac{1}{2}\hat{q}_i(t) + \sum_{i=1}^{n}L_{ii}e_i^2(t) - \sum_{i=1}^{n}\beta_i\chi_i(t) + \sum_{i=1}^{n}\delta_i(\frac{\sigma_i}{2}\hat{q}_i(t) - L_{ii}e_i^2(t))\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n}\frac{1}{2}(1-\sigma_i)\hat{q}_i(t) - \sum_{i=1}^{n}\beta_i\chi_i(t) + \sum_{i=1}^{n}(\delta_i - 1)(\frac{\sigma_i}{2}\hat{q}_i(t) - L_{ii}e_i^2(t))\\
&\leq -\sum_{i=1}^{n}\frac{1}{2}(1-\sigma_i)\hat{q}_i(t) - \sum_{i=1}^{n}\beta_i\chi_i(t) + \sum_{i=1}^{n}\frac{1-\delta_i}{\theta_i}\chi_i(t)\\
&= -\sum_{i=1}^{n}\frac{1}{2}(1-\sigma_i)\hat{q}_i(t) - \sum_{i=1}^{n}\Big(\beta_i - \frac{1-\delta_i}{\theta_i}\Big)\chi_i(t)\\
&\leq -(1-\sigma_{\max})\sum_{i=1}^{n}\frac{1}{2}\hat{q}_i(t) - k_d\sum_{i=1}^{n}\chi_i(t)\\
&= -\frac{1}{2}(1-\sigma_{\max})\hat{x}^\top(t)L\hat{x}(t) - k_d\sum_{i=1}^{n}\chi_i(t).
\end{aligned}
$$

Similar to the derivation process to get (3.24), we have

$$
\begin{aligned}
x^\top(t)Lx(t) \leq& 2\hat{x}^\top(t)L\hat{x}(t) + 2\|L\|\|e(t)\|^2\\
\leq& 2\hat{x}^\top(t)L\hat{x}(t) + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\sum_{i=1}^{n}\hat{q}_i(t) + \frac{2\|L\|}{\min_i\{\theta_i L_{ii}\}}\sum_{i=1}^{n}\chi_i(t)\\
=& \Big(2 + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\Big)\hat{x}^\top(t)L\hat{x}(t) + \frac{2\|L\|}{\min_i\{\theta_i L_{ii}\}}\sum_{i=1}^{n}\chi_i(t)\\
\leq& k_x\hat{x}^\top(t)L\hat{x}(t) + \frac{2\|L\|}{\min_i\{\theta_i L_{ii}\}}\sum_{i=1}^{n}\chi_i(t),
\end{aligned}
\tag{3.30}
$$

where

$$
k_x = \max\Big\{2 + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}, \frac{2(1-\sigma_{\max})\|L\|}{k_d\min_i\{\theta_i L_{ii}\}}\Big\}.
$$

Then,

$$
-\frac{1}{2}(1-\sigma_{\max})\hat{x}^\top(t)L\hat{x}(t) \leq -\frac{1}{2k_x}(1-\sigma_{\max})x^\top(t)Lx(t) + \frac{k_d}{2}\sum_{i=1}^{n}\chi_i(t).
$$

Thus,

$$
\begin{aligned}
\dot{F}(x(t),\chi(t)) \leq& -\frac{1}{2k_x}(1-\sigma_{\max})x^\top(t)Lx(t) - \frac{k_d}{2}\sum_{i=1}^{n}\chi_i(t)\\
\leq& -\frac{\rho_2(L)}{2k_x}(1-\sigma_{\max})x^\top(t)K_n x(t) - \frac{k_d}{2}\sum_{i=1}^{n}\chi_i(t)\\
=& -\frac{\rho_2(L)}{k_x}(1-\sigma_{\max})V(t) - \frac{k_d}{2}\sum_{i=1}^{n}\chi_i(t)
\end{aligned}
$$

$$\leq k_F F(x(t), \chi(t)),$$

where $k_F = \min\left\{\frac{\rho_2(L)}{k_x}(1 - \sigma_{\max}), \frac{k_d}{2}\right\}$. Hence,

$$V(x(t)) < F(x(t), \chi(t)) \leq F(x(0), \chi(0))e^{-k_F t}, \ \forall t \geq 0. \tag{3.31}$$

This implies that system (3.1)–(3.2) reaches average consensus exponentially.
(ii) The way to exclude Zeno behavior is the same as the proof in Theorem 3.1. □

Remark 3.8. *The triggering law (3.27) is dynamic and it is also distributed. One can easily check that every agent does not need to continuously access its neighbors' states when implementing the static and dynamic triggering laws (3.22) and (3.27).*

Remark 3.9. *The static triggering law (3.22) can be seen as a limit case of the dynamic triggering law (3.27) when θ_i grows large. Thus, from the analysis in Remark 3.7, we can conclude that the distributed triggering law (6) in [17] is a special case of the dynamic triggering law (3.27).*

If we choose β_i large enough, then $k_F = \frac{(1-\sigma_{\max}) \min_i L_{ii}}{2 \min_i L_{ii} + \|L\| \sigma_{\max}} \rho_2(L)$. Hence, in this case, from (3.25) and (3.31), we know that the trajectories of the multi-agent system (3.1)–(3.2) under static triggering law (3.22) and dynamic triggering law (3.27) have the same guaranteed decay rate given by (3.25).

Remark 3.10. *In [65], the authors propose three distributed triggering laws for multi-agent systems with event-triggered control and directed topologies. With some modifications, similar to this chapter, the three distributed triggering laws in [65] can be extended to dynamic triggering laws as the one in Theorems 3.1 and 3.2. In other words, the results in Theorems 3.1 and 3.2 can be extended to the case that the underlying graph is directed and has a directed spanning tree. Moreover, the results in Theorems 3.1 and 3.2 also can most likely be extended to general linear and even nonlinear multi-agent systems. However, in the general linear case, the triggering laws are not distributed anymore since global information, such as the eigenvalues of the Laplacian matrix, is needed. Actually, to the best of our knowledge, in all the existing papers that consider event-triggered control for general linear multi-agent systems, the use of the eigenvalues of the Laplacian matrix cannot be avoided. And for the nonlinear case, some standard continuity assumptions, such as upper and lower Lipschtiz continuity assumptions, for the nonlinear dynamics are normally required.*

3.3 Self-triggered algorithm

When applying the dynamic triggering law (3.27) in Theorem 3.2, although each agent avoids to continuously monitor its neighbors' states, agent i still needs to continuously sense its own state since it has to continuously monitor the triggering law (3.27) and continuously listen to $x_j(t_k^j)$, $k = 1, 2, \ldots,$ $j \in \mathcal{N}_i$, since it does not know the triggering times of its neighbors, t_k^j, $k = 1, 2, \ldots,$ $j \in \mathcal{N}_i$, in advance. The way to avoid continuous sensing is straight forward since the control input of each agent is piece-wise constant and the state of each agent can be predicted by simple calculation as (3.32) in the following. The challenge is to avoid continuous listening. If every agent $i \in \mathcal{I}$, at its current triggering time t^i_k, can predict (determine) its next triggering time t^i_{k+1} and broadcast it to its neighbors, then at time t^i_k agent i knows agent j's latest triggering time $t^j_{k_j(t^i_k)}$ which is before t^i_k and its next triggering time $t^j_{k_j(t^i_k)+1}$ which is after t^i_k, for $j \in \mathcal{N}_i$. In this case, agent i only needs listen to and receive information at $\{t^j_k\}_{k=1}^{\infty}$, $j \in \mathcal{N}_i$ since it knows these time instants in advance. In this case, each agent only needs to to sense its state information and broadcast its triggering information at its own triggering times, and to listen to and receive incoming information from its neighbors at their triggering times. Inspired by this, in the following we will propose a self-triggered algorithm such that at time t^i_k each agent i could determine t^i_{k+1} in advance. The idea is explained below.

From $\dot{x}_i(t) = u_i(t) = -\sum_{j=1}^n L_{ij}x_j(t^j_{k_j(t)}) = -\sum_{j=1}^n L_{ij}u_{ij}(t)$ with $u_{ij}(t) = x_j(t^j_{k_j(t)}) - x_i(t^i_{k_i(t)})$, we have

$$x_i(t) = x_i(t^i_k) + \int_{t^i_k}^t u_i(s)ds = x_i(t^i_k) - \int_{t^i_k}^t \sum_{j=1}^n L_{ij}u_{ij}(s)ds, t \in [t^i_k, t^i_{k+1}). \tag{3.32}$$

Thus for $t \in [t^i_k, t^i_{k+1})$, we have

$$|e_i(t)| = |x_i(t^i_k) - x_i(t)| = \Big|\sum_{j=1}^n \int_{t^i_k}^t L_{ij}u_{ij}(s)ds\Big|. \tag{3.33}$$

Here we need to highlight that $u_{ij}(t)$ may not be a constant for all $t \in [t^i_k, t^i_{k+1})$ since $x_j(t^j_{k_j(t)})$ may not be a constant for all $t \in [t^i_k, t^i_{k+1})$. So at time t^i_k, we do not know the value of $|e_i(t)|$ for all $t \in (t^i_k, t^i_{k+1})$ in advance. However, if at time t^i_k we could estimate the upper-bound of $u_{ij}(t)$, then we could estimate the upper-bound of $|e_i(t)|$. In this case, we can estimate t^i_{k+1} at time t^i_k.

In order to estimate the upper-bound of $u_{ij}(t)$, we first need to simplify the dynamic triggering laws (3.11) and (3.2) in Theorem 3.1 and Theorem 3.2. As Remark 3.5 pointed out, if we choose $\delta_i = 0$ in (3.10) and $\sigma_i = 0$ in (3.11), then $\eta_i(t) = \eta_i(0)e^{-\beta_i t}$ and now the inequality in (3.11) is $|e_i(t)| \ge \alpha_i e^{-\frac{\beta_i}{2}t}$ with $\alpha_i = \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}} > 0$. Here, α_i can be chosen as any positive real numbers since $\eta_i(0)$ can be chosen as any positive real numbers. Then from Theorem 3.1, we derive the following corollary[1].

Corollary 3.1. *Consider the multi-agent system (3.1)–(3.2). Suppose that the underlying graph $\mathcal{G}$ is undirected. Given $\alpha > 0$, $\beta > 0$ and the first triggering time $t^i_1 = 0$, agent i determines the triggering times $\{t^i_k\}_{k=2}^{\infty}$ by*

$$t^i_{k+1} = \min\Big\{t : |e_i(t)| \ge \frac{\alpha}{\sqrt{L_{ii}}}e^{-\frac{\beta}{2}t}, \ t \ge t^i_k\Big\}. \tag{3.34}$$

[1]If we choose $\delta_i = 0$ in (3.26) and $\sigma_i = 0$ in (3.27), then Corollary 3.1 is also a special case of Theorem 3.2.

Then, (i) average consensus is achieved exponentially if and only if $\mathcal{G}$ is connected; and (ii) there is no Zeno behavior.

Remark 3.11. *The design parameters α and β can be distributively chosen for each agent in the above corollary, but their effects on inter-event times and decay rate are not clear in theory. The reason that we require every agent to choose the same design parameters here is that it is convenient to design the self-triggered algorithm in the following.*

Next, let us upper-bound $|x_i(t) - x_j(t)|$ which will be used later. From the way to determine the triggering times in (3.34), we have

$$|e_i(t)| \leq \frac{\alpha}{\sqrt{L_{ii}}} e^{-\frac{\beta}{2}t}, \ \forall t \geq 0. \tag{3.35}$$

From (3.4) and (3.35), we have

$$\begin{aligned}\dot{V}(x(t)) \leq & -\sum_{i=1}^{n} \frac{1}{2} q_i(t) + \sum_{i=1}^{n} L_{ii} e_i^2(t) \leq -\frac{1}{2} x^\top(t) L x(t) + \sum_{i=1}^{n} \alpha^2 e^{-\beta t} \\ \leq & -\frac{1}{2} \rho_2(L) x^\top(t) K_n x(t) + n\alpha^2 e^{-\beta t} = -\rho_2(L) V(x(t)) + n\alpha^2 e^{-\beta t}.\end{aligned}$$

Then

$$\frac{dV(t)e^{\rho_2(L)t}}{dt} \leq n\alpha^2 e^{(\rho_2(L)-\beta)t}.$$

Then

$$V(x(t)) \leq \begin{cases} V(0)e^{-\rho_2(L)t} + \frac{n\alpha^2}{\rho_2(L)-\beta}(e^{-\beta t} - e^{-\rho_2(L)t}), & \text{if } \rho_2(L) \neq \beta, \\ V(0)e^{-\rho_2(L)t} + n\alpha^2 t e^{-\rho_2(L)t}, & \text{if } \rho_2(L) = \beta. \end{cases}$$

From the factor that for any given $\varepsilon > 0$, $e^{\varepsilon t} \geq 1 + \varepsilon t$ holds, we have

$$V(x(t)) \leq k_1 e^{-\rho_2(L)t} + k_2 e^{-k_3 t}, \ \forall t \geq 0,$$

where

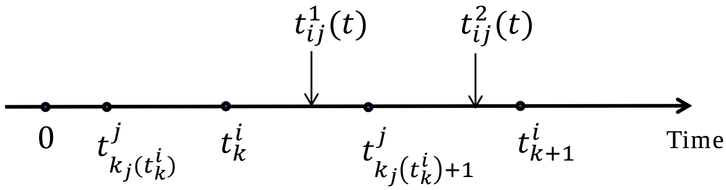
$$k_1 = \begin{cases} V(x(0)) - \frac{n\alpha^2}{\rho_2(L)-\beta}, & \text{if } \rho_2(L) \neq \beta, \\ V(x(0)) - \frac{n\alpha^2}{\varepsilon}, & \text{if } \rho_2(L) = \beta, \end{cases}$$

$$k_2 = \begin{cases} \frac{n\alpha^2}{\rho_2(L)-\beta}, & \text{if } \rho_2(L) \neq \beta, \\ \frac{n\alpha^2}{\varepsilon}, & \text{if } \rho_2(L) = \beta, \end{cases}$$

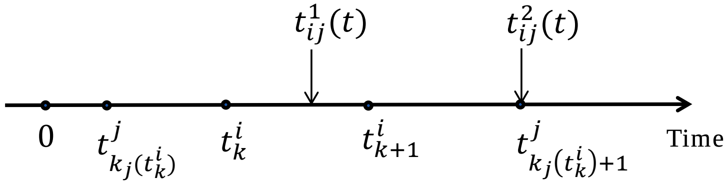
$$k_3 = \begin{cases} \beta, & \text{if } \rho_2(L) \neq \beta, \\ \beta - \varepsilon, & \text{if } \rho_2(L) = \beta, \end{cases}$$

with $\varepsilon \in (0, \beta)$ is a design parameter. Then, from (3.3), we have

$$\sum_{i=1}^{n} |x_i(t) - \bar{x}(0)|^2 = 2V(x(t)) \leq 2(k_1 e^{-\rho_2(L)t} + k_2 e^{-k_3 t}), \ \forall t \geq 0.$$



(a) The case of $t^j_{k_j(t^i_k)+1} < t^i_{k+1}$.



(b) The case of $t^j_{k_j(t^i_k)+1} \geq t^i_{k+1}$.

Figure 3.1: Illustration of the relation of t^i_k, t^i_{k+1}, $t \in [t^i_k, t^i_{k+1})$, $t^j_{k_j(t^i_k)}$, $t^j_{k_j(t^i_k)+1}$, $t^1_{ij}(t)$ and $t^2_{ij}(t)$.

Thus,

$$\begin{aligned}|x_i(t) - x_j(t)| \leq & |x_i(t) - \bar{x}(0)| + |x_j(t) - \bar{x}(0)| \\ & \leq \sqrt{2(|x_i(t) - \bar{x}(0)|^2 + |x_j(t) - \bar{x}(0)|^2)} \leq f^x(t), \ \forall t \geq 0, \end{aligned} \tag{3.36}$$

where $f^x(t) = 2\sqrt{k_1 e^{-\rho_2(L)t} + k_2 e^{-k_3 t}}$.

Now, we upper-bound $u_{ij}(t)$ as follows

$$\begin{aligned}|u_{ij}(t)| = & |x_j(t^j_{k_j(t)}) - x_i(t^i_{k_i(t)})| = |x_j(t^j_{k_j(t)}) - x_j(t) + x_j(t) - x_i(t) + x_i(t) - x_i(t^i_{k_i(t)})| \\ \leq & |x_j(t^j_{k_j(t)}) - x_j(t)| + |x_j(t) - x_i(t)| + |x_i(t) - x_i(t^i_{k_i(t)})| \\ \leq & \Big(\frac{\alpha}{\sqrt{L_{ii}}} + \frac{\alpha}{\sqrt{L_{jj}}}\Big) e^{-\frac{\beta}{2}t} + f^x(t), \ \forall t \geq 0. \end{aligned} \tag{3.37}$$

Finally, let us upper-bound $e_i(t)$. For $t \in [t^i_k, t^i_{k+1})$, denote

$$t^1_{ij}(t) = \min\left\{t, t^j_{k_j(t^i_k)+1}\right\}, \ t^2_{ij}(t) = \max\left\{t, t^j_{k_j(t^i_k)+1}\right\}. \tag{3.38}$$

Figure 3.1 illustrates the relation of t^i_k, t^i_{k+1}, $t \in [t^i_k, t^i_{k+1})$, $t^j_{k_j(t^i_k)}$, $t^j_{k_j(t^i_k)+1}$, $t^1_{ij}(t)$ and $t^2_{ij}(t)$.

From the definition of $u_{ij}(t)$ and $t^1_{ij}(t)$, we know that $u_{ij}(t)$ is constant for all $t \in [t^i_k, t^1_{ij}(t)]$. And for $t > t^1_{ij}(t)$, $u_{ij}(t)$ can be upper-bounded by (3.37). Thus, from (3.33),

Algorithm 3.1

1: Choose $\alpha > 0, \beta > 0$ and $\varepsilon \in (0,\beta)$;
2: Agent $i \in \mathcal{I}$ sends L_{ii} to its neighbors;
3: Initialize $t^i_1 = 0$ and $k = 1$;
4: At time $s = t^i_k$, agent i senses its own state $x_i(t^i_k)$, and updates its control input $u_i(t^i_k)$ by (3.2), and determines t^i_{k+1} by (3.40)[1], and broadcasts its triggering information $\{t^i_{k+1}, x_i(t^i_k)\}$ to its neighbors;
5: At agent i's neighbors' triggering times which are between $[t^i_k, t^i_{k+1}]$, agent i receives triggering information for its neighbors[2] and updates its control input $u_i(\cdot)$ by (3.2);
6: resets $k = k + 1$, and goes back to Step 4.

for $t \in [t^i_k, t^i_{k+1})$ we have

$$|e_i(t)| = \Big|\sum_{j=1}^{n}\int_{t^i_k}^{t} L_{ij}u_{ij}(s)ds\Big| = \Big|\sum_{j=1}^{n} L_{ij}\Big\{\int_{t^i_k}^{t^1_{ij}} u_{ij}(s)ds + \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}} u_{ij}(s)ds\Big\}\Big| \le g_i(t), \quad (3.39)$$

where

$$g_i(t) = \Big|\sum_{j=1}^{n} L_{ij}(t^1_{ij} - t^i_k)u_{ij}(t^i_k)\Big| - \sum_{j=1, j\neq i}^{n} L_{ij}\int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}} \Big[\Big(\frac{\alpha}{\sqrt{L_{ii}}} + \frac{\alpha}{\sqrt{L_{jj}}}\Big)e^{-\frac{\beta}{2}s} + f^x(s)\Big]ds.$$

Hence, a necessary condition to guarantee (3.34), i.e.,

$$|e_i(t)| \ge \frac{\alpha}{\sqrt{L_{ii}}}e^{-\frac{\beta}{2}t}, \ \forall t \in [t^i_k, t^i_{k+1}),$$

is

$$g_i(t) \ge \frac{\alpha}{\sqrt{L_{ii}}}e^{-\frac{\beta}{2}t}, \ \forall t \in [t^i_k, t^i_{k+1}).$$

Since $\frac{\alpha}{\sqrt{L_{ii}}}e^{-\frac{\beta}{2}t}$ decreases with respect to t, $g_i(t)$ increases with respect to t during $[t^i_k, t^i_{k+1})$ and $g_i(t^i_k) = 0$, then given t^i_k, agent i can estimate t^i_{k+1} by solving

$$g_i(t) = \frac{\alpha}{\sqrt{L_{ii}}}e^{-\frac{\beta}{2}t}, \ t \ge t^i_k. \quad (3.40)$$

In other words, if at time t^i_k agent i knows $t^j_{k_j(t^i_k)}$, $t^j_{k_j(t^i_k)+1}$, $x_j(t^j_{k_j(t^i_k)})$, L_{jj}, $\forall j \in \mathcal{N}_i$, then it can determine its next triggering time t^i_{k+1} by solving (3.40). The above implement idea is summarized in Algorithm 3.1.

The following theorem proves that consensus is achieved exponentially and there is no Zeno behavior when every agent performs Algorithm 3.1.

[1] Agent i uses $t^j_{k_j(t^i_k)}$ to replace $t^j_{k_j(t^i_k)+1}$ to determine t^i_{k+1} by (3.40) when $t^i_k = t^j_{k_j(t^i_k)}$.

[2] In other words, agent i onlys listen to incoming information at its neighbors' triggering times. Thus continuous listening is avoided.

Table 3.1: Summary of the communication requirements for agent i when dynamic triggering laws (3.11) and (3.27), and Algorithm 3.1 are performed.

	Law (3.11)	Law (3.27)	Algorithm 3.1
Broadcasting time	All $t \geq 0$	$\{t_k^i\}_{k=1}^{\infty}$	$\{t_k^i\}_{k=1}^{\infty}$
Listening time	All $t \geq 0$	All $t \geq 0$	$\{t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$
Receiving time	All $t \geq 0$	$\{t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$	$\{t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$
Information broadcasted	$\{x_i(t), t \geq 0\}$	$\{x_i(t_k^i)\}_{k=1}^{\infty}$	$\{t_{k+1}^i, x_i(t_k^i)\}_{k=1}^{\infty}$
Zeno behavior	No	No	No

Theorem 3.3. *Consider the multi-agent system (3.1)–(3.2). Suppose that the underlying graph $\mathcal{G}$ is undirected. If all agents perform Algorithm 3.1, then, (i) average consensus is achieved exponentially if and only if $\mathcal{G}$ is connected; and (ii) there is no Zeno behavior.*

Proof. The necessity is straightforward.

Under Algorithm 3.1, we have $|e_i(t)| \leq \frac{\alpha}{\sqrt{L_{ii}}} e^{-\frac{\beta}{2}t}$ for all $i \in \mathcal{I}$ and $t \geq 0$. Then from Corollary 3.1, we know that consensus is achieved exponentially.

The method of the exclusion of Zeno behavior is similar to the corresponding proof of Theorem 3.1. □

Remark 3.12. *In order to perform Algorithm 3.1, the global parameters $V(0)$, n, and $\rho_2(L)$ are needed to be known in advance, which man be a drawback.*

Remark 3.13. *Self-triggered control approaches are also proposed in [16, 65, 66, 69, 70, 72, 73]. However, one potential drawback of these papers and other papers using a similar approach is that continuous listening is still needed. One can verify that continuous sensing, broadcasting, listening, and receiving are avoided under Algorithm 3.1. Although these are also avoided in [80–83] by combining event-triggered control with periodic sampling, periodic sensing and listening are still needed. Moreover, it is not clear how to show that the average inter-event time is strictly larger than the required sampling period in theory.*

Table 3.1 summarizes the required exchange of information by agent $i \in \mathcal{I}$ if the dynamic triggering laws (3.11) and (3.27), and Algorithm 3.1 are performed.

3.4 Simulations

In this section, a numerical example is given to demonstrate the presented results. Consider a connected undirected graph in Figure 2.2 (a). We choose an arbitrary initial state $x(0) = [6.2945, 8.1158, -7.4603, 8.2675]^\top$. Then the average initial state is $\bar{x}(0) = 3.8044$.

Figure 3.2 (a) shows the state evolutions of the multi-agent system (3.1)–(3.2) under the static triggering law (3.6) with $\sigma_i = 0.5$. Figure 3.2 (b) shows the corresponding triggering times for each agent.

Figure 3.3 (a) shows the state evolutions of the multi-agent system (3.1)–(3.2) under the dynamic triggering law (3.11) with $\sigma_i = 0.5$, $\eta_i(0) = 10$, $\beta_i = 1$, $\delta_i = 1$ and $\theta_i = 1$. Figure 3.3 (b) shows the corresponding triggering times for each agent.

Figure 3.4 (a) shows the state evolutions of the multi-agent system (3.1)–(3.2) under the static triggering law (3.22) with $\sigma_i = 0.5$. Figure 3.4 (b) shows the corresponding triggering times for each agent.

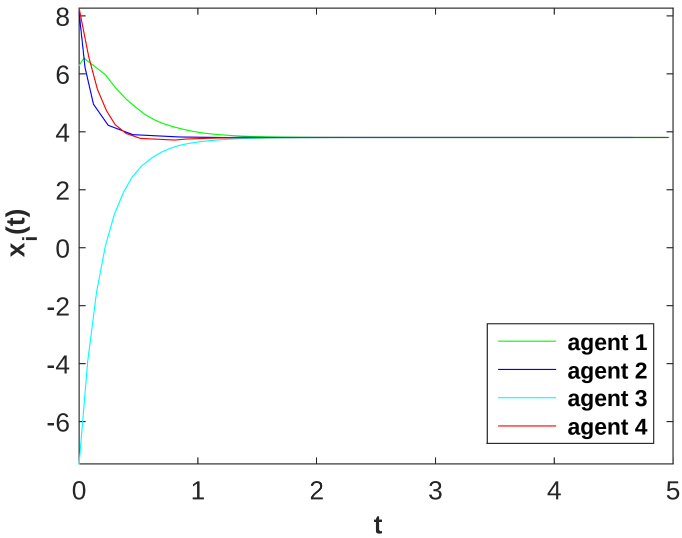
Figure 3.5 (a) shows the state evolutions of the multi-agent system (3.1)–(3.2) under the dynamic triggering law (3.27) with $\sigma_i = 0.5$, $\chi_i(0) = 10$, $\beta_i = 1$, $\delta_i = 1$ and $\theta_i = 1$. Figure 3.5 (b) shows the corresponding triggering times for each agent.

Figure 3.6 (a) shows the state evolutions of the multi-agent system (3.1)–(3.2) when each agent performs Algorithm 3.1 with $\alpha = 10, \beta = 1$ and $\varepsilon = \frac{\beta}{2}$. Figure 3.6 (b) shows the corresponding triggering times for each agent. And the smallest inter-event time is 0.009 in this simulation.

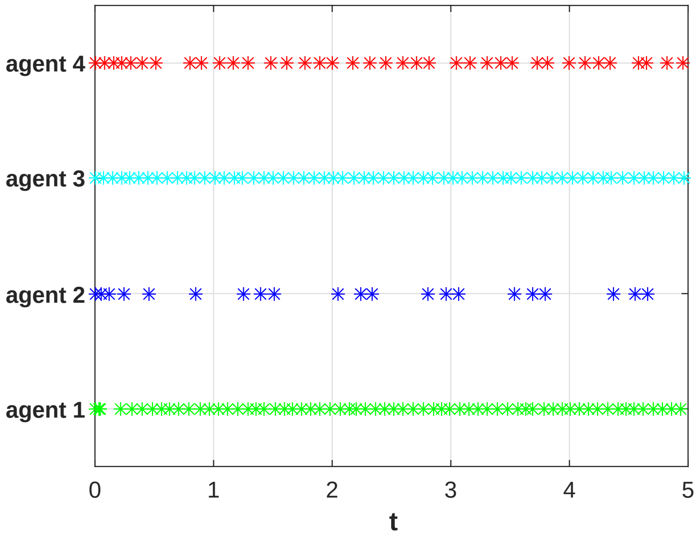
It can be seen that average consensus is achieved when performing the four triggering laws and Algorithm 3.1 proposed in this chapter. Moreover, as stated in Theorem 3.1, Theorem 3.2 and Theorem 3.3, from the simulations we can also see that there is no Zeno behavior under the dynamic triggering law (3.11), the dynamic triggering law (3.27) and Algorithm 3.1. It can also be seen that the average inter-event time under the dynamic triggering law (3.11) and the dynamic triggering law (3.27) are larger than that determined by Algorithm 3.1. Although there is also no Zeno behavior under the static triggering laws (3.6) and (3.22) in the simulations, it is still not clear if this could be proven in theory.

3.5 Summary

In this chapter, we presented two dynamic triggering laws and one self-triggered algorithm for multi-agent systems with event-triggered control over undirected graphs. We showed that, some existing triggering laws are special cases of the proposed dynamic triggering laws and average consensus is achieved exponentially if and only if the communication graph is connected. In addition, Zeno behavior was excluded by proving that the triggering time sequence of each agent is divergent. Moreover, each agent only needs to sense and broadcast at its own triggering times, and to listen to and receive incoming information from its neighbors at their triggering times. Thus continuous listening is avoided. Future research directions include considering the influence of parameters in the proposed dynamic triggering laws.

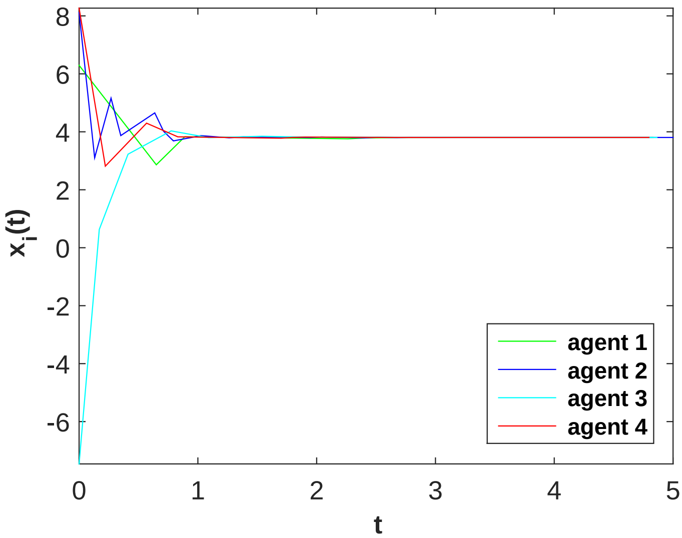


(a)



(b)

Figure 3.2: (a) The state evolutions of the multi-agent system (3.1)–(3.2) under the static triggering law (3.6). (b) The triggering times for each agent.



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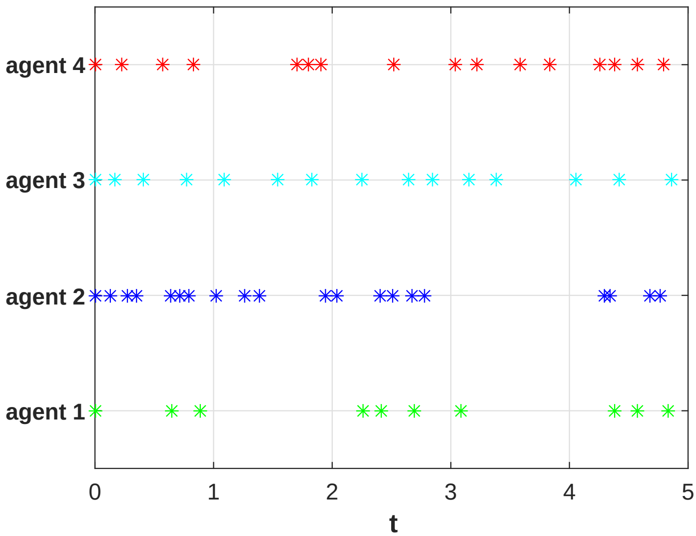
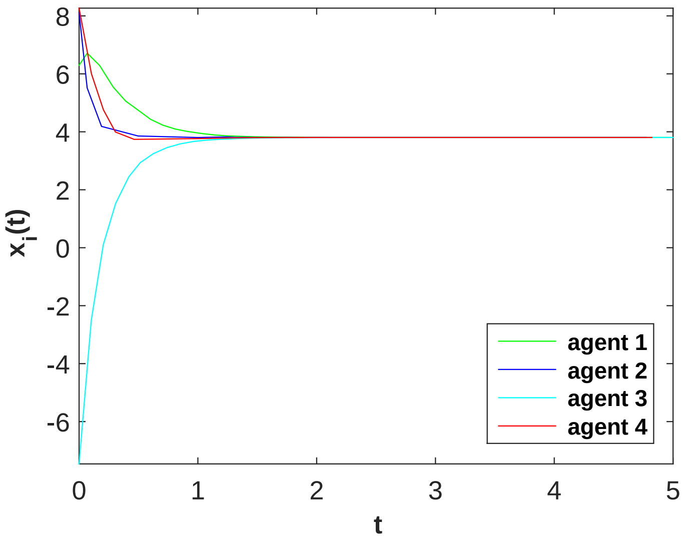


(b)

Figure 3.3: (a) The state evolutions of the multi-agent system (3.1)–(3.2) under the dynamic triggering law (3.11). (b) The triggering times for each agent.



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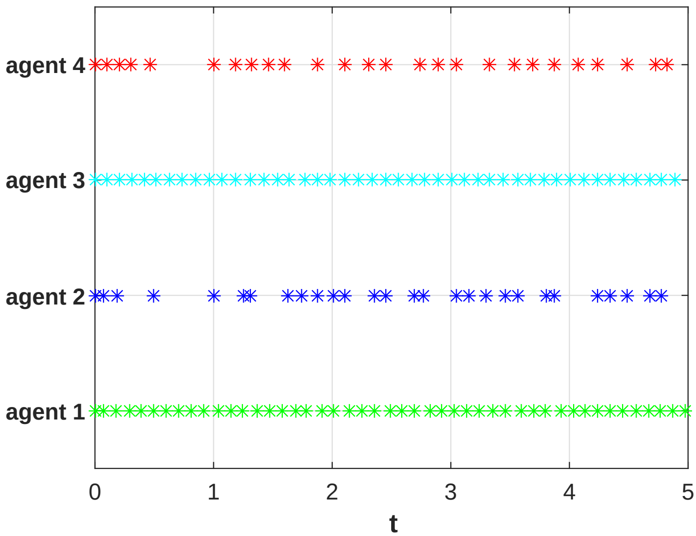
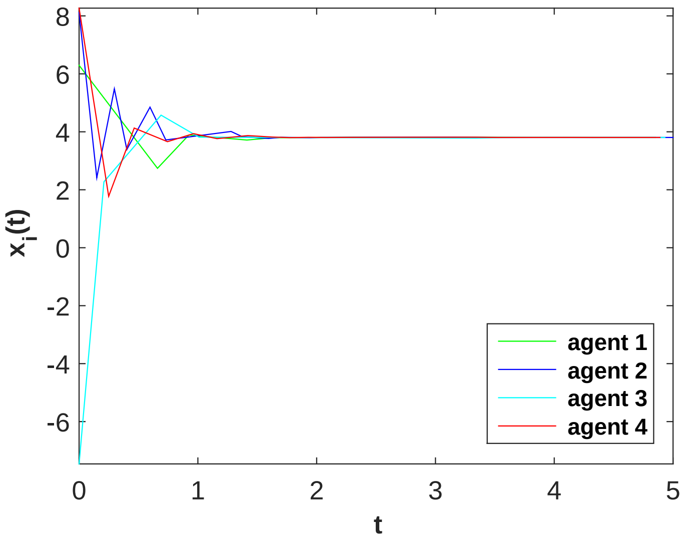


(b)

Figure 3.4: (a) The state evolutions of the multi-agent system (3.1)–(3.2) under the static triggering law (3.22). (b) The triggering times for each agent.



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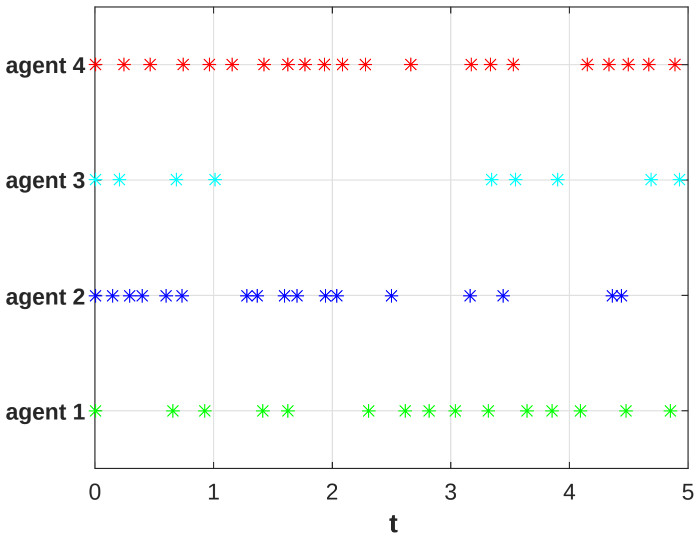
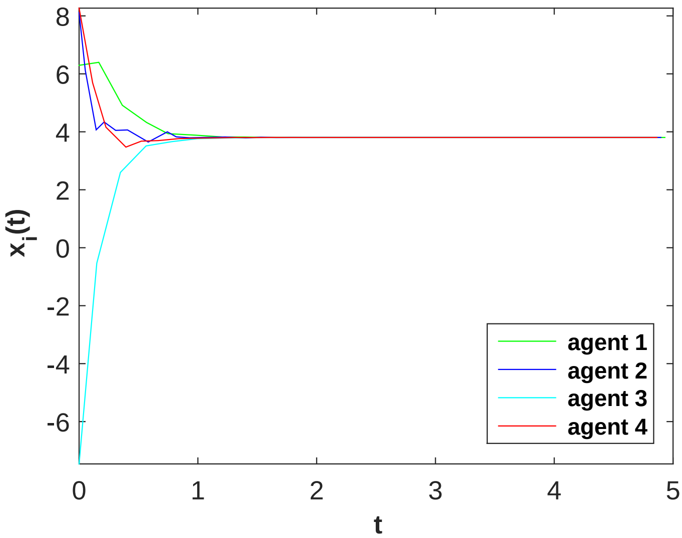
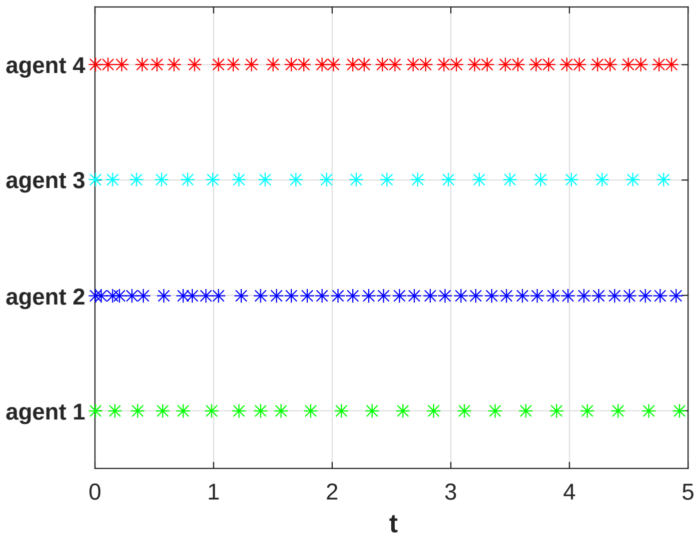


(b)

Figure 3.5: (a) The state evolutions of the multi-agent system (3.1)–(3.2) under the dynamic triggering law (3.27). (b) The triggering times for each agent.



(a)



(b)

Figure 3.6: (a) The state evolutions of the multi-agent system (3.1)–(3.2) when performing Algorithm 3.1. (b) The triggering times for each agent.

Chapter 4

Multi-agent systems with input saturation

In almost all real applications, actuators have bounds. However, there are few event-triggered papers that take saturation into consideration. In fact, even for a single-agent system with input saturation and event-triggered control, the stability problem is challenging. [99] addresses the influence of actuator saturation on event-triggered control. [100] studies a global stabilization of multiple integrator system using event-triggered bounded control. Consensus problem with input saturation and event-triggered control is challenging since the constraints lead to nonlinearities in the closed-loop dynamics. [67] proposes a distributed event-triggered control strategy to achieve consensus for multi-agent systems subject to input saturation through output feedback. Different from this chapter, the underlying graph they consider is undirected and they do not exclude Zeno behavior in their analysis. [74] investigates the event-triggered semi-global consensus problem for general linear multi-agent systems subject to input saturation. However, the underlying graph is assumed to be undirected and in order to determine the triggering times, each agent needs to continuously measure its neighbors' states, i.e., continuous communication is still needed.

In this chapter, we solve the (global) consensus problem for multi-agent systems with input saturation over digraphs. More specifically, we first show that the multi-agent systems achieve consensus if and only if the digraph has a directed spanning tree. In other words, the existence of a directed spanning tree is a necessary and sufficient condition for consensus for both multi-agent systems with and without input saturation, despite that the saturation gives rise to a more complex nonlinear dynamic behavior. We then consider event-triggered control and propose a distributed triggering law, which leads to consensus under the same necessary and sufficient directed spanning tree condition. By distributed, we mean that the event-triggered control input together with the triggering law do not require any a priori knowledge of global network parameters. The triggering law is a special kind of dynamic triggering law, and is free from Zeno behavior, and is inspired by the Lyapunov function we use in the proof of the first consensus result. The Lyapunov function is different from the one in [58, 59]. As a result, continuous broadcasting, receiving, and updating are avoided. However, continuous sensing is needed since each agent has to continuously monitor the triggering law and continuous listening is also needed since the triggering times are determined during runtime and not known in advance. Then, inspired by the idea of self-triggered algorithm in Section 3.3, we also propose one self-triggered algorithm to avoid continuous sensing and listening.

The remainder of this chapter is organized as follows. Section 4.1 reviews the consensus problem for the first-order continuous-time multi-agent systems with input saturation. Section 4.2 shows that the underlying digraph having a directed spanning tree is a necessary and sufficient condition for consensus. Section 4.3 and Section 4.4 use event-triggered and self-triggered control to solve the same problem, respectively. Simulations are given in Section 4.5. The chapter is concluded in Section 4.6. Section 4.7 gives the proof of the main results.

4.1 Problem formulation

We consider a set of n agents modeled as single integrators with input saturation:

$$\dot{x}_i(t) = \mathrm{sat}_h(u_i(t)),\ i \in \mathcal{I},\ t \geq 0, \tag{4.1}$$

where $x_i(t) \in \mathbb{R}^p$ is the state and $u_i(t) \in \mathbb{R}^p$ is the control input of agent i, respectively, $p > 0$ is the state dimension, and $\mathrm{sat}_h(\cdot)$ is the saturation function defined in (1.16).

Definition 4.1. *We say consensus*[1] *for the multi-agent system (4.1) is achieved if*

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0,\ \forall i, j \in \mathcal{I},\ \forall x_l(0) \in \mathbb{R}^p,\ l \in \mathcal{I}.$$

We again consider the distributed consensus protocol

$$u_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t), \tag{4.2}$$

where L_{ij} is the element of the Laplacian matrix L. In this chapter, we assume that the underlying graph $\mathcal{G}$ is directed.

Our first goal in this chapter is to solve the following problem.

Problem 4.1. *Prove that consensus for the multi-agent system (4.1) is achieved if and only if the digraph $\mathcal{G}$ has a directed spanning tree.*

Remark 4.1. *For the ease of presentation, we focus on the case where all the agents have the same saturation level. The analysis can be readily extended to the case where the agents have different saturation levels.*

The following properties about the saturation function are useful for our analysis.

Lemma 4.1. *For any real constants a and b,*

$$\frac{1}{2}a^2 \geq \int_0^a sat_h(s)ds \geq \frac{1}{2}(sat_h(a))^2,\ \text{and } (a-b)^2 \geq (sat_h(a) - sat_h(b))^2.$$

[1] In some literatures, such consensus is also referred to as global consensus.

Lemma 4.2. *Suppose that L is the Laplacian matrix associated with a digraph $\mathcal{G}$ that has a directed spanning tree. For $x_1, \dots, x_n \in \mathbb{R}^p$, define $\pi_i = sat_h(-\sum_{j=1}^n L_{ij}x_j)$. Then $\pi_1 = \dots = \pi_n$ if and only if $x_1 = \dots = x_n$.*

Proof. The sufficiency is straightforward. Let us show the necessity. Let $\mu_i = -\sum_{j=1}^n L_{ij}x_j$. From $\pi_1 = \dots = \pi_n$, we know that for any $l = 1, \dots, p$, $c_l(\mu_i) > 0,\ \forall i \in \mathcal{I}$, or $c_l(\mu_i) < 0,\ \forall i \in \mathcal{I}$, or $c_l(\mu_i) = 0,\ \forall i \in \mathcal{I}$, where $c_l(\mu_i)$ is the l-th component of μ_i.

From Lemma 2 in [58], we know that neither $c_l(\mu_i) > 0,\ \forall i \in \mathcal{I}$ nor $c_l(\mu_i) < 0,\ \forall i \in \mathcal{I}$ holds. Thus $-\sum_{j=1}^n L_{ij}c_l(x_j) = c_l(\mu_i) = 0,\ \forall i \in \mathcal{I}$. From Lemma 2.1, we know $\text{rank}(L) = n-1$. Thus, we have $c_l(x_i) = c_l(x_j),\ \forall i, j \in \mathcal{I}$. Hence $x_1 = \dots = x_n$. □

4.2 Consensus

In this section, we will show that consensus is achieved even in the presence of input saturation if $\mathcal{G}$ has a directed spanning tree. The mathematical analysis is inspired by [101].

In the following, we show a necessary and sufficient condition to consensus for system (4.1)–(4.2).

Theorem 4.1. *Consider the multi-agent system (4.1)–(4.2). Consensus is achieved if and only if the digraph $\mathcal{G}$ has a directed spanning tree.*

The necessity in Theorem 4.1 is a direct result of Lemma 2.3. We illustrate the main idea of the proof of sufficiency here, while the detailed proof is given in Appendix 4.7.1. We first consider the case where $\mathcal{G}$ is strongly connected, i.e., $M = 1$ in (2.2), and show that consensus is achieved. We next consider the case $\mathcal{G}$ has a directed spanning tree but it is not strongly connected, i.e., $M \geq 2$. From the first case ($M = 1$), it follows that all agents in SCC_M achieve consensus since SCC_M is either strongly connected or of dimension one. Then, we consider SCC_{M-1} and note that all agents in SCC_{M-1}, which is either strongly connected or of dimension one, achieve the same consensus value as those in SCC_M, since the agents in SCC_M and SCC_{M-1} are not influenced by $\text{SCC}_1, \dots, \text{SCC}_{M-2}$ and the consensus problem of this subsystem can be treated as a leader–follower problem where agents in SCC_M are leaders and agents in SCC_{M-1} are followers. Notice that $\text{SCC}_1, \dots, \text{SCC}_{M-2}$, are either strongly connected or of dimension one. By applying a similar analysis, consensus of $\text{SCC}_m, \text{SCC}_{m+1}, \dots, \text{SCC}_M$ can be treated as a leader–follower consensus problem with agents in $\text{SCC}_M, \text{SCC}_{M-1}, \dots, \text{SCC}_{m+1}$ being leaders and agents in SCC_m being followers. Therefore, the result follows.

Remark 4.2. *The proof of Theorem 4.1 is based on the Lyapunov function*

$$V(x(t)) = \sum_{i=1}^{n} \xi_i \sum_{l=1}^{p} \int_0^{-\sum_{j=1}^n L_{ij}c_l(x_j(t))} sat_h(s)ds, \tag{4.3}$$

where $x(t) = [x_1^\top(t), \dots, x_n^\top(t)]^\top$ and $\xi^\top = [\xi_1, \dots, \xi_n]$ was defined in Lemma 2.1. It is different from the one used in [58]. In addition, our Lyapunov function facilitates the design of event-triggered control as shown in Section 4.3.

Remark 4.3. *When $h \to \infty$, i.e., the multi-agent system is free from saturation, Theorem 4.1 corresponds to the well known result for the consensus problem of multi-agent systems without saturation [36, 37]. The main differences between the case with and without saturation are the convergence speed and the consensus value. For the saturated case, the convergence speed is slower and the consensus value is not fully determined by the Laplacian matrix L and the initial states of the agents. From the proof of Theorem 4.1, we know that the saturation is no longer active after a finite time $T_2 \geq 0$ which depends on the initial value of each agent, the saturation level, and the network topology. Thus after T_2 the convergence speed is exponential and the consensus value is determined by the state of each agent at T_2.*

4.3 Event-triggered control

To avoid continuous exchange of information among agents and update of actuators, we equip the consensus protocol (4.2) with an event-triggered communication scheme. The control signal is only updated when the triggering condition is satisfied. It results in the following multi-agent system with input saturation and event-triggered control input

$$\dot{x}_i(t) = \text{sat}_h(\hat{u}_i(t)),\ i \in \mathcal{I},\ t \geq 0, \tag{4.4}$$

$$\hat{u}_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t^j_{k_j(t)}). \tag{4.5}$$

Note that the consensus protocol (4.5) only updates at the triggering times and is constant between two consecutive triggering times. For simplicity, let $\hat{x}_i(t) = x_i(t^i_{k_i(t)})$, and $e_i(t) = \hat{x}_i(t) - x_i(t)$.

Our second goal in this chapter is to solve the following problem.

Problem 4.2. *Propose methods to determine the triggering times such that consensus is reached, while continuous exchange of information, continuous update of actuators, and Zeno behavior are avoided.*

This problem is solved by the following theorem.

Theorem 4.2. *Consider the multi-agent system (4.4)–(4.5). Given $\alpha_i > 0$, $\beta_i > 0$ and the first triggering time $t^i_1 = 0$, agent i determines the triggering times $\{t^i_k\}^{\infty}_{k=2}$ by*

$$t^i_{k+1} = \min\left\{t :\ \|e_i(t)\|^2 \geq \alpha_i e^{-\beta_i t},\ t \geq t^i_k\right\}. \tag{4.6}$$

Then, (i) there is no Zeno behavior; and (ii) consensus is achieved if and only if the underlying digraph $\mathcal{G}$ has a directed spanning tree.

The proof is given in Section 4.7.2.

Remark 4.4. *The event-triggered control input (4.5) together with the triggering law (4.6) is fully distributed. That is, each agent only requires its own state information and its neighbors' state information, without any a priori knowledge of any global parameter, such as the eigenvalue of the Laplacian matrix. This is different from [18, 66].*

4.4 Self-triggered algorithm

When performing the event-triggered control input (4.5) together with the triggering law (4.6), each agent needs to broadcast its state to its neighbors at its triggering times, and to receive and to update its input at its neighbors' triggering times. Thus, continuous broadcasting, receiving, and updating are avoided. However, continuous sensing is needed since each agent has to continuously monitor the triggering law and continuous listening is also needed since the triggering times are determined during runtime and not known in advance. Inspired by the idea of self-triggered algorithm in Section 3.3, if each agent can predict its next triggering time and broadcast it to its neighbors at the current triggering time, then each agent only needs to sense and broadcast at its own triggering times, and to listen to and receive incoming information from its neighbors at their triggering times. In the following we will propose a self-triggered algorithm such that at time t^i_k each agent i could estimate t^i_{k+1}. The idea is illustrated as follows.

From $\dot{x}_i(t) = \mathrm{sat}_h(\hat{u}_i(t))$, we have

$$x_i(t) = x_i(t^i_k) + \int_{t^i_k}^{t} \mathrm{sat}_h(\hat{u}_i(s))ds, t \in [t^i_k, t^i_{k+1}]. \tag{4.7}$$

Thus for $t \in [t^i_k, t^i_{k+1})$, we have

$$\|e_i(t)\| = \|x_i(t^i_k) - x_i(t)\| = \left\| \int_{t^i_k}^{t} \mathrm{sat}_h(\hat{u}_i(s))ds \right\|. \tag{4.8}$$

Here we need to highlight that $\mathrm{sat}_h(\hat{u}_i(t))$ may be not a constant vector for all $t \in [t^i_k, t^i_{k+1})$ since $x_j(t^j_{k_j(t)})$ may be not a constant vector for all $t \in [t^i_k, t^i_{k+1})$ which is due to that agent j may trigger at some time instants in this interval. So at time t^i_k we do not know what is the value of $\|e_i(t)\|$ for all $t \in (t^i_k, t^i_{k+1})$. However, we know $\mathrm{sat}_h(\hat{u}_i(t))$ is a constant vector for $t \in [t^i_k, T^1_i(t^i_k))$, where

$$T^1_i(t^i_k) = \min\left\{t^i_{k_j(t^i_k)+1}, j \in \mathcal{N}^{in}_i\right\}, \tag{4.9}$$

i.e., $T^1_i(t^i_k)$ is the first triggering time of all agent i's neighbors after time t^i_k. Although, at time t^i_k, agent i does not know $\mathrm{sat}_h(\hat{u}_i(t))$ for $t > T^1_i(t^i_k)$, it knows $|c_l(\mathrm{sat}_h(\hat{u}_i(t)))| \le h$, $l = 1 \dots, p$. Hence

$$\|e_i(t)\| = \left\| \int_{t^i_k}^{t} \mathrm{sat}_h(\hat{u}_i(s))ds \right\| = \left\| \int_{t^i_k}^{T^2_i(t)} \mathrm{sat}_h(\hat{u}_i(s))ds + \int_{T^2_i(t)}^{t} \mathrm{sat}_h(\hat{u}_i(s))ds \right\| \le \varrho_i(t), \tag{4.10}$$

where

$$T^2_i(t) = \min\left\{T^1_i(t^i_k), t\right\}, \text{ for } t \in [t^i_k, t^i_{k+1}), \tag{4.11}$$

Algorithm 4.1

1: Agent $i \in \mathcal{I}$ chooses $\alpha_i > 0$ and $\beta_i > 0$;
2: Initialize $t_1^i = 0$ and $k = 1$;
3: At time $s = t_k^i$, agent i senses $x_i(t_k^i)$, and updates $u_i(t_k^i)$ by (4.5), and determines t_{k+1}^i by (4.13)[1], and broadcasts its triggering information $\{t_{k+1}^i, x_i(t_k^i)\}$ to its neighbors;
4: At agent i's neighbors' triggering times which are between $[t_k^i, t_{k+1}^i]$, agent i receives triggering information for its neighbors[2] and updates its $u_i(\cdot)$ by (4.5);
5: resets $k = k + 1$, and goes back to Step 3.

and

$$\varrho_i(t) = (T_i^2(t) - t_k^i)\|\mathrm{sat}_h(\hat{u}_i(t_k^i))\| + (t - T_i^2(t))h\sqrt{p}, \text{ for } t \in [t_k^i, t_{k+1}^i). \tag{4.12}$$

Then, a necessary condition to guarantee the inequality in (4.6), i.e.,

$$\|e_i(t)\|^2 \geq \alpha_i e^{-\beta_i t}, \ \forall t \in [t_k^i, t_{k+1}^i),$$

holds is

$$\varrho_i(t) \geq \sqrt{\alpha_i} e^{-\frac{\beta_i}{2}t}, \ \forall t \in [t_k^i, t_{k+1}^i).$$

Since $\sqrt{\alpha_i} e^{-\frac{\beta_i}{2}t}$ decreases with respect to t, $\varrho_i(t)$ increases with respect to t during $[t_k^i, t_{k+1}^i)$ and $\varrho_i(t_k^i) = 0$, then given t_k^i, agent i can estimate t_{k+1}^i by solving

$$\varrho_i(t) = \sqrt{\alpha_i} e^{-\frac{\beta_i}{2}t}, \ t \geq t_k^i. \tag{4.13}$$

In other words, if at time t_k^i agent i knows $t_{k_j(t_k^i)}^j$, $t_{k_j(t_k^i)+1}^j$, $x_j(t_{k_j(t_k^i)}^j)$, $\forall j \in \mathcal{N}_i$, then it can estimate its next triggering time t_{k+1}^i by solving (4.13). The above implement idea is summarized in Algorithm 4.1.

The following theorem shows that consensus is achieved and there is no Zeno behavior when every agent performs Algorithm 4.1.

Theorem 4.3. *Consider the multi-agent system (4.4)–(4.5). If all agents perform Algorithm 4.1, then, (i) there is no Zeno behavior; and (ii) consensus is achieved if and only if the underlying digraph $\mathcal{G}$ has a directed spanning tree.*

Proof. The method of the exclusion of Zeno behavior is similar to the way in the proof of Theorem 4.2. Under Algorithm 4.1, we have $\|e_i(t)\|^2 \leq \alpha_i e^{-\beta_i t}$ for all $i \in \mathcal{I}$ and $t \geq 0$. Then from Theorem 4.2, we know that consensus is achieved. □

Remark 4.5. *In order to perform Algorithm 4.1, no global parameters are used, i.e., Algorithm 4.1 is distributed.*

[1] Agent i uses $t_{k_j(t_k^i)}^j$ to replace $t_{k_j(t_k^i)+1}^j$ to determine t_{k+1}^i by (4.13) when $t_k^i = t_{k_j(t_k^i)}^j$.

[2] In other words, agent i only listen to incoming information at its neighbors' triggering times. Thus continuous listening is avoided.

4.5 Simulations

In this section, simulations are given to demonstrate the theoretical results. Consider again the digraph and the corresponding multi-agent system in Figure 2.1. Let the saturation level be $h = 10$. We choose an arbitrary initial state $x(0) = [6.2945, 8.1158, -7.4603, 8.2675, 2.6472, -8.0492, -4.4300]^\top$.

Figure 4.1 (a) shows the state evolutions of the multi-agent system (4.1)–(4.2) and Figure 4.1 (b) shows the saturated input of each agent. We see that consensus is achieved, even if some agents are saturated initially.

We next consider the case with event-triggered control input. Figure 4.2 (a) shows the state evolutions of the multi-agent system (4.4)–(4.5) under the triggering law (4.6) with $\alpha_i = 10$ and $\beta_i = 1$. Figure 4.2 (b) shows the saturated input of each agent. Figure 4.3 shows the corresponding triggering times for each agent. We see that consensus is achieved also in this case. Moreover, from Figure 4.3, we see that each agent only needs to broadcast its state to its neighbors at its triggering times. Thus continuous broadcasting and receiving are avoided.

Figure 4.4 (a) shows the state evolutions of the multi-agent system (4.4)–(4.5) when each agent performs Algorithm 4.1 with $\alpha_i = 10$ and $\beta_i = 1$. Figure 4.4 (b) shows the saturated input of each agent. Figure 4.5 shows the corresponding triggering times for each agent. From Figure 4.4 (a) and (b), we see that consensus is achieved and $\text{sat}_h(u_i(t))$ is within the saturation level. Moreover, from Figure 4.5, we see that each agent only needs to sense and broadcast at its triggering times. Thus continuous sensing, broadcasting, receiving, and listening are avoided. Note however that both the event-triggered control and self-triggered control give rise to a less smooth state evolutions because of the large variability in the control action.

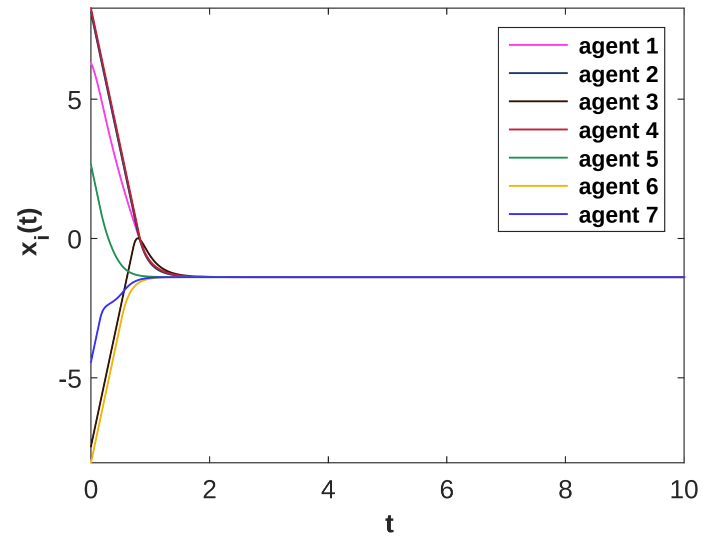
4.6 Summary

In this chapter, we studied consensus problem for multi-agent systems with input saturation constraints over digraphs. We showed that consensus is achieved if and only if the underlying directed communication topology has a directed spanning tree by using a Laypunov function. Moreover, we considered event-triggered control and presented a distributed triggering law and a self-triggered algorithm to reduce the overall need of communication and system updates. We showed that consensus is still achieved under the same connectivity condition. Furthermore, Zeno behavior was excluded. Future research directions include considering more general systems such as double integrator systems and comparing the convergence speed between the saturation and non-saturation cases.

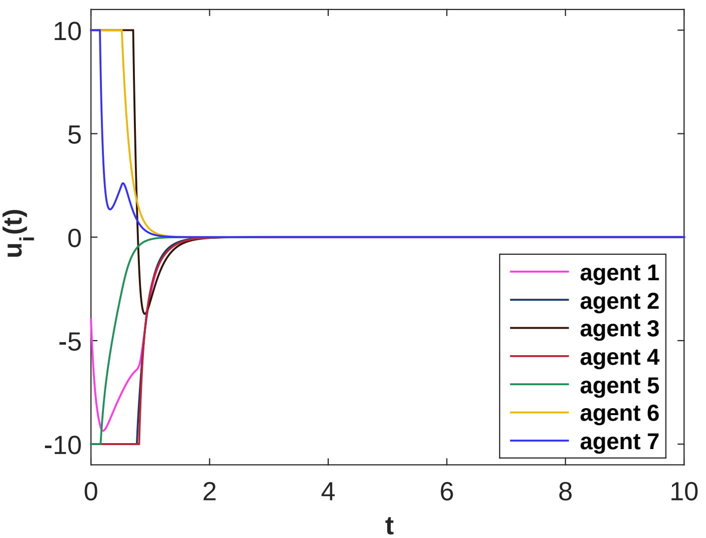
4.7 Appendices

4.7.1 Proof of sufficiency of Theorem 4.1

The proof of sufficiency follows the structure outlined after the theorem stated in Section 4.2. More specifically, we first show consensus for the case where $M = 1$ in (2.2) which

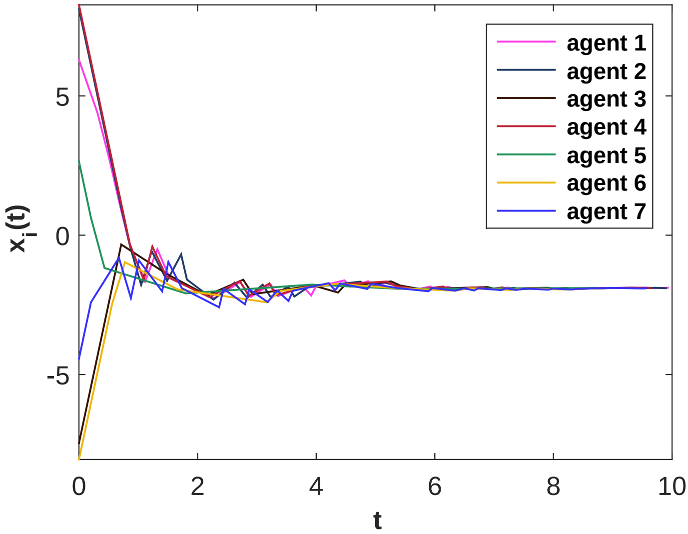


(a)

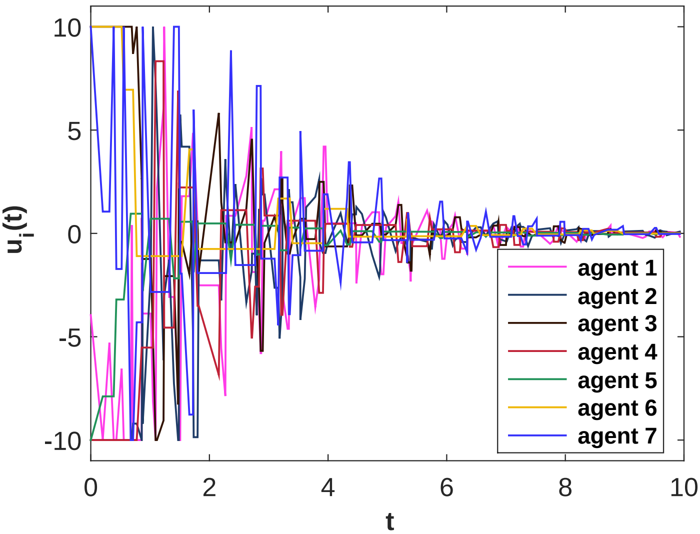


(b)

Figure 4.1: (a) The state evolutions of the multi-agent system (4.1)–(4.2). (b) The saturated input of each agent.



(a)



(b)

Figure 4.2: (a) The state evolutions of the multi-agent system (4.4)–(4.5) under the triggering law (4.6). (b) The saturated input of each agent.

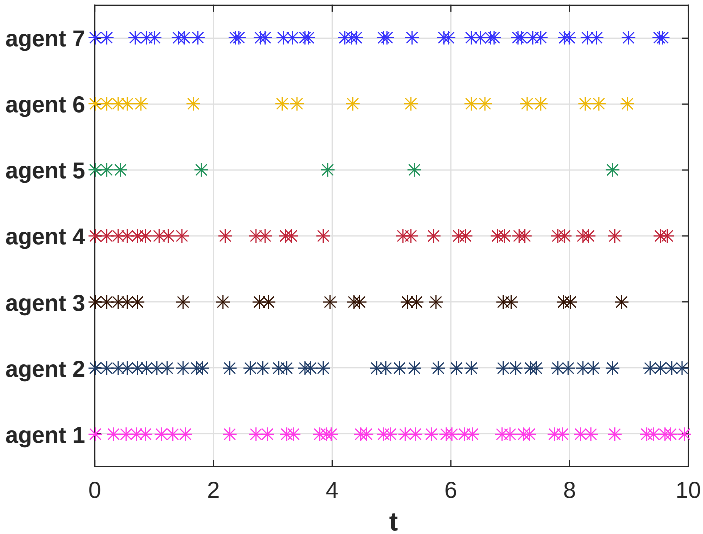


Figure 4.3: The triggering times for each agent in the multi-agent system (4.4)–(4.5) under the triggering law (4.6).

corresponds to only one SCC. Then, we consider the case $M = 2$ in (2.2), and show that the agents in SCC_1 and SCC_2 reach consensus. We finally argue that the general case where $M > 2$ follows in a similar way.

(i) In this part, we consider the situation where $\mathcal{G}$ is strongly connected, i.e., $M = 1$ in (2.2).

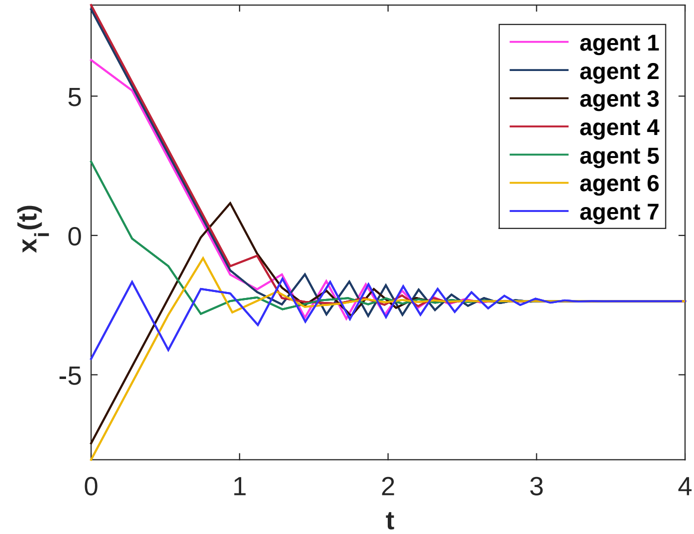
We first prove that consensus is achieved. Consider the Lyapunov candidate (4.3) introduced in Remark 4.2. From Lemma 2.1, we have $\xi_i > 0, i \in \mathcal{I}$, since $\mathcal{G}$ is strongly connected. From Lemma 4.1, we know that

$$V_{il}(x(t)) := \int_0^{-\sum_{j=1}^n L_{ij}c_l(x_j(t))} \text{sat}_h(s)ds \geq 0,$$

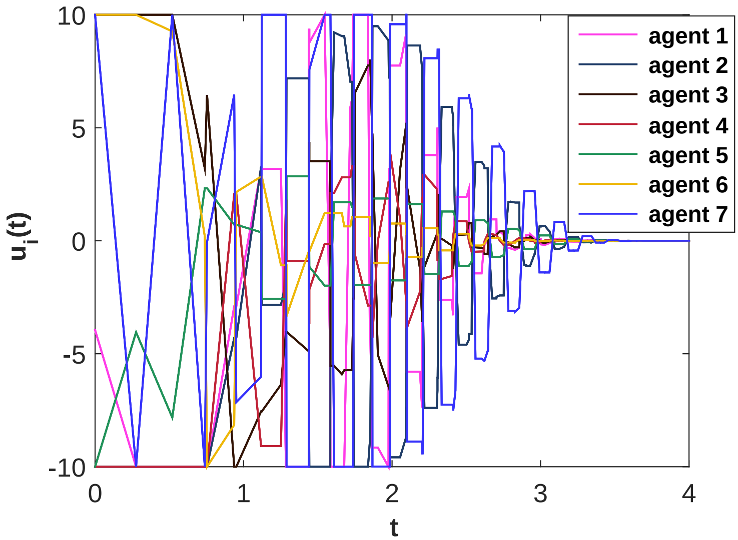
and $V_{il}(x(t)) = 0$ if and only if $-\sum_{j=1}^n L_{ij}c_l(x_j(t)) = 0$. Thus, we know that

$$V(x(t)) = \sum_{i=1}^n \xi_i \sum_{l=1}^p V_{il}(x) \geq 0,$$

and $V(x(t)) = 0$ if and only if $-\sum_{j=1}^n L_{ij}c_l(x_j(t)) = 0$ for all $i \in \mathcal{I}$ and $l = 1, \ldots, p$. This is furthermore equivalent to $x_1(t) = \cdots = x_n(t)$ since $\text{rank}(L) = n - 1$. Hence, we have $V(x(t)) \geq 0$ and $V(x(t)) = 0$ if and only if $x_1(t) = \cdots = x_n(t)$.



(a)



(b)

Figure 4.4: (a) The state evolutions of the multi-agent system (4.4)–(4.5) when each agent performs Algorithm 4.1. (b) The saturated input of each agent.

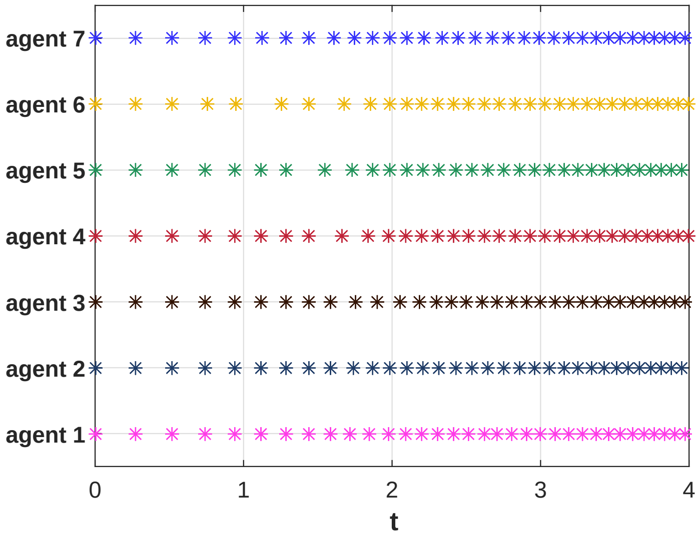


Figure 4.5: The triggering times for each agent in the multi-agent system (4.4)–(4.5) when each agent performs Algorithm 4.1.

The derivative of $V(x)$ along the trajectories of (4.1)–(4.2) is

$$\begin{aligned}\dot{V}(x(t)) &= \sum_{i=1}^{n} \xi_i \sum_{l=1}^{p} [\text{sat}_h(-\sum_{j=1}^{n} L_{ij} c_l(x_j(t)))][-\sum_{j=1}^{n} L_{ij} c_l(\dot{x}_j(t))] \\ &= \sum_{i=1}^{n} \xi_i \sum_{l=1}^{p} [\text{sat}_h(c_l(u_i(t)))][-\sum_{j=1}^{n} L_{ij}\text{sat}_h(c_l(u_j(t)))] \\ &= \sum_{i=1}^{n} \xi_i [\text{sat}_h(u_i(t))]^\top \sum_{j=1}^{n} -L_{ij}\text{sat}_h(u_j(t)) \\ &= -\sum_{i=1}^{n} \xi_i q_i^s(t),\end{aligned} \tag{4.14}$$

where

$$q_i^s(t) = -\frac{1}{2}\sum_{j=1}^{n} L_{ij}\|\text{sat}_h(u_j(t)) - \text{sat}_h(u_i(t))\|^2 \geq 0,$$

and the last equality of (4.14) holds since

$$-\sum_{i=1}^{n} \xi_i q_i^s(t) = \sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{n} \xi_i L_{ij}\|\text{sat}_h(u_j(t)) - \text{sat}_h(u_i(t))\|^2$$

$$
\begin{aligned}
=&\sum_{i=1}^{n}\frac{1}{2}\sum_{j=1}^{n}\xi_i L_{ij}\Big[\|\mathrm{sat}_h(u_j(t))\|^2+\|\mathrm{sat}_h(u_i(t))\|^2\Big]\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i L_{ij}[\mathrm{sat}_h(u_j(t))]^\top\mathrm{sat}_h(u_i(t))\\
=&\frac{1}{2}\sum_{j=1}^{n}\|\mathrm{sat}_h(u_j(t))\|^2\sum_{i=1}^{n}\xi_i L_{ij}+\frac{1}{2}\sum_{i=1}^{n}\xi_i\|\mathrm{sat}_h(u_i(t))\|^2\sum_{j=1}^{n}L_{ij}\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i L_{ij}[\mathrm{sat}_h(u_j(t))]^\top\mathrm{sat}_h(u_i(t))\\
=&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i L_{ij}[\mathrm{sat}_h(u_j(t))]^\top\mathrm{sat}_h(u_i(t)),
\end{aligned}
\tag{4.15}
$$

where we have used $\xi^\top L=0$ and $L\mathbf{1}_n=0$ in (4.15).

From (4.14), we know that $\dot V(x(t))\le 0$ and $\dot V(x(t))=0$ if and only if $\mathrm{sat}_h(u_i(t))=\mathrm{sat}_h(u_j(t)),\ \forall i,j\in\mathcal{I}$. It follows from Lemma 4.2 that, this is equivalent to $x_i(t)=x_j(t),\ \forall i,j\in\mathcal{I}$. Thus, by LaSalle Invariance Principle [102], we have

$$
\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0,\ \forall i,j\in\mathcal{I},
\tag{4.16}
$$

i.e., consensus is achieved.

We next show that the input of each agent enters into the saturation level in finite time.

Since $-\sum_{j=1}^{n}L_{ij}c_l(x_j(t)),\ i\in\mathcal{I},\ l=1,\dots,p$ are continuous with respect to t, it then follows from (4.16) that there exists a constant $T_1\ge 0$ such that

$$
|c_l(u_i(t))|=\Big|-\sum_{j=1}^{n}L_{ij}c_l(x_j(t))\Big|\le h,\forall t\ge T_1.
$$

In other words the saturation function in (4.1) is not active after T_1. Thus,

$$
\dot x_i(t)=-\sum_{j=1}^{n}L_{ij}x_j(t),t\ge T_1.
\tag{4.17}
$$

Finally, we estimate the convergence speed, which will be used later. Consider the following function

$$
\tilde V(x(t))=\frac{1}{2}x^\top(t)(U\otimes I_p)x(t).
\tag{4.18}
$$

From Lemma 2.2, we know that $\tilde V(x(t))\ge 0$. The derivative of $\tilde V(x(t))$ along the trajectories of system (4.17) satisfies

$$
\dot{\tilde V}(x(t))=x^\top(t)(U\otimes I_p)\dot x(t)
$$

$$
\begin{aligned}
&= x^\top(t)(U \otimes I_p)(-L \otimes I_p)x(t) = -x^\top(t)(R \otimes I_p)x(t) \\
&\le -\frac{\rho_2(R)}{\rho(U)} x^\top(t)(U \otimes I_p)x(t) = -2\frac{\rho_2(R)}{\rho(U)}\tilde{V}(x(t)),\ \forall t \ge T_1.
\end{aligned}
$$

Thus

$$
\tilde{V}(x(t)) \le \tilde{V}(x(T_1))e^{-2\frac{\rho_2(R)}{\rho(U)}(t-T_1)},\ \forall t \ge T_1.
$$

Noting that $\tilde{V}(x(t))$ is continuous with respect to t, there exists a positive constant C_1 such that

$$
\tilde{V}(x(t)) \le C_1,\ \forall t \in [0, T_1].
$$

Then

$$
\tilde{V}(x(t)) \le C_2 e^{-2\frac{\rho_2(R)}{\rho(U)}t},\ \forall t \ge 0, \tag{4.19}
$$

where $C_2 = \max\left\{\tilde{V}(x(T_1)), C_1 e^{2\frac{\rho_2(R)}{\rho(U)}T_1}\right\}$.

Moreover, from Lemma 2.2, we know that

$$
\begin{aligned}
\sum_{j=1}^{n} \|u_j(t)\|^2 &= x^\top(t)(L^\top L \otimes I_p)x(t) \\
&\le \frac{\rho(L^\top L)}{\rho_2(U)} x^\top(t)(U \otimes I_p)x(t) = 2\frac{\rho(L^\top L)}{\rho_2(U)}\tilde{V}(x(t)) \\
&\le 2\frac{\rho(L^\top L)}{\rho_2(U)} C_2 e^{-2\frac{\rho_2(R)}{\rho(U)}t},\ \forall t \ge 0. \qquad (4.20)
\end{aligned}
$$

(ii) In this part, we consider the case where $M \ge 2$, but we first introduce some notations which will be used later.

Let $N_0 = 0$, $N_l = \sum_{m=1}^{l} n_m$, $l = 1, \ldots, M$, where n_m is the dimension of $L^{m,m}$. Then the i-th agent in SCC$_m$ is the $N_{m-1} + i$-th agent of the whole graph. In the following, we exchangeably use v_i^m and $v_{N_{m-1}+i}$ to denote this agent. Accordingly, denote $x_i^m(t) = x_{N_{m-1}+i}(t)$, $\hat{x}_i^m(t) = \hat{x}_{N_{m-1}+i}(t)$, $u_i^m(t) = u_{N_{m-1}+i}(t)$ and define $u^m(t) = [(u_1^m)^\top(t), \ldots, (u_{n_m}^m)^\top(t)]^\top$.

In the following we only consider the case where $M = 2$. The case where $M > 2$ can be treated in a similar manner, as discussed in the proof sketch in Section 4.2.

First, note that the agents in SCC$_2$ do not depend on any agents in SCC$_1$. Thus, SCC$_2$ can be treated as a strongly connected digraph. Then, from the analysis in (i), we have

$$
\lim_{t\to+\infty} \|x_i^2(t) - x_j^2(t)\| = 0,\ i, j = 1\ldots, n_2,
$$

and that there exists a constant $T_2 \ge 0$ such that

$$
|c_l(u_i^2(t))| = \left| -\sum_{j=1}^{n_2} L_{ij}^{2,2} c_l(x_j^2(t)) \right| \le h,\ \forall t \ge T_2. \tag{4.21}
$$

In addition, similar to (4.20), we have

$$\|u^2(t)\|^2 = \sum_{j=1}^{n_2} \|u_j^2(t)\|^2 \le C_3 e^{-C_4 t},\ t \ge 0,$$

where C_3 and C_4 are two positive constants.

Second, let us consider SCC_1. Similar to $V(x)$ defined in (4.3), define

$$V_1(x(t)) = \sum_{i=1}^{n_1} \xi_i^1 \sum_{l=1}^{p} \int_0^{c_l(u_i^1(t))} \text{sat}_h(s) ds, \tag{4.22}$$

$$V_2(x(t)) = \sum_{i=1}^{n_2} \xi_i^2 \sum_{l=1}^{p} \int_0^{c_l(u_i^2(t))} \text{sat}_h(s) ds. \tag{4.23}$$

From the definition of the component operator $c_l(\cdot)$, we know $c_l(u_i^1(t)) = -\sum_{j=1}^{n_1} L_{ij}^{1,1} c_l(x_i^1(t)) - \sum_{j=1}^{n_2} L_{ij}^{1,2} c_l(x_i^2(t))$ and $c_l(u_i^2(t)) = -\sum_{j=1}^{n_2} L_{ij}^{2,2} c_l(x_i^2(t))$. From Lemma 4.1, we have $V_1(x) \ge 0$ and $V_2(x) \ge 0$.

Similar to (4.14),

$$\dot{V}_2(x(t)) = \sum_{i=1}^{n_2} -\xi_i^2 q_i^2(t),$$

where

$$q_i^2(t) = -\frac{1}{2} \sum_{j=1}^{n} L_{ij}^{2,2} \|\text{sat}_h(u_j^2(t)) - \text{sat}_h(u_i^2(t))\|^2 \ge 0.$$

Moreover, similar to the analysis of $\dot{V}(x(t))$ in (i), we know that $\dot{V}_2(x(t)) = 0$ if and only if $x_i^2(t) = x_j^2(t), \forall i, j = 1, \dots, n_2$.

The derivative of $V_1(x(t))$ along the trajectories of (4.1)–(4.2) satisfies

$$\begin{aligned}
\dot{V}_1(x(t)) &= \sum_{i=1}^{n_1} \xi_i^1 \sum_{l=1}^{p} \text{sat}_h(c_l(u_i^1(t))) c_l(\dot{u}_i^1(t)) \\
&= \sum_{i=1}^{n_1} \xi_i^1 \sum_{l=1}^{p} c_l(\text{sat}_h(u_i^1(t))) \Big[-\sum_{j=1}^{n_1} L_{ij}^{1,1} c_l(\text{sat}_h(u_j^1(t))) - \sum_{j=1}^{n_2} L_{ij}^{1,2} c_l(\text{sat}_h(u_j^2(t))) \Big] \\
&= \sum_{i=1}^{n_1} \xi_i^1 [\text{sat}_h(u_i^1(t))]^\top \Big[-\sum_{j=1}^{n_1} L_{ij}^{1,1} \text{sat}_h(u_j^1(t)) - \sum_{j=1}^{n_2} L_{ij}^{1,2} \text{sat}_h(u_j^2(t)) \Big] \\
&= -[\text{sat}_h(u^1(t))]^\top (Q^1 \otimes I_p) \text{sat}_h(u^1(t)) - \sum_{i=1}^{n_1} \xi_i^1 [\text{sat}_h(u_i^1(t))]^\top \sum_{j=1}^{n_2} L_{ij}^{1,2} \text{sat}_h(u_j^2(t)) \\
&\le -\rho_2(Q^1) \|\text{sat}_h(u^1(t))\|^2 + \frac{\rho_2(Q^1)}{2} \sum_{i=1}^{n_1} \|\text{sat}_h(u_i^1(t))\|^2
\end{aligned}$$

$$+\frac{1}{2\rho_2(Q^1)}\sum_{i=1}^{n_1}\left\|\xi_i^1\sum_{j=1}^{n_2}L_{ij}^{1,2}\mathrm{sat}_h(u_j^2(t))\right\|^2$$

$$\leq -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(u^1(t))\|^2 + \frac{n_1n_2\max\{(L_{ij}^{1,2})^2\}}{2\rho_2(Q^1)}\|\mathrm{sat}_h(u^2(t))\|^2$$

$$\leq -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(u^1(t))\|^2 + \frac{n_1n_2\max\{(L_{ij}^{1,2})^2\}}{2\rho_2(Q^1)}C_3e^{-C_4t},\ t\geq 0,$$

where the first inequality holds since $Q^1 > 0$ which could be found in Lemma 2.4.

Let us treat $y_i(t) = e^{-C_4t}, t \geq 0, i \in \mathcal{I}$, as an additional state of each agent, and let $y(t) = [y_1(t), \ldots, y_n(t)]^\top$. Consider a Lyapunov candidate:

$$V_3(x(t), y(t)) = V_1(x(t)) + V_2(x(t)) + \frac{2n_1n_2\max\{(L_{ij}^{1,2})^2\}}{2\rho_2(Q^1)C_4n}C_3\sum_{i=1}^{n}y_i(t).$$

The derivative of $V_3(t)$ along the trajectories of (4.1)–(4.2) is

$$\dot{V}_3(x(t), y(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) - \frac{2n_1n_2\max\{(L_{ij}^{1,2})^2\}}{2\rho_2(Q^1)n}C_3\sum_{i=1}^{n}y_i(t).$$

Then, we have

$$\dot{V}_3(x(t), y(t)) \leq -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(u^1(t))\|^2 + \sum_{i=1}^{n_2}-\xi_i^2q_i^2(t) - \frac{n_1n_2\max\{(L_{ij}^{1,2})^2\}}{2\rho_2(Q^1)n}C_3\sum_{i=1}^{n}y_i(t),\ t\geq 0.$$

By LaSalle Invariance Principle, similar to the analysis in (i), we have

$$\lim_{t\to\infty}\|x_j(t) - x_i(t)\| = 0,\ \forall i, j \in \mathcal{I}.$$

Thus, consensus is achieved. Moreover, similar to the analysis in (i), we can show that after a finite time $T_2 \geq 0$ the saturation is no longer active.

4.7.2 Proof of Theorem 4.2

(i) Similar to the proof of excluding Zeno behavior in Theorem 3.1, we prove that there is no Zeno behavior by contradiction. Suppose there exists Zeno behavior. Then there exists an agent v_i, such that $\lim_{k\to\infty} t_k^i = T_0$ for some constant T_0. Let $\varepsilon_0 = \frac{\sqrt{\alpha_i}}{2\sqrt{p}h}e^{-\frac{1}{2}\beta_iT_0} > 0$. Then from the property of limits, there exists a positive integer $N(\varepsilon_0)$ such that

$$t_k^i \in [T_0 - \varepsilon_0, T_0],\ \forall k \geq N(\varepsilon_0). \tag{4.24}$$

Also noting $\|\mathrm{sat}_h(s)\| \leq h\sqrt{p}$ for any $s \in \mathbb{R}^p$, we have

$$\|\mathrm{sat}_h(\hat{u}_i(t))\| \leq h\sqrt{p}.$$

Noting

$$\left|\frac{d\|e_i(t)\|}{dt}\right| \le \|\dot{x}_i(t)\| = \|\mathrm{sat}_h(\hat{u}_i(t))\| \le h\sqrt{p},$$

and $\|\hat{x}_i(t_k^i) - x_i(t_k^i)\| = 0$ for any triggering time t_k^i, we conclude that one necessary condition to guarantee $\|e_i(t)\|^2 \ge \alpha_i e^{-\beta_i t}, t \ge t_k^i$ is

$$(t - t_k^i)h\sqrt{p} \ge \sqrt{\alpha_i}e^{-\frac{1}{2}\beta_i t},\ t \ge t_k^i. \tag{4.25}$$

Then

$$t^i_{N(\varepsilon_0)+1} - t^i_{N(\varepsilon_0)} \ge \frac{\sqrt{\alpha_i}}{\sqrt{p}h}e^{-\frac{1}{2}\beta_i t^i_{N(\varepsilon_0)+1}} \ge \frac{\sqrt{\alpha_i}}{\sqrt{p}h}e^{-\frac{1}{2}\beta_i T_0} = 2\varepsilon_0,$$

which contradicts (4.24). Therefore, there is no Zeno behavior.

(ii) (Necessity) Necessity follows from Lemma 2.3.

(Sufficiency) (ii-1) In this part, we consider the situation where $\mathcal{G}$ is strongly connected, i.e., $M = 1$ in (2.2).

We first show that consensus is achieved. Let $f_i(t) = \mathrm{sat}_h(\hat{u}_i(t)) - \mathrm{sat}_h(u_i(t))$. The derivative of $V(x)$, as defined in (4.3), but along the trajectories of (4.4)–(4.5), satisfies

$$\begin{aligned}
\dot{V}(x(t)) &= \sum_{i=1}^{n}\xi_i\sum_{l=1}^{p}[\mathrm{sat}_h(-\sum_{j=1}^{n}L_{ij}c_l(x_j(t)))][-\sum_{j=1}^{n}L_{ij}c_l(\dot{x}_j(t))]\\
&= \sum_{i=1}^{n}\xi_i\sum_{l=1}^{p}[\mathrm{sat}_h(c_l(u_i(t)))][-\sum_{j=1}^{n}L_{ij}\mathrm{sat}_h(c_l(\hat{u}_j(t)))]\\
&= -\sum_{i=1}^{n}\xi_i[\mathrm{sat}_h(u_i(t))]^\top\sum_{j=1}^{n}L_{ij}\mathrm{sat}_h(\hat{u}_j(t))\\
&= -\sum_{i=1}^{n}\xi_i[\mathrm{sat}_h(u_i(t))]^\top\sum_{j=1}^{n}L_{ij}[\mathrm{sat}_h(u_j(t)) - f_j(t)]\\
&= -\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_iL_{ij}[\mathrm{sat}_h(u_i(t))]^\top\mathrm{sat}_h(u_j(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_iL_{ij}[f_j(t)]^\top\mathrm{sat}_h(u_i(t))\\
&\overset{*}{=} \sum_{i=1}^{n}\frac{\xi_i}{2}\sum_{j=1}^{n}L_{ij}\|\mathrm{sat}_h(u_i(t)) - \mathrm{sat}_h(u_j(t))\|^2\\
&\quad - \sum_{i=1}^{n}\sum_{j=1,j\ne i}^{n}\xi_iL_{ij}[f_i(t)]^\top[\mathrm{sat}_h(u_j(t)) - \mathrm{sat}_h(u_i(t))]\\
&\le \sum_{i=1}^{n}\frac{\xi_i}{2}\sum_{j=1}^{n}L_{ij}\|\mathrm{sat}_h(u_j(t)) - \mathrm{sat}_h(u_i(t))\|^2\\
&\quad + \sum_{i=1}^{n}\sum_{j=1,j\ne i}^{n}\Big\{-\xi_iL_{ij}\frac{1}{4}\|\mathrm{sat}_h(u_j(t)) - \mathrm{sat}_h(u_i(t))\|^2 - \xi_iL_{ij}\|f_i(t)\|^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{\xi_i}{4} \sum_{j=1}^{n} L_{ij} \|\mathrm{sat}_h(u_j(t)) - \mathrm{sat}_h(u_i(t))\|^2 + \sum_{i=1}^{n} \xi_i L_{ii} \|f_i(t)\|^2 \\
&= -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \sum_{i=1}^{n} \xi_i L_{ii} \|f_i(t)\|^2 \\
&= -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \sum_{i=1}^{n} \xi_i L_{ii} \|\mathrm{sat}_h(\hat{u}_i(t)) - \mathrm{sat}_h(u_i(t))\|^2 \\
&\overset{**}{\leq} -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \sum_{i=1}^{n} \xi_i L_{ii} \|\hat{u}_i(t) - u_i(t)\|^2 \\
&= -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \sum_{i=1}^{n} \xi_i L_{ii} \Big\| \sum_{j=1}^{n} L_{ij} e_j(t) \Big\|^2 \\
&\leq -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \max_{i \in \mathcal{I}} \Big\{ \xi_i L_{ii} \Big\} e^\top(t) (L^\top L \otimes I_p) e(t) \\
&\leq -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \max_{i \in \mathcal{I}} \Big\{ \xi_i L_{ii} \Big\} \rho(L^\top L) \sum_{i=1}^{n} \|e_i(t)\|^2, \qquad (4.26)
\end{aligned}
$$

where the equality denoted by $\overset{*}{=}$ holds due to (4.15) and the inequality denoted by $\overset{**}{\leq}$ holds due to Lemma 4.1.

Let us treat $z_i(t) = e^{-\beta_i t}, t \geq 0$ as an additional state to agent $v_i, i \in \mathcal{I}$, and let $z(t) = [z_1(t), \ldots, z_n(t)]^\top$. Consider a Lyapunov candidate:

$$
W(x(t), z(t)) = V(x(t)) + 2 \max_{i} \Big\{ \xi_i L_{ii} \Big\} \rho(L^\top L) \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} z_i(t).
$$

The derivative of $W(x(t), z(t))$ along the trajectories of (4.4)–(4.5) and $\dot{z}_i(t) = -\beta_i z_i(t)$ is

$$
\begin{aligned}
\dot{W}(x(t), z(t)) =& \dot{V}(x) - 2 \max_{i} \Big\{ \xi_i L_{ii} \Big\} \rho(L^\top L) \sum_{i=1}^{n} \alpha_i e^{-\beta_i t} \\
\leq& -\sum_{i=1}^{n} \frac{\xi_i}{2} q_i^s(t) + \max_{i} \Big\{ \xi_i L_{ii} \Big\} \rho(L^\top L) \sum_{i=1}^{n} \|e_i(t)\|^2 - 2 \max_{i} \Big\{ \xi_i L_{ii} \Big\} \rho(L^\top L) \sum_{i=1}^{n} \alpha_i e^{-\beta_i t} \\
\leq& -\sum_{i=1}^{n} \frac{\xi_i}{4} q_i^s(t) - \max_{i} \Big\{ \xi_i L_{ii} \Big\} \rho(L^\top L) \sum_{i=1}^{n} \alpha_i e^{-\beta_i t} \leq 0.
\end{aligned}
$$

By LaSalle Invariance Principle, similar to the proof of Theorem 4.1, we have

$$
\lim_{t \to \infty} \|x_j(t) - x_i(t)\| = 0, \; i, j \in \mathcal{I}, \qquad (4.27)
$$

i.e., consensus is achieved.

We next show that the input of each agent enters into the saturation level in finite time.

Since $c_l(\hat{u}_i(t)) = -\sum_{j=1}^{n} L_{ij}c_l(x_j(t)) - \sum_{j=1}^{n} L_{ij}c_l(e_j(t))$, (4.6), $-\sum_{j=1}^{n} L_{ij}c_l(x_j(t)), i \in \mathcal{I}, l = 1, \dots, p$ are continuous with respect to t, it then follows from (4.27) that there exists a constant $T_3 \geq 0$ such that

$$|c_l(\hat{u}_i(t))| \leq \Big| -\sum_{j=1}^{n} L_{ij}c_l(x_j(t))\Big| + \Big| -\sum_{j=1}^{n} L_{ij}c_l(e_j(t))\Big| \leq h, \ \forall t \geq T_3. \tag{4.28}$$

In other words, the saturation function in (4.4) is no longer active after T_3. Thus, the multi-agent system (4.4) with event-triggered control input (4.5) reduces to

$$\dot{x}_i(t) = -\sum_{j=1}^{n} L_{ij}\hat{x}_j(t), \ t \geq T_3. \tag{4.29}$$

Finally, we estimate the convergence speed, which will be used later. Similar to the proof of Theorem 2 in [65], we conclude that there exist $C_5 > 0$ and $C_6 > 0$ such that

$$\tilde{V}(x(t)) \leq C_5 e^{-C_6 t}, \ \forall t \geq T_3,$$

where $\tilde{V}(x(t))$ is defined in (4.18). Similar to (4.19), we have

$$\tilde{V}(x(t)) \leq C_7 e^{-C_6 t}, \ \forall t \geq 0, \tag{4.30}$$

where C_7 is a positive constant.

Moreover, similar to the analysis for obtaining (4.20), we have

$$\begin{aligned}\sum_{i=1}^{n} \|\hat{u}_i(t)\|^2 =& \sum_{i=1}^{n} \|u_i(t) - \sum_{j=1}^{n} L_{ij}e_j(t)\|^2 \\ \leq& 2\sum_{i=1}^{n} \|u_i(t)\|^2 + 2\rho(L^\top L)\sum_{i=1}^{n} \|e_i(t)\|^2 \\ \leq& C_9 e^{-C_8 t}, \ \forall t \geq 0,\end{aligned} \tag{4.31}$$

where C_9 and C_8 are two positive constants.

(ii-2) In this part, we consider the situation where $\mathcal{G}$ has a directed spanning tree but it is not strongly connected, i.e., $M \geq 2$ in (2.2). For simplicity, we only consider the case where $M = 2$. The general case can be treated in a similar manner. We use the same notation as in the proof of Theorem 4.1. For simplicity, let $\hat{u}_i^m(t) = \hat{u}_{N_{m-1}+i}(t)$, $e_i^m(t) = e_{N_{m-1}+i}(t)$, $f_i^m(t) = f_{N_{m-1}+i}(t)$, $\alpha_i^m = \alpha_{N_{m-1}+i}$, $\beta_i^m = \beta_{N_{m-1}+i}$, and $\hat{u}^m(t) = [(\hat{u}_1^m)^\top(t), \dots, (\hat{u}_{n_m}^m)^\top(t)]^\top$.

First, let us consider SCC_2 and note that no agent in SCC_2 is dependent on any agent in SCC_1. Thus, SCC_2 can be treated as a strongly connected digraph. Then, from the analysis in (ii-1), we have that

$$\lim_{t\to\infty} \|x_i^2(t) - x_j^2(t)\| = 0, \ i, j = 1 \dots, n_2,$$

and that there exists a constant $T_4 \geq 0$ such that

$$|c_l(\hat{u}_i^2(t))| = \Big| - \sum_{j=1}^{n_2} L_{ij}^{2,2} c_l(\hat{x}_j^2(t))\Big| \leq h, \ \forall t \geq T_4. \tag{4.32}$$

In addition, similar to (4.31), we have

$$\|\hat{u}^2(t)\|^2 = \sum_{j=1}^{n_2} \|\hat{u}_j^2(t)\|^2 \leq C_{11} e^{-C_{10}t}, \ t \geq 0,$$

where C_{11} and C_{10} are two positive constants.

Second, let us consider SCC_1. Similar to (4.26), the derivative of $V_2(x(t))$, as defined in (4.23), but along the trajectories of system (4.4)–(4.5), satisfies

$$\dot{V}_2(x(t)) \leq -\sum_{i=1}^{n_2} \frac{\xi_i^2}{2} q_i^2(t) + d_1 \sum_{i=1}^{n_2} \|e_i^2(t)\|^2,$$

where

$$d_1 = \max_{i \in \mathcal{I}} \left\{\xi_i^2 L_{ii}^{2,2}\right\} \rho((L^{2,2})^\top L^{2,2}).$$

The derivative of $V_1(x(t))$, as defined in (4.22), but along the trajectories of system (4.4)–(4.5), satisfies

$$\begin{aligned}
\dot{V}_1(x(t)) &= \sum_{i=1}^{n_1} \xi_i^1 \sum_{l=1}^{p} \mathrm{sat}_h(c_l(u_i^1(t))) c_l(\dot{u}_i^1(t)) \\
&= \sum_{i=1}^{n_1} \xi_i^1 \sum_{l=1}^{p} c_l(\mathrm{sat}_h(u_i^1(t)))\Big[- \sum_{j=1}^{n_1} L_{ij}^{1,1} c_l(\mathrm{sat}_h(\hat{u}_j^1(t))) - \sum_{j=1}^{n_2} L_{ij}^{1,2} c_l(\mathrm{sat}_h(\hat{u}_j^2(t)))\Big] \\
&= \sum_{i=1}^{n_1} \xi_i^1 [\mathrm{sat}_h(u_i^1(t))]^\top \Big[- \sum_{j=1}^{n_1} L_{ij}^{1,1} \mathrm{sat}_h(\hat{u}_j^1(t)) - \sum_{j=1}^{n_2} L_{ij}^{1,2} \mathrm{sat}_h(\hat{u}_j^2(t))\Big] \\
&= \sum_{i=1}^{n_1} \xi_i^1 [\mathrm{sat}_h(\hat{u}_i^1(t)) - f_i^1(t)]^\top \Big[- \sum_{j=1}^{n_1} L_{ij}^{1,1} \mathrm{sat}_h(\hat{u}_j^1(t)) - \sum_{j=1}^{n_2} L_{ij}^{1,2} \mathrm{sat}_h(\hat{u}_j^2(t))\Big] \\
&= -[\mathrm{sat}_h(\hat{u}^1(t))]^\top (Q^1 \otimes I_p) \mathrm{sat}_h(\hat{u}^1(t)) + \sum_{i=1}^{n_1} \xi_i^1 [\mathrm{sat}_h(\hat{u}_i^1(t))]^\top \sum_{j=1}^{n_2} L_{ij}^{1,2} \mathrm{sat}_h(\hat{u}_j^2(t)) \\
&\quad + \sum_{i=1}^{n_1} \xi_i^1 [f_i^1(t)]^\top \Big[\sum_{j=1}^{n_1} L_{ij}^{1,1} \mathrm{sat}_h(\hat{u}_j^1(t)) + \sum_{j=1}^{n_2} L_{ij}^{1,2} \mathrm{sat}_h(\hat{u}_j^2(t))\Big] \\
&\leq -\rho_2(Q^1) \|\mathrm{sat}_h(\hat{u}^1(t))\|^2 + \frac{\rho_2(Q^1)}{4} \sum_{i=1}^{n_1} \|\mathrm{sat}_h(\hat{u}_i^1(t))\|^2 \\
&\quad + \frac{1}{\rho_2(Q^1)} \sum_{i=1}^{n_1} \Big\| \xi_i^1 \sum_{j=1}^{n_2} L_{ij}^{1,2} \mathrm{sat}_h(\hat{u}_j^2(t)) \Big\|^2 + \frac{\rho_2(Q^1)}{4} \sum_{j=1}^{n_1} \|\mathrm{sat}_h(\hat{u}_j^1(t))\|^2
\end{aligned}$$

$$+\frac{1}{\rho_2(Q^1)}\sum_{j=1}^{n_1}\Big\|\sum_{i=1}^{n_1}\xi_i^1 L_{ij}^{1,1} f_i^1(t)\Big\|^2+\sum_{i=1}^{n_1}\frac{1}{4}\|f_i^1(t)\|^2+\sum_{i=1}^{n_1}\Big\|\xi_i^1\sum_{j=1}^{n_2}L_{ij}^{1,2}\mathrm{sat}_h(\hat{u}_j^2(t))\Big\|^2$$

$$\leq -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(\hat{u}^1(t))\|^2+d_2\sum_{i=1}^{n_1}\|f_i^1(t)\|^2+d_3\|\mathrm{sat}_h(\hat{u}^2(t))\|^2, \tag{4.33}$$

where

$$d_2=\frac{1}{4}+(n_1)^2\max_{i\in\{1,\dots,n_1\}}\Big\{(\xi_i^1 L_{ij}^{1,1})^2\Big\}\frac{1}{\rho_2(Q^1)},$$
$$d_3=2n_1n_2\max_{i\in\{1,\dots,n_1\}}\Big\{(\xi_i^1 L_{ij}^{1,2})^2\Big\}\Big(\frac{1}{\rho_2(Q^1)}+1\Big).$$

Similar to the analysis to get (4.26), from (4.33), we have

$$\dot{V}_1(x(t))\leq -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(\hat{u}^1(t))\|^2+d_4\sum_{i=1}^{n_1}\|e_i^1(t)\|^2+d_4\sum_{i=1}^{n_2}\|e_i^2(t)\|^2+d_3\|\mathrm{sat}_h(\hat{u}^2(t))\|^2,$$

where

$$d_4=d_2\rho(L^\top L).$$

Let us treat $\eta_i^r(t)=e^{-\beta_i^r y},\ t\geq 0$, as an additional state of agent $v_i^r,\ r=1,2,\ i=1,\dots,n_2$, $\theta_i^2(t)=e^{-C_{10}t},\ t\geq 0$, as an additional state of agent $v_i^2,\ i=1,\dots,n_2$, and $\theta_i^1(t)=0,\ t\geq 0$, as an additional state of agent $v_i^1,\ i=1,\dots,n_1$. Let $\eta(t)=[\eta_1^1(t),\dots,\eta_{n_1}^1(t),\eta_1^2(t),\dots,\eta_{n_2}^1(t)]^\top$ and $\theta=[\theta_1^1(t),\dots,\theta_{n_1}^1(t),\theta_1^2(t),\dots,\theta_{n_2}^1(t)]^\top$.

Consider the following Lyapunov candidate:

$$W_r(x(t),\eta(t),\theta(t))=V_1(x(t))+V_2(x(t))+2\frac{C_{11}}{C_{10}}d_3\sum_{i=1}^{n_2}\theta_i^2(t)$$
$$+2\sum_{i=1}^{n_2}\frac{(d_1+d_4)\alpha_i^2}{\beta_i^2}\eta_i^2(t)+2\sum_{i=1}^{n_1}\frac{d_4\alpha_i^1}{\beta_i^1}\eta_i^1(t).$$

The derivative of $W_r(t)$ along the trajectories of system (4.4)–(4.5) satisfies

$$\dot{W}_r(x(t),\eta(t),\theta(t))=\dot{V}_1(x(t))+\dot{V}_2(x(t))-2C_{11}d_3\sum_{i=1}^{n_2}\theta_i^2(t)$$
$$-2\sum_{i=1}^{n_2}(d_1+d_4)\alpha_i^2\eta_i^2(t)-2\sum_{i=1}^{n_1}d_4\alpha_i^1\eta_i^1(t).$$

Then, for any $t\geq T_4$, we have

$$\dot{W}_r(x(t),\eta(t),\theta(t))\leq -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(u^1(t))\|^2+\sum_{i=1}^{n_2}-\frac{\xi_i^2}{2}q_i^2(t)$$

$$- C_{11} d_3 \sum_{i=1}^{n_2} \theta_i^2(t) - \sum_{i=1}^{n_2} (d_1 + d_4) \alpha_i^2 \eta_i^2(t) - \sum_{i=1}^{n_1} d_4 \alpha_i^1 \eta_i^1(t).$$

By LaSalle Invariance Principle again, we have

$$\lim_{t \to \infty} \|x_j(t) - x_i(t)\| = 0, \ i, j \in \mathcal{I}.$$

Thus, consensus is achieved. Moreover, similar to the analysis in (ii-1), we can show that after a finite time the saturation is no longer active.

Chapter 5

Event-triggered formation control with connectivity preservation

In this chapter, we study formation control for multi-agent systems with connectivity preservation and event-triggered control. We propose distributed triggering laws for agents to determine their triggering times and one corresponding algorithm for each agent to avoid continuous monitoring of its own triggering law. The advantages of this algorithm are that absolute measurements of states are avoided and it is only at its triggering times that each agent needs to update its control input by sensing the relative states, to broadcast its triggering information, including current triggering time and control input at this time, to its neighbors. The main disadvantage is that continuous listening is still needed. To overcome this, we then present two self-triggered algorithms. Two types of system dynamics, single integrators and double integrators, are considered. We show that under the proposed event-triggered and self-triggered algorithms all agents converge to pre-specified formations exponentially with connectivity preservation. In addition, Zeno behavior can be excluded by proving that the inter-event times are lower bounded by a positive constant for single integrators and the triggering time sequence of each agent is divergent for double integrators. Two related existing papers are [68], [75]. However, [68] does not explicitly exclude Zeno behavior, but it is well known that such behavior can be problematic, see [19]. And it is under the assumption that no agent exhibits Zeno behavior, that [75] proves asymptotic rendezvous can be achieved.

The rest of this chapter is organized as follows. In Section 5.1, we review the formation problem and related preliminaries. In Section 5.2, we consider formation control for first-order continuous-time multi-agent systems with connectivity preservation and event-triggered control. We then extend the results to second-order systems in Section 5.3. Simulations are given in Section 5.4. Finally, the chapter is concluded in Section 5.5.

5.1 Formation control problem

Consider a connected undirected graph $\mathcal{G}$ with n vertices and n_e edges. Let $B(\mathcal{G})$ denotes its incidence matrix as defined in Section 2.2 and $d_{ij} \in \mathbb{R}^p$ the desired internode displacement of edge $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$. Denote $\Phi = \{(\tau_1^\top, \dots, \tau_n^\top)^\top \in \mathbb{R}^{np} | \tau_i - \tau_j = d_{ij}, \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$. We call the set of desired internode displacements $\{d_{ij}, \ (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ a formation associated with $\mathcal{G}$ and we say it is feasible if $\Phi \neq \emptyset$.

Definition 5.1. *Consider a multi-agent system with n agents whose underlying graph is $\mathcal{G}$. Let $x_i(t) \in \mathbb{R}^p$ denotes the position of agent i at time $t \geq 0$. The multi-agent system converges to a desired formation $\{d_{ij}, \ (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ if*

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = d_{ij}, \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}).$$

In practice, agents normally have limited communication capabilities and one agent cannot exchange information with the agents that outside its communication radius. For simplicity we assume all agents have the same communication radius $\Delta > 0$. Figure 5.1 (a) shows the initial positions of three agents and each agent has the same communication radius Δ; and (b) shows the desired formation $\{d_{12}, d_{13}, d_{23}\}$. We say the graph $\mathcal{G}$ and the multi-agent system are consistent if $\|x_i(t) - x_j(t)\| \leq \Delta$ for all $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ and all times $t \geq 0$. Namely, the communication channels are kept for all time. Notice here that we assume the following.

Assumption 5.1. *The desired formation $\{d_{ij}, \ (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ is feasible and $\|d_{ij}\| < \Delta$, $\forall (v_i, v_j) \in \mathcal{E}(\mathcal{G})$.*

Definition 5.2. *A group of agents are said to converge to the desired formation with connectivity preservation if they converge to the formation while the graph $\mathcal{G}$ remains consistent with their dynamics.*

Remark 5.1. *We do not assume new edges are created, while we only show that old edges are maintained.*

Our goal in this chapter is to solve the following problem.

Problem 5.1. *Propose distributed event-triggered control input and determine the corresponding triggering times for first-order and second-order multi-agent systems such that a desired formation is achieved with connectivity preservation, while the use of absolute state information, continuous exchange of information, continuous update of actuators, and Zeno behavior are avoided.*

5.2 Single integrators

In this section, we consider the case when the dynamics of agents is modeled as single integrators given by

$$\dot{x}_i(t) = u_i(t), \ i \in \mathcal{I}, \ t \geq 0, \tag{5.1}$$

where $x_i(t) \in \mathbb{R}^p$ is the position and $u_i(t) \in \mathbb{R}^p$ is the control input of agent i, respectively.

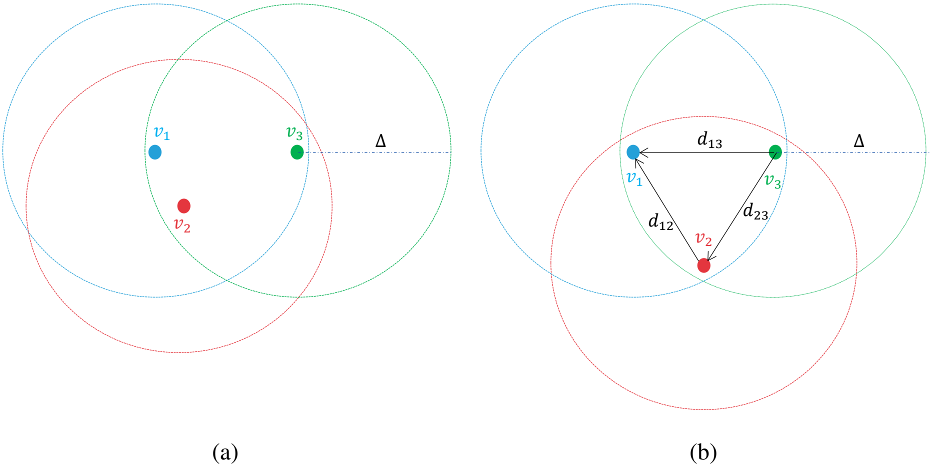


Figure 5.1: (a) The initial positions of three agents. (b) The desired formation $\{d_{12}, d_{13}, d_{23}\}$.

From Assumption 5.1, we know $\Phi \neq \emptyset$. Choose any $(\tau_1^\top, \ldots, \tau_n^\top)^\top \in \Phi$. Let $y_i(t) = x_i(t) - \tau_i$ for $i \in \mathcal{I}$ and $y(t) = [y_1^\top(t), \ldots, y_n^\top(t)]^\top$. Then, we can rewrite the above multi-agent system as

$$\dot{y}_i(t) = u_i(t), \; i \in \mathcal{I}, \; t \geq 0. \tag{5.2}$$

At time t, for $\|y_i(t) - y_j(t)\| < \Delta - \|d_{ij}\|$, the edge-tension function ν_{ij} (introduced first in [25]) is defined as

$$\nu_{ij}(\Delta, y(t)) = \begin{cases} \frac{\|y_i(t)-y_j(t)\|^2}{\Delta-\|d_{ij}\|-\|y_i(t)-y_j(t)\|}, & \text{if } (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise,} \end{cases}$$

with

$$\frac{\partial \nu_{ij}(\Delta, y(t))}{\partial y_i} = \begin{cases} \frac{2\Delta-2\|d_{ij}\|-\|y_i(t)-y_j(t)\|}{(\Delta-\|d_{ij}\|-\|y_i(t)-y_j(t)\|)^2}(y_i(t) - y_j(t)), & \text{if } (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

We denote as $\omega_{ij}(t)$ the weight coefficient of the partial derivative of ν_{ij} with respect to y_i as above, i.e.,

$$\omega_{ij}(t) = \begin{cases} \frac{2\Delta-2\|d_{ij}\|-\|y_i(t)-y_j(t)\|}{(\Delta-\|d_{ij}\|-\|y_i(t)-y_j(t)\|)^2}, & \text{if } (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\omega_{ij}(t)$ can also be written as a function of $x_i(t)$ and $x_j(t)$ since $y_i(t) - y_j(t) = x_i(t) - x_j(t) - d_{ij}$.

Let L_ω denotes the Laplacian matrix associated with $\mathcal{G}$ after assigning the above weight $\omega_{ij}(t)$ to edge $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$. Then, from Lemma 2.6, we have

$$L_\omega = B(\mathcal{G})WB(\mathcal{G})^\top,$$

where $W = \mathrm{Diag}([\omega(e_1), \cdots, \omega(e_{n_e})])$, where $\omega(e_k) = \omega_{ij}$ with e_k being the label of edge (v_i, v_j).

In order to reduce the overall need of communication and system updates, we use the following event-triggered control input

$$u_i(t) = \sum_{j\in N_i} -\omega_{ij}(t^i_{k_i(t)})(y_i(t^i_{k_i(t)}) - y_j(t^i_{k_i(t)})) \tag{5.3}$$

$$= \sum_{j\in N_i} -\omega_{ij}(t^i_{k_i(t)})(x_i(t^i_{k_i(t)}) - x_j(t^i_{k_i(t)}) - d_{ij}). \tag{5.4}$$

One can see that the above control input only updates at the triggering times.

5.2.1 Event-triggered approach

In the following theorem, we will give triggering laws to determine the triggering times such that the formation with connectivity preservation can be established and Zeno behavior can be excluded.

Theorem 5.1. *Given a graph $\mathcal{G}$ which is undirected and connected, and a desired formation associated with $\mathcal{G}$ which satisfies Assumption 5.1. Consider the multi-agent system (5.1) with event-triggered control input (5.4) associated with $\mathcal{G}$. Assume that at the initial time,*

$$\|x_i(0) - x_j(0) - d_{ij}\| = \|y_i(0) - y_j(0)\| < \Delta - \|d_{ij}\|, \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}). \tag{5.5}$$

Given $\alpha > 0$, and $0 < \beta < \beta_0$ with $\beta_0 = \frac{\rho_2(B(\mathcal{G})B(\mathcal{G})^\top)}{\Delta_0}$ and $\Delta_0 = \max_{(v_i,v_j)\in\mathcal{E}(\mathcal{G})} \Delta - \|d_{ij}\|$, and given the first triggering time $t^i_1 = 0$, agent i determines the triggering times $\{t^i_k\}^\infty_{k=2}$ by

$$t^i_{k+1} = \min\left\{t : \ \|e_i(t)\| \ge \alpha e^{-\beta t}, \ t \ge t^i_k\right\} \tag{5.6}$$

where

$$e_i(t) = \sum_{j\in N_i} \omega_{ij}(t)(x_i(t) - x_j(t) - d_{ij}) - \sum_{j\in N_i} \omega_{ij}(t^i_{k_i(t)})(x_i(t^i_{k_i(t)}) - x_j(t^i_{k_i(t)}) - d_{ij}).$$

Then the multi-agent system (5.1) with event-triggered control input (5.4) converges to the desired formation exponentially with connectivity preservation, and there is no Zeno behavior.

Proof. This theorem holds if we can prove that

(i) $\|x_i(t) - x_j(t)\| \le \Delta, \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \ \forall t \ge 0;$

(ii) $\lim_{t\to\infty}(x_i(t) - x_j(t)) = d_{ij},\ \forall(v_i, v_j) \in \mathcal{E}(\mathcal{G})$, exponentially;

(iii) there exists a constant $\epsilon_i > 0$, such that $t^i_{k+1} - t^i_k \geq \epsilon_i$, $\forall i \in \mathcal{I}$ and $\forall k = 1, 2, \ldots$

(i) We define the total tension energy of $\mathcal{G}$ as

$$v(\Delta, y(t)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i} v_{ij}(\Delta, y(t)). \tag{5.7}$$

The time derivative of $v(\Delta, y(t))$ along the trajectories of the multi-agent system (5.2)–(5.3) is

$$\begin{aligned}
\dot{v}(\Delta, y(t)) &= \sum_{i=1}^{n}\sum_{j\in N_i}\Big[\frac{\partial v_{ij}(\Delta, y)}{\partial y_i}\Big]^\top\Big|_{y=y(t)}\dot{y}_i(t)\\
&= \sum_{i=1}^{n}\sum_{j\in N_i}[\omega_{ij}(t)(y_i(t) - y_j(t))]^\top \sum_{j\in N_i} -\omega_{ij}(t^i_{k_i(t)})(y_i(t^i_{k_i(t)}) - y_j(t^i_{k_i(t)}))\\
&= \sum_{i=1}^{n}\sum_{j\in N_i}[\omega_{ij}(t)(y_i(t) - y_j(t))]^\top\Big[e_i(t) - \sum_{j\in N_i}\omega_{ij}(t)(y_i(t) - y_j(t))\Big]\\
&= \sum_{i=1}^{n}\sum_{j\in N_i}[\omega_{ij}(t)(y_i(t) - y_j(t))]^\top\Big(- \sum_{j\in N_i}\omega_{ij}(t)(y_i(t) - y_j(t))\Big)\\
&\quad + \sum_{i=1}^{n}\sum_{j\in N_i}[\omega_{ij}(t)(y_i(t) - y_j(t))]^\top e_i(t)\\
&\leq - \|L_\omega y(t)\|^2 + \sum_{i=1}^{n}\Big\|\sum_{j\in N_i}\omega_{ij}(t)(y_i(t) - y_j(t))\Big\|^2 + \frac{1}{4}\sum_{i=1}^{n}\|e_i(t)\|^2\\
&= \frac{1}{4}\sum_{i=1}^{n}\|e_i(t)\|^2,
\end{aligned}$$

From (5.6), we know that

$$\|e_i(t)\| \leq \alpha e^{-\beta t},\ \forall t \geq 0.$$

Hence

$$\dot{v}(\Delta, y(t)) \leq \frac{n\alpha^2}{4}e^{-2\beta t},\ \forall t \geq 0.$$

Thus

$$v(\Delta, y(t)) \leq v(\Delta, y(0)) + \frac{n\alpha^2}{8\beta}[1 - e^{-2\beta t}] \leq k_v,\ \forall t \geq 0,$$

where

$$k_v = v(\Delta, y(0)) + \frac{n\alpha^2}{8\beta} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\frac{\|x_i(0)-x_j(0)-d_{ij}\|^2}{\Delta-\|d_{ij}\|-\|x_i(0)-x_j(0)-d_{ij}\|} + \frac{n\alpha^2}{8\beta}. \tag{5.8}$$

Then, for any $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ and $t \geq 0$, we have

$$v_{ij}(\Delta, y(t)) = \frac{\|y_i(t)-y_j(t)\|^2}{\Delta-\|d_{ij}\|-\|y_i(t)-y_j(t)\|} \leq 2v(\Delta, y(t)) \leq 2k_v.$$

Hence

$$\|y_i(t)-y_j(t)\| \leq k_{ij}, \tag{5.9}$$

where

$$k_{ij} = -k_v + \sqrt{k_v^2 + 2k_v(\Delta-\|d_{ij}\|)} < \Delta - \|d_{ij}\|. \tag{5.10}$$

Then, we have

$$\begin{aligned}\|x_i(t)-x_j(t)\| =& \|x_i(t)-\tau_i-(x_j(t)-\tau_j)+d_{ij}\|\\ =& \|y_i(t)-y_j(t)+d_{ij}\| \leq \|y_i(t)-y_j(t)\| + \|d_{ij}\| \leq k_{ij} + \|d_{ij}\| < \Delta,\end{aligned}$$

and thus connectivity maintenance is established.

(ii) Let $e(t) = [e_1^\top(t), \ldots, e_n^\top(t)]^\top$, $\bar{y}(t) = \frac{1}{n}\sum_{i=1}^{n} y_i(t)$ and $\delta(t) = y(t) - \mathbf{1}_n \otimes \bar{y}(t) = (K_n \otimes I_p)y(t)$. We consider the Lyapunov candidate

$$V(y(t)) = \frac{1}{2}\delta^\top(t)\delta(t) = \frac{1}{2}y^\top(t)(K_n \otimes I_p)y(t). \tag{5.11}$$

Then its derivative along the trajectories of the multi-agent system (5.2)–(5.3) is

$$\begin{aligned}\dot{V}(y(t)) =& y^\top(t)(K_n \otimes I_p)\dot{y}(t) = y^\top(t)(K_n \otimes I_p)[-(L_\omega \otimes I_p)y(t) + e(t)]\\ =& -y^\top(t)(L_\omega \otimes I_p)y(t) + \delta^\top(t)e(t).\end{aligned}$$

For $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$, define

$$f_{ij}(l) = \frac{2\Delta - 2\|d_{ij}\| - l}{(\Delta - \|d_{ij}\| - l)^2},\ l \in [0, \Delta - \|d_{ij}\|). \tag{5.12}$$

We can easily check that $f_{ij}(l)$ is an increasing function on $[0, \Delta - \|d_{ij}\|)$. Then from (5.9) and (5.10) we have

$$\omega_{ij}(t) \leq f_{ij}(k_{ij}),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ \forall t \geq 0, \tag{5.13}$$

and

$$\omega_{ij}(t) \geq f_{ij}(0) = \frac{2}{\Delta - \|d_{ij}\|} \geq \frac{2}{\Delta_0}.$$

Then,

$$W = \mathrm{Diag}([\omega(e_1), \cdots, \omega(e_m)]) \geq \frac{2}{\Delta_0} I_m,$$

and

$$L_\omega = B(\mathcal{G}) W B(\mathcal{G})^\top \geq \frac{2}{\Delta_0} B(\mathcal{G}) I_m B(\mathcal{G})^\top \geq \frac{2\rho_2(B(\mathcal{G})B(\mathcal{G})^\top)}{\Delta_0} K_n = 2\beta_0 K_n.$$

Thus

$$\begin{aligned}\dot{V}(y(t)) =& -y^\top(t)(L_\omega \otimes I_p)y(t) + \delta^\top(t)e(t)\\ \leq& -2\beta_0 y^\top(t)(K_n \otimes I_p)y(t) + \beta_0 \delta^\top(t)\delta(t) + \frac{1}{4\beta_0}\|e(t)\|^2\\ =& -2\beta_0 V(y(t)) + \frac{1}{4\beta_0}\|e(t)\|^2 \leq -2\beta_0 V(y(t)) + \frac{n\alpha^2}{4\beta_0} e^{-2\beta t},\end{aligned}$$

where the first inequality holds since $W \geq \frac{2}{\Delta_0} I_m$ and the second inequality holds since Lemma 2.7. Hence

$$V(y(t)) \leq V(y(0))e^{-2\beta_0 t} + \frac{n\alpha^2}{8\beta_0(\beta_0 - \beta)}[e^{-2\beta t} - e^{-2\beta_0 t}] < k_V e^{-2\beta t},$$

where

$$k_V = V(y(0)) + \frac{n\alpha^2}{8\beta_0(\beta_0 - \beta)}. \tag{5.14}$$

Thus

$$\begin{aligned}\|y_i(t) - y_j(t)\|^2 &\leq 2\|y_i(t) - \bar{y}(t)\|^2 + 2\|\bar{y}(t) - y_j(t)\|^2\\ &\leq 4V(y(t)) < 4k_V e^{-2\beta t}, \ \forall i, j \in \mathcal{I}.\end{aligned} \tag{5.15}$$

Hence

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = \lim_{t\to\infty}(y_i(t) - \tau_i - (y_j(t) - \tau_j)) = d_{ij},$$

exponentially.

(iii) For $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$, define

$$g_{ij}(l) = \frac{2(\Delta - \|d_{ij}\|)^2}{(\Delta - \|d_{ij}\| - l)^3}, \ l \in [0, \Delta - \|d_{ij}\|), \tag{5.16}$$

$$h_{ij}(l) = \frac{3\Delta - 3\|d_{ij}\| - l}{(\Delta - \|d_{ij}\| - l)^3}, \ l \in [0, \Delta - \|d_{ij}\|). \tag{5.17}$$

We can easily check that both $g_{ij}(l)$ and $h_{ij}(l)$ are increasing functions on $[0, \Delta - \|d_{ij}\|)$. From (5.13), we have

$$\|\dot{y}_i(t)\| = \|e_i(t) - \sum_{j\in N_i} \omega_{ij}(t)(y_i(t) - y_j(t))\| \leq \|e_i(t)\| + \sum_{j\in N_i} \omega_{ij}(t)\|(y_i(t) - y_j(t))\| \tag{5.18}$$

$$< \alpha e^{-\beta t} + \sum_{j\in N_i} 2 f_{ij}(k_{ij}) \sqrt{k_V} e^{-\beta t}. \tag{5.19}$$

From

$$\begin{aligned}\dot{e}_i(t) &= \sum_{j\in N_i} [\dot{\omega}_{ij}(t)(y_i(t) - y_j(t)) + \omega_{ij}(t)(\dot{y}_i(t) - \dot{y}_j(t))] \\ &= \sum_{j\in N_i} \Big\{ h_{ij}(\|y_i(t) - y_j(t)\|) \frac{(y_i(t) - y_j(t))^\top}{\|y_i(t) - y_j(t)\|} (\dot{y}_i(t) - \dot{y}_j(t))(y_i(t) - y_j(t)) \\ &\quad + \omega_{ij}(t)(\dot{y}_i(t) - \dot{y}_j(t)) \Big\},\end{aligned} \tag{5.20}$$

we have

$$\begin{aligned}\frac{d\|e_i(t)\|}{dt} &\leq \|\dot{e}_i(t)\| \\ &\leq \sum_{j\in N_i} \Big\{ \Big\| h_{ij}(\|y_i(t) - y_j(t)\|) \frac{(y_i(t) - y_j(t))^\top}{\|y_i(t) - y_j(t)\|} (\dot{y}_i(t) - \dot{y}_j(t))(y_i(t) - y_j(t)) \Big\| \\ &\quad + \|\omega_{ij}(t)(\dot{y}_i(t) - \dot{y}_j(t))\| \Big\} \\ &\leq \sum_{j\in N_i} \Big\{ h_{ij}(\|y_i(t) - y_j(t)\|) \|\dot{y}_i(t) - \dot{y}_j(t)\| \|y_i(t) - y_j(t))\| + \omega_{ij}(t) \|\dot{y}_i(t) - \dot{y}_j(t)\| \Big\} \\ &= \sum_{j\in N_i} g_{ij}(\|y_i(t) - y_j(t)\|) \|\dot{y}_i(t) - \dot{y}_j(t)\| \qquad (5.21) \\ &\leq \sum_{j\in N_i} g_{ij}(\|y_i(t) - y_j(t)\|)(\|\dot{y}_i(t)\| + \|\dot{y}_j(t)\|) \qquad (5.22) \\ &\leq \sum_{j\in N_i} g_{ij}(k_{ij})[\|\dot{y}_i(t)\| + \|\dot{y}_j(t)\|] < c_i e^{-\beta t}, \qquad (5.23)\end{aligned}$$

where

$$c_i = \sum_{j\in N_i} g_{ij}(k_{ij}) \Big[2\alpha + \sum_{l\in N_i} 2 f_{il}(k_{il}) \sqrt{k_V} + \sum_{l\in N_j} 2 f_{jl}(k_{jl}) \sqrt{k_V} \Big]. \tag{5.24}$$

Thus, a necessary condition to guarantee the inequality in (5.6), i.e.,

$$\alpha e^{-\beta t} \leq \|e_i(t)\| = \int_{t_k^i}^{t} \frac{d\|e_i(s)\|}{ds} ds, \ \forall t \in [t_k^i, t_{k+1}^i),$$

is

$$\begin{aligned}\alpha e^{-\beta t} \le &\int_{t_k^i}^{t} c_i e^{-\beta s} ds = \frac{c_i}{\beta}[e^{-\beta t_k^i} - e^{-\beta t}] \\ \Leftrightarrow &(c_i + \alpha\beta)e^{-\beta t} \le c_i e^{-\beta t_k^i} \Leftrightarrow (c_i + \alpha\beta)e^{-\beta(t-t_k^i)} \le c_i \\ \Rightarrow &(c_i + \alpha\beta)[1 - \beta(t - t_k^i)] \le c_i \Leftrightarrow t - t_k^i \ge \epsilon_i,\end{aligned} \tag{5.25}$$

where

$$\epsilon_i = \frac{\alpha}{c_i + \alpha\beta} > 0. \tag{5.26}$$

In other words, for all $t \in [t_k^i, t_k^i + \epsilon_i]$, $\|e_i(t)\| \le \alpha e^{-\beta t}$ holds. Hence $t_{k+1}^i \ge t_k^i + \epsilon_i$. □

Apparently, in order to monitor the inequality in the triggering law (5.6), each agent needs to continuously sense the relative positions to its neighbors. This may be a drawback. In the following we will give an event-triggered algorithm to avoid this. In other words, the following algorithm is an implementation of Theorem 5.1, but it only requires agents to sense, broadcast and receive at the triggering times. The idea is illustrated as follows.

Each agent $i \in \mathcal{I}$, at any time $s \ge 0$, knows its last triggering time $t_{k_i(s)}^i$ and its control input $u_i(s) = u_i(t_{k_i(s)}^i)$ which is a constant until it determines its next triggering time. If agent i also knows the relative position $x_i(s) - x_j(s)$ and $u_j(s) = u_j(t_{k_j(s)}^j)$ which is a constant until agent j determines its next triggering time, for $j \in \mathcal{N}_i$, then agent i can predict

$$x_i(t) - x_j(t) = x_i(s) - x_j(s) + (t - s)[u_i(t_{k_i(s)}^i) - u_j(t_{k_j(s)}^j)],\ t \ge s, \tag{5.27}$$

until $t \le \min\left\{t_{k_i(s)+1}^i, t_{k_j(s)+1}^j\right\}$. This means continuous sensing, broadcasting and receiving are not needed any more. The above implement idea is summarized in Algorithm 5.1.

Remark 5.2. *In order to implement Algorithm 5.1, β_0 should be known first. However β_0 is a global parameter since it relates to $\rho_2(B(\mathcal{G})B(\mathcal{G})^\top)$ and Δ_0. We can lower bound β_0 by $\frac{4}{n(n-1)\Delta}$ since $\Delta_0 < \Delta$ and $\rho_2(B(\mathcal{G})B(\mathcal{G})^\top) \ge \frac{4}{n(n-1)}$, see [103].*

5.2.2 Self-triggered algorithms

When applying Algorithm 5.1, although continuous sensing, broadcasting and sensing are avoided, each agent still needs to continuously listen to incoming information from its neighbors since the triggering times are not known in advance. If every agent $i \in \mathcal{I}$, at its current triggering time t_k^i, can predict its next triggering time t_{k+1}^i and broadcast it to its neighbors, then at time t_k^i agent i knows agent j's latest triggering time $t_{k_j(t_k^i)}^j$ which is before t_k^i and its next triggering time $t_{k_j(t_k^i)+1}^j$ which is after t_k^i, for $j \in \mathcal{N}_i$. In this case, agent i only needs to listen to and receive information at $\{t^j\}_{k=1}^{\infty}$, $j \in \mathcal{N}_i$ since it knows these time instants in advance. Thus, each agent only needs to sense and broadcast at its own triggering times, and to listen to and receive the incoming information from its neighbors at

Algorithm 5.1

1: Choose $\alpha > 0$ and $0 < \beta < \beta_0$;
2: Initialize $t_1^i = 0$ and $k = 1$;
3: Agent $i \in \mathcal{I}$ sends $\{d_{ij}, (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ to its neighbors;
4: Agent i continuously listens to whether there is broadcasting from its neighbors and receives the broadcasted information if it occurs;
5: At time $s = t_k^i$, agent i senses the relative position $x_i(s) - x_j(s)$ and predicts future relative position $x_i(t) - x_j(t),\ t \geq s,\ \forall j \in \mathcal{N}_i$ by (5.27);
6: Agent i substitutes these relative positions into $e_i(t)$ and finds out τ_{k+1}^i which is the smallest solution of equation $\|e_i(t)\| = \alpha e^{-\beta t}$;
7: **if** there is broadcasting from its neighbors at $t_0 \in (s, \tau_{k+1}^i)$, i.e., there exists $j \in \mathcal{N}_i$ such that agent j broadcasts its triggering information at $t_0 \in (s, \tau_{k+1}^i)$[1] **then**
8: agent i receives information at t_0, and updates $s = t_0$, and goes back to Step 5;
9: **else**
10: agent i determines $t_{k+1}^i = \tau_{k+1}^i$, and updates its control input $u_i(t_{k+1}^i)$ by sensing the relative positions to its neighbors, and broadcasts its triggering information $\{t_{k_i(t)}^i, u_i(t_{k_i(t)}^i)\}$ to its neighbors, and resets $k = k + 1$, and goes back to Step 5;
11: **end if**

their triggering times. Inspired by this, in the following we will propose two self-triggered algorithms such that at time t_k^i each agent i could estimate t_{k+1}^i in a more precise way than $t_k^i + \epsilon_i$. The idea is explained below.

From (5.9) and (5.15), we have

$$\|y_i(t) - y_j(t)\| < \hat{k}_{ij}(t),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ \forall t \geq 0, \tag{5.28}$$

where

$$\hat{k}_{ij}(t) = \min\left\{k_{ij}, 2\sqrt{k_V} e^{-\beta t}\right\}.$$

Then, from (5.18), we have

$$\|u_i(t)\| = \|\dot{y}_i(t)\| \leq \theta_i(t),\ \forall i \in \mathcal{I},\ \forall t \geq 0, \tag{5.29}$$

where

$$\theta_i(t) = \alpha e^{-\beta t} + \sum_{j \in N_i} f_{ij}(\hat{k}_{ij}(t))\hat{k}_{ij}(t).$$

From (5.2), we have $\dot{y}_i(t) - \dot{y}_j(t) = u_i(t) - u_j(t)$. Then,

$$y_i(t) - y_j(t) = y_i(t_k^i) - y_j(t_k^i) + \int_{t_k^i}^{t} [u_i(s) - u_j(s)]ds,\ t \geq t_k^i.$$

[1]This kind of situation can only occur at most finite times during (s, τ_{k+1}^i) since $|\mathcal{N}_i|$ is finite and there is no Zeno behavior.

Agent i can determine $y_i(t_k^i) - y_j(t_k^i) = x_i(t_k^i) - x_j(t_k^i) - d_{ij}$ for $j \in \mathcal{N}_i$ by sensing the relative position to its neighbors at time t_k^i.

The control input $u_i(s)$ is a constant during $[t_k^i, t_{k+1}^i)$ and $u_j(s)$ is a constant during $[t^j_{k_j(t_k^i)}, t^j_{k_j(t_k^i)+1})$. At time t_k^i, agent i already knows $t^j_{k_j(t_k^i)}$ and $u_j(t^j_{k_j(t_k^i)})$, for $j \in \mathcal{N}_i$. If at time t_k^i, agent i also knows $t^j_{k_j(t_k^i)+1}$, then at time t_k^i it knows $u_j(s) \equiv u_j(t^j_{k_j(t_k^i)})$, for $s \in [t_k^i, t^j_{k_j(t_k^i)+1})$. In other words, same as (3.38), for $t \in [t_k^i, t_{k+1}^i)$, if denote

$$t_{ij}^1(t) = \min\left\{t, t^j_{k_j(t_k^i)+1}\right\},\ t_{ij}^2(t) = \max\left\{t, t^j_{k_j(t_k^i)+1}\right\}, \tag{5.30}$$

then at time t_k^i, agent i knows $u_j(s) \equiv u_j(t^j_{k_j(t_k^i)})$, for $s \in [t_k^i, t_{ij}^1(t))$ but does not know $u_j(s)$, for $s \geq t_{ij}^2(t)$. Figure 3.1 illustrates the relation of t_k^i, t_{k+1}^i, $t \in [t_k^i, t_{k+1}^i)$, $t^j_{k_j(t_k^i)}$, $t^j_{k_j(t_k^i)+1}$, $t_{ij}^1(t)$ and $t_{ij}^2(t)$.

Then,

$$y_i(t) - y_j(t) = z_{ij}(t_k^i, t) - \int_{t^j_{k_j(t_k^i)+1}}^{t_{ij}^2(t)} u_j(s)ds,\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t_k^i, t_{k+1}^i), \tag{5.31}$$

where

$$z_{ij}(t_k^i, t) = y_i(t_k^i) - y_j(t_k^i) + (t - t_k^i)u_i(t_k^i) - (t_{ij}^1(t) - t_k^i)u_j(t^j_{k_j(t_k^i)}).$$

Thus

$$\|y_i(t) - y_j(t)\| \leq \|z_{ij}(t_k^i, t)\| + \int_{t^j_{k_j(t_k^i)+1}}^{t_{ij}^2(t)} \|u_j(s)\|ds,\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t_k^i, t_{k+1}^i).$$

Then, from (5.29), we have

$$\|y_i(t) - y_j(t)\| \leq \check{k}_{ij}(t),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t_k^i, t_{k+1}^i), \tag{5.32}$$

where

$$\check{k}_{ij}(t) = \|z_{ij}(t_k^i, t)\| + \int_{t^j_{k_j(t_k^i)+1}}^{t_{ij}^2(t)} \theta_j(s)ds,\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t_k^i, t_{k+1}^i).$$

Then, from (5.28) and (5.32), we have

$$\|y_i(t) - y_j(t)\| \leq \tilde{k}_{ij}(t),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t_k^i, t_{k+1}^i), \tag{5.33}$$

where

$$\tilde{k}_{ij}(t) = \min\left\{\hat{k}_{ij}(t), \check{k}_{ij}(t)\right\},\ t \in [t_k^i, t_{k+1}^i). \tag{5.34}$$

Thus, from (5.20), (5.21), (5.29), (5.31) and (5.33), we have

$$\|e_i(t)\| \le \varphi_i(t),\ t \in [t_k^i, t_{k+1}^i),$$

where

$$\begin{aligned}\varphi_i(t) =& \Big\| \sum_{j\in N_i} \int_{t_k^i}^{t_{ij}^1(t)} \Big\{ h_{ij}(\|z_{ij}(t_k^i, s)\|) \frac{(z_{ij}(t_k^i, s))^\top}{\|z_{ij}(t_k^i, s)\|}(u_i(t_k^i) - u_j(t^j_{k_j(t_k^i)}))z_{ij}(t_k^i, s) \\ &+ f_{ij}(\|z_{ij}(t_k^i, s)\|)(u_i(t_k^i) - u_j(t^j_{k_j(t_k^i)}))\Big\} ds \Big\| \\ &+ \sum_{j\in N_i} \int_{t_{ij}^1(t)}^{t} g_{ij}(\tilde{k}_{ij}(s))\|u_i(t_k^i)\| ds + \sum_{j\in N_i} \int_{t^j_{k_j(t_k^i)+1}}^{t_{ij}^2(t)} g_{ij}(\tilde{k}_{ij}(s))\theta_j(s) ds \\ =& \Big\| \sum_{j\in N_i} \Big\{ f_{ij}(\|z_{ij}(t_k^i, t_{ij}^1(t))\|) z_{ij}(t_k^i, t_{ij}^1(t)) - f_{ij}(\|z_{ij}(t_k^i, t_k^i)\|) z_{ij}(t_k^i, t_k^i) \Big\} \Big\| \\ &+ \sum_{j\in N_i} \int_{t_{ij}^1(t)}^{t} g_{ij}(\tilde{k}_{ij}(s))\|u_i(t_k^i)\| ds + \sum_{j\in N_i} \int_{t^j_{k_j(t_k^i)+1}}^{t_{ij}^2(t)} g_{ij}(\tilde{k}_{ij}(s))\theta_j(s) ds,\ t \in [t_k^i, t_{k+1}^i).\end{aligned} \tag{5.35}$$

Hence, a necessary condition to guarantee the inequality in (5.6), i.e.,

$$\alpha e^{-\beta t} \le \|e_i(t)\|,\ \forall t \in [t_k^i, t_{k+1}^i),$$

is

$$\alpha e^{-\beta t} \le \varphi_i(t),\ \forall t \in [t_k^i, t_{k+1}^i).$$

Since $\alpha e^{-\beta t}$ decreases with respect to t, $\varphi_i(t)$ increases with respect to t during $[t_k^i, t_{k+1}^i)$ and $\varphi_i(t_k^i) = 0$, then given t_k^i, agent i can estimate t_{k+1}^i by the solution to

$$\alpha e^{-\beta t} = \varphi_i(t),\ t \ge t_k^i. \tag{5.36}$$

In conclusion, if at time t_k^i agent i knows $u_i(t_k^i)$, $t^j_{k_j(t_k^i)}$, $t^j_{k_j(t_k^i)+1}$, $u_j(t^j_{k_j(t_k^i)})$, $\forall j \in \mathcal{N}_i$, then it can predict its next triggering time t_{k+1}^i by solving (5.36). The above implement idea is summarized in Algorithm 5.2.

Actually, broadcasting, receiving and listening can be ruled out except at the beginning, and each agent only needs to sense the relative positions to its neighbors and update its control input at its triggering times. The idea is illustrated as follows.

From (5.28), (5.18) and (5.22), we have

$$\frac{d\|e_i(t)\|}{dt} < \hat{c}_i(t), \tag{5.37}$$

where

$$\hat{c}_i(t) = \sum_{j\in N_i} g_{ij}(\hat{k}_{ij}(t))\Big[2\alpha + \sum_{l\in N_i} f_{il}(\hat{k}_{il}(t))\hat{k}_{il}(t) + \sum_{l\in N_j} f_{jl}(\hat{k}_{jl}(t))\hat{k}_{jl}(t)\Big].$$

Algorithm 5.2

1: Choose $\alpha > 0$ and $0 < \beta < \beta_0$;
2: Agent $i \in \mathcal{I}$ sends $\{d_{ij}, (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ to its neighbors;
3: Initialize $t_1^i = 0$ and $k = 1$;
4: At time $s = t_k^i$, agent i updates its control input $u_i(t_k^i)$ by sensing the relative positions to its neighbors, and determines t_{k+1}^i by (5.36)[1], and broadcasts its triggering information $\{t_{k+1}^i, u_i(t_k^i)\}$ to its neighbors;
5: At agent i's neighbors' triggering times which are between $[t_k^i, t_{k+1}^i]$, agent i receives triggering information for its neighbors[2];
6: resets $k = k + 1$, and goes back to Step 4.

Algorithm 5.3

1: Choose $\alpha > 0$ and $0 < \beta < \beta_0$;
2: Agent $i \in \mathcal{I}$ sends $\{d_{ij}, (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ to its neighbors;
3: Initialize $t_1^i = 0$ and $k = 1$;
4: At time $s = t_k^i$, agent i updates its control input $u_i(t_k^i)$ by sensing the relative positions to its neighbors, and determines t_{k+1}^i by (5.38), and resets $k = k + 1$, and repeats this step.

Then, similar to the way to determine ξ_i in (5.26), if t_k^i is known, then agent i can estimate t_{k+1}^i by

$$\int_{t_k^i}^{t_{k+1}^i} \hat{c}_i(t)dt = \alpha e^{-\beta t_{k+1}^i}. \tag{5.38}$$

The above implement idea is summarized in Algorithm 5.3.

The following theorem shows that the formation with connectivity preservation can be established and Zeno behavior can be excluded.

Theorem 5.2. *Under the same settings as Theorem 5.1. All agents perform Algorithm 5.2 or Algorithm 5.3, then the multi-agent system (5.1) with event-triggered control input (5.4) converges to the formation exponentially with connectivity preservation, and there is no Zeno behavior.*

Proof. Under both Algorithm 5.2 and Algorithm 5.3, $\|e_i(t)\| \leq \alpha e^{-\beta t}$ holds for all $i \in \mathcal{I}$ and $t \geq 0$. Then from Theorem 5.1, we know that the formation is achieved exponentially and

[1] Agent i uses $t^j_{k_j(t_k^i)}$ to replace $t^j_{k_j(t_k^i)+1}$ to determine t_{k+1}^i by (5.36) when $t_k^i = t^j_{k_j(t_k^i)}$, i.e., when agent i does not know $t^j_{k_j(t_k^i)+1}$ at time t_k^i. This situation could occur, for example when two adjacent agents trigger at the same time.

[2] In other words, agent i only listens to incoming information at its neighbors' triggering times. Thus continuous listening is avoided. This is the main difference with Algorithm 5.1.

the connectivity is preserved. The method of the exclusion of Zeno behavior is similar to the way in the proof of Theorem 5.1. □

Remark 5.3. *In order to perform Algorithm 5.2 and Algorithm 5.3, the global parameters n, β_0, k_v defined in (5.8) and k_V defined in (5.14) are needed to be known in advance. Firstly, from Remark 5.2, we can estimate β_0 by $\frac{4}{n^2\Delta}$. Secondly, one way to avoid using k_v is by choosing an arbitrary small $\varepsilon > 0$. Then, from (5.10), we have*

$$\hat{k}_{ij}(\varepsilon) := \Delta - \|d_{ij}\| - \varepsilon \geq k_{ij}. \tag{5.39}$$

Thus, $\hat{k}_{ij}(\varepsilon)$ can be used to replace k_{ij} since $f_{ij}(\cdot)$ defined in (5.12) and $g_{ij}(\cdot)$ defined in (5.16) are increasing functions. Thirdly, k_V can be estimated if we know the upper bound of $V(y(0))$ defined in (5.11). From the underlying graph $\mathcal{G}$ is connected, we have $\|y_i(0) - y_j(0)\| < (n-1)\Delta$, $\forall i, j \in \mathcal{I}$. Then $\|y_i(0) - \bar{y}(0)\| < \Delta$, $\forall i \in \mathcal{I}$. Hence $V(y(0)) < \frac{1}{2}n\Delta^2$. Thus, the only global parameter that is needed to perform Algorithm 5.2 and Algorithm 5.3 is n the number of agents.

The comparison of the inter-event times determined by Algorithm 5.1, Algorithm 5.2 and Algorithm 5.3 is shown as below.

Property 5.1. *Consider the multi-agent system (5.1) with event-triggered control input (5.4). For agent i, assume t^i_k has been determined, let $t^{i,E1}_{k+1}$, $t^{i,S1}_{k+1}$ and $t^{i,S2}_{k+1}$ be the next triggering time determined by Algorithm 5.1, Algorithm 5.2 and Algorithm 5.3 respectively, then $t^{i,E1}_{k+1} \geq t^{i,S2}_{k+1} \geq t^i_k + \epsilon_i$ and $t^{i,S1}_{k+1} \geq t^{i,S2}_{k+1} \geq t^i_k + \epsilon_i$.*

Proof. From (5.24) and (5.37), we know $c_i e^{-\beta t} \geq \hat{c}_i(t), \forall t \geq 0$ since (5.28), and $f_{ij}(\cdot)$ defined in (5.12) and $g_{ij}(\cdot)$ defined in (5.16) are increasing functions. Thus $t^{i,S2}_{k+1} \geq t^i_k + \epsilon_i$.

From (5.35) and (5.37), we know $\varphi_i(t) \leq \int_{t^i_k}^t \hat{c}_i(s)ds$, for $t \geq t^i_k$ since (5.34), and $f_{ij}(\cdot)$ and $g_{ij}(\cdot)$ are increasing functions. Thus $t^{i,S1}_{k+1} \geq t^{i,S2}_{k+1}$.

From (5.37), we know $t^{i,E1}_{k+1} \geq t^{i,S2}_{k+1}$. □

Remark 5.4. *Property 5.1 has to be considered carefully, since it only shows that for given t^i_k, the next triggering time determined by Algorithm 5.1 or Algorithm 5.2 is larger than that determined by Algorithm 5.3. However, we cannot say anything on further triggering times because generally $t^{i,E1}_{k+1} \neq t^{i,S2}_{k+1}$ and $t^{i,S1}_{k+1} \neq t^{i,S2}_{k+1}$, and thus we cannot apply this property again. Moreover, we cannot to compare $t^{i,E1}_{k+1}$ and $t^{i,S1}_{k+1}$ since $u_j(\cdot)$ are different when we perform Algorithm 5.1 and Algorithm 5.2.*

Table 5.1 summary the required exchange of information by agent $i \in \mathcal{I}$ if Algorithms 5.1–5.3 are performed.

Table 5.1: Summary of the communication requirements for agent i when Algorithms 5.1–5.3 are performed.

	Algorithm 5.1	Algorithm 5.2	Algorithm 5.3
Sensing time	$\{t_k^i, t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$	$\{t_k^i\}_{k=1}^{\infty}$	$\{t_k^i\}_{k=1}^{\infty}$
Broadcasting time	$\{t_k^i\}_{k=1}^{\infty}$	$\{t_k^i\}_{k=1}^{\infty}$	$t_1^i = 0$
Listening time	All $t \geq 0$	$\{t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$	$t_1^i = 0$
Receiving time	$\{t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$	$\{t_k^j, j \in \mathcal{N}_i\}_{k=1}^{\infty}$	$t_1^i = 0$
Information sensed	$\{x_i(t_k^i) - x_j(t_k^i)\}_{k=1}^{\infty}$, $\{x_i(t_k^j) - x_j(t_k^j), j \in \mathcal{N}_i\}_{k=1}^{\infty}$	$\{x_i(t_k^i) - x_j(t_k^i)\}_{k=1}^{\infty}$	$\{x_i(t_k^i) - x_j(t_k^i)\}_{k=1}^{\infty}$
Information broadcasted	$\{t_k^i, u_i(t_k^i)\}_{k=1}^{\infty}, d_{ij}, j \in \mathcal{N}_i$	$\{t_k^i, u_i(t_k^i)\}_{k=1}^{\infty}$, $d_{ij}, j \in \mathcal{N}_i$	$d_{ij}, j \in \mathcal{N}_i$
Zeno behavior	No	No	No

5.3 Double integrators

In this section, we extend the results in above section to the case where the dynamics of agents is modeled as double integrators given by

$$\begin{cases} \dot{x}_i(t) = r_i(t), \\ \dot{r}_i(t) = u_i^d(t), \ i \in \mathcal{I}, \ t \geq 0, \end{cases} \tag{5.40}$$

where $x_i(t) \in \mathbb{R}^p$ still denotes the position of agent i at time t, $r_i(t) \in \mathbb{R}^p$ denotes the speed and $u_i^d(t) \in \mathbb{R}^p$ is the control input. We can also rewrite (5.40) as

$$\begin{cases} \dot{y}_i(t) = r_i(t), \\ \dot{r}_i(t) = u_i^d(t), \ i \in \mathcal{I}, \ t \geq 0. \end{cases} \tag{5.41}$$

Denote

$$B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, z_i(t) = \begin{bmatrix} y_i(t) \\ r_i(t) \end{bmatrix},$$

then we can rewrite (5.41) as

$$\dot{z}_i(t) = (B_1 \otimes I_p) z_i(t) + (B_2 \otimes I_p) u_i^d(t). \tag{5.42}$$

One can easily check that (B_1, B_2) is controllable and (I_2, B_1) is observable. Hence, from [104], we know that there exist positive constants k_0, k_1 and k_2 such that

$$P > 0, \ \frac{1}{2}(PB_1 + B_1^\top P) - \beta_1 PB_2 B_2^\top P + 2I_2 \leq 0, \tag{5.43}$$

with $P = \begin{bmatrix} k_0 & k_1 \\ k_1 & k_2 \end{bmatrix}$ and $0 < \beta_1 \leq \beta_0$. Similar to (2.5), we have

$$\rho(P) \geq P \geq \rho_2(P). \tag{5.44}$$

Similar to the event-triggered control input (5.4), we use the following event-triggered control input

$$\begin{aligned} u_i^d(t) =& -k_1 \sum_{j \in N_i} \omega_{ij}(t^i_{k_i(t)})(y_i(t^i_{k_i(t)}) - y_j(t^i_{k_i(t)})) \\ & - k_2 \sum_{j \in N_i} \omega_{ij}(t^i_{k_i(t)})(r_i(t^i_{k_i(t)}) - r_j(t^i_{k_i(t)})) - k_3 r_i(t^i_{k_i(t)}) \end{aligned} \tag{5.45}$$

$$\begin{aligned} =& -k_1 \sum_{j \in N_i} \omega_{ij}(t^i_{k_i(t)})(x_i(t^i_{k_i(t)}) - x_j(t^i_{k_i(t)}) - d_{ij}) \\ & - k_2 \sum_{j \in N_i} \omega_{ij}(t^i_{k_i(t)})(r_i(t^i_{k_i(t)}) - r_j(t^i_{k_i(t)})) - k_3 r_i(t^i_{k_i(t)}), \end{aligned} \tag{5.46}$$

where k_3 is a constant which will be determined later. Here we should highlight that this control input needs absolute speed information because of the term $k_3 r_i(t^i_{k_i(t)})$. Later we will show that no agent needs to sense absolute speed if each agent knows its initial speed.

5.3.1 Event-triggered approach

Similar to Theorem 5.1, we have the following results.

Theorem 5.3. *Given a graph $\mathcal{G}$ which is undirected and connected, and a desired formation associated with $\mathcal{G}$ which satisfies Assumption 5.1. Given $0 < \beta_1 \leq \beta_0$ with β_0 defined in Theorem 5.1, determine P by (5.43). Consider the multi-agent system (5.40) with event-triggered control input (5.46) associated with $\mathcal{G}$. Assume the initial position satisfies (5.5) for all $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ and every agent knows its initial speed[1]. Given $0 < k_3 < \frac{4}{k_2 + \sqrt{k_1^2 + k_2^2}}$, $\alpha_d > 0$, $0 < \beta_d < \frac{2 - k_4}{\rho(P)}$ with $k_4 = k_3 \frac{k_2 + \sqrt{k_1^2 + k_2^2}}{2} < 2$, and the first triggering time $t_1^i = 0$, agent i determines the triggering times $\{t_k^i\}_{k=2}^{\infty}$ by*

$$t_{k+1}^i = \min\{t : \ \|E_i(t)\| \geq \alpha_d e^{-\beta_d t}, \ t \geq t_k^i\}, \tag{5.47}$$

where

$$\begin{aligned} E_i(t) =& k_1 e_i(t) + k_2 e_i^r(t) + k_3(r_i(t) - r_i(t^i_{k_i(t)})), \\ e_i^r(t) =& \sum_{j \in N_i} \omega_{ij}(t)(r_i(t) - r_j(t)) - \sum_{j \in N_i} \omega_{ij}(t^i_{k_i(t)})(r_i(t^i_{k_i(t)}) - r_j(t^i_{k_i(t)})). \end{aligned}$$

Then the multi-agent system (5.40) with event-triggered control input (5.46) converges to the formation exponentially with connectivity preservation, and there is no Zeno behavior.

[1] In real applications, initial speed normally is zero.

Proof. This theorem holds if we can prove that

(i) $\|x_i(t) - x_j(t)\| \leq \Delta,\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ \forall t \geq 0$;

(ii) $\lim_{t\to\infty}(x_i(t) - x_j(t)) = d_{ij},\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G})$, exponentially;

(iii) there is no Zeno behavior.

(i) We define the total tension energy of $\mathcal{G}$ as

$$v_d(\Delta, y(t)) = k_1 v(\Delta, y(t)) + \frac{1}{2}\sum_{i=1}^{n} \|r_i(t)\|^2. \tag{5.48}$$

Then time derivative of $v_d(\Delta, y(t))$ along the trajectories of the multi-agent system (5.41) with event-triggered control input (5.45) is

$$\begin{aligned}
\dot{v}_d(\Delta, y(t)) =& k_1 \sum_{i=1}^{n}\sum_{j\in N_i}\Big[\frac{\partial v_{ij}(\Delta, y)}{\partial y_i}\Big]^\top\Big|_{y=y(t)}\dot{y}_i(t) + \sum_{i=1}^{n} r_i^\top(t)\dot{r}_i(t)\\
=& \sum_{i=1}^{n} r_i^\top(t)\Big\{k_1\sum_{j\in N_i}[\omega_{ij}(t)(y_i(t) - y_j(t))] + u_i^d(t)\Big\}\\
=& \sum_{i=1}^{n} r_i^\top(t)\Big\{E_i(t) - k_2\sum_{j\in N_i}\omega_{ij}(t)(r_i(t) - r_j(t)) - k_3 r_i(t)\Big\}\\
\leq& \frac{1}{4k_3}\sum_{i=1}^{n}\|E_i(t)\|^2 - \sum_{i=1}^{n} r_i^\top(t)k_2\sum_{j\in N_i}\omega_{ij}(t)(r_i(t) - r_j(t))\\
=& \frac{1}{4k_3}\sum_{i=1}^{n}\|E_i(t)\|^2 - k_2 r^\top(t)L_\omega r(t),
\end{aligned} \tag{5.49}$$

From (5.47), we know that

$$\|E_i(t)\| \leq \alpha_d e^{-\beta_d t}, \forall t \geq 0. \tag{5.50}$$

Hence

$$\dot{v}_d(\Delta, y(t)) \leq \frac{n\alpha_d^2}{4k_3}e^{-2\beta_d t}, \forall t \geq 0.$$

Thus

$$v_d(\Delta, y(t)) \leq v_d(\Delta, y(0)) + \frac{n\alpha_d^2}{8k_3\beta_d}[1 - e^{-2\beta_d t}] \leq k_v^d, \forall t \geq 0, \tag{5.51}$$

where

$$k_v^d = v_d(\Delta, y(0)) + \frac{n\alpha_d^2}{8k_3\beta_d}$$

$$=\frac{k_1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\frac{\|x_i(0)-x_j(0)-d_{ij}\|^2}{\Delta-\|d_{ij}\|-\|x_i(0)-x_j(0)-d_{ij}\|}+\frac{1}{2}\sum_{i=1}^{n}\|r_i(0)\|^2+\frac{n\alpha_d^2}{8k_3\beta_d}. \tag{5.52}$$

Then, for any $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ and $t \geq 0$, we have

$$v_{ij}(\Delta, y(t)) = \frac{\|y_i(t)-y_j(t)\|^2}{\Delta-\|d_{ij}\|-\|y_i(t)-y_j(t)\|} \leq \frac{2}{k_1}v_d(\Delta, y(t)) \leq \frac{2}{k_1}k_v^d.$$

Hence

$$\|y_i(t)-y_j(t)\| \leq k_{ij}^d, \tag{5.53}$$

where

$$k_{ij}^d = -\frac{k_v^d}{k_1}+\sqrt{\Big(\frac{k_v^d}{k_1}\Big)^2+2\frac{k_v^d}{k_1}(\Delta-\|d_{ij}\|)} < \Delta-\|d_{ij}\|. \tag{5.54}$$

Then, we have

$$\begin{aligned}\|x_i(t)-x_j(t)\| =&\|x_i(t)-\tau_i-(x_j(t)-\tau_j)+d_{ij}\| = \|y_i(t)-y_j(t)+d_{ij}\|\\ \leq&\|y_i(t)-y_j(t)\|+\|d_{ij}\| \leq k_{ij}^d+\|d_{ij}\| < \Delta,\end{aligned}$$

and thus connectivity maintenance is guaranteed.

(ii) Note $B_2^\top P = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then we can rewrite the control input (5.45) as

$$u_i^d(t) = -(B_2^\top P\otimes I_p)\sum_{j\in N_i}\omega_{ij}(t)(z_i(t)-z_j(t))+E_i(t)-k_3(B_2^\top\otimes I_p)z_i(t).$$

Let $z(t) = [z_1^\top(t), \ldots, z_n^\top(t)]^\top$ and $\bar{z}(t) = \frac{1}{n}\sum_{i=1}^n z_i(t)$. We consider the following Lyapunov candidate

$$V_d(z(t)) = \frac{1}{2}[z(t)-\mathbf{1}_n\bar{z}(t)]^\top(I_n\otimes P\otimes I_p)[z(t)-\mathbf{1}_n\bar{z}(t)] = \frac{1}{2}z^\top(t)(K_n\otimes P\otimes I_p)z(t).$$

The last equality holds since

$$z^\top(t)(K_n\otimes I_2\otimes I_p)z(t) = [z(t)-\mathbf{1}_n\bar{z}(t)]^\top(I_n\otimes I_2\otimes I_p)[z(t)-\mathbf{1}_n\bar{z}(t)]$$

due to $K_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. Then the derivative of $V_d(z(t))$ along the trajectories of (5.42) is

$$\begin{aligned}\dot{V}_d(z(t)) =&z^\top(t)(K_n\otimes P\otimes I_p)\dot{z}(t)\\ =&z^\top(t)(K_n\otimes P\otimes I_p)\Big\{(I_n\otimes B_1\otimes I_p)z(t)+(I_n\otimes B_2\otimes I_p)u^d(t)\Big\}\\ =&z^\top(t)(K_n\otimes P\otimes I_p)\Big\{(I_n\otimes B_1\otimes I_p)z(t)\\ &+(I_n\otimes B_2\otimes I_p)[-(L_\omega\otimes B_2^\top P\otimes I_p)z(t)+E(t)-k_3(I_n\otimes B_2^\top\otimes I_p)z(t)]\Big\}\end{aligned}$$

$$
\begin{aligned}
=&z^\top(t)(K_n \otimes \frac{PB_1 + B_1^\top P}{2} \otimes I_p)z(t) - z^\top(t)(L_\omega \otimes PB_2B_2^\top P \otimes I_p)z(t)\\
&- k_3 z^\top(t)(K_n \otimes PB_2B_2^\top \otimes I_p)z(t) + z^\top(t)(K_n \otimes PB_2 \otimes I_p)E(t),
\end{aligned}
$$

where $u^d(t) = [(u_1^d)^\top(t), \ldots, (u_n^d)^\top(t)]$ and $E(t) = [E_1^\top(t), \ldots, E_n^\top(t)]$. From $PB_2B_2^\top P \geq 0$ and $L_\omega \geq 2\beta_0 K_n \geq 2\beta_1 K_n$ (see Lemma 2.7), we have

$$
-z^\top(t)(L_\omega \otimes PB_2B_2^\top P \otimes I_p)z(t) \leq -2\beta_1 z^\top(t)(K_n \otimes PB_2B_2^\top P \otimes I_p)z(t). \tag{5.55}
$$

Noting

$$
\frac{PB_2B_2^\top + B_2B_2^\top P}{2} = \begin{bmatrix} 0 & \frac{k_1}{2} \\ \frac{k_1}{2} & k_2 \end{bmatrix},
$$

one can easily check that $\rho(\frac{PB_2B_2^\top + B_2B_2^\top P}{2}) = \frac{k_2 + \sqrt{k_1^2 + k_2^2}}{2}$. Noting $k_4 = k_3 \frac{k_2 + \sqrt{k_1^2 + k_2^2}}{2}$, we have

$$
-k_3 z^\top(t)(K_n \otimes PB_2B_2^\top \otimes I_p)z(t) \leq k_4 z^\top(t)(K_n \otimes I_2 \otimes I_p)z(t). \tag{5.56}
$$

Then from (5.55), (5.56) and the following inequality

$$
z^\top(t)(K_n \otimes PB_2 \otimes I_p)E(t) \leq \beta_1 z^\top(t)(K_n \otimes PB_2B_2^\top P \otimes I_p)z(t) + \frac{1}{4\beta_1}\|E(t)\|^2,
$$

we get

$$
\begin{aligned}
\dot{V}_d(z(t)) \leq& z^\top(t)\Big(K_n \otimes \Big[\frac{PB_1 + B_1^\top P}{2} - \beta_1 PB_2B_2^\top P\Big] \otimes I_p\Big)z(t)\\
&+ k_4 z^\top(t)(K_n \otimes I_2 \otimes I_p)z(t) + \frac{1}{4\beta_1}\|E(t)\|^2\\
\leq& -(2 - k_4)z^\top(t)(K_n \otimes I_2 \otimes I_p)z(t) + \frac{1}{4\beta_1}\|E(t)\|^2\\
\leq& -\frac{2(2 - k_4)}{\rho(P)}V_d(z(t)) + \frac{n\alpha_d^2}{4\beta_1}e^{-2\beta_d t},
\end{aligned}
$$

where the second inequality holds since (5.43) and the last inequality holds since (5.44). Hence

$$
V_d(z(t)) \leq V_d(z(0))e^{-\frac{2(2-k_4)}{\rho(P)}t} + \frac{\rho(P)n\alpha_d^2[e^{-2\beta_d t} - e^{-\frac{2(2-k_4)}{\rho(P)}t}]}{8\beta_1[(2 - k_4) - \beta_d\rho(P)]}.
$$

Thus

$$
\begin{aligned}
&\|y_i(t) - y_j(t)\|^2 + \|r_i(t) - r_j(t)\|^2\\
=&\|z_i(t) - z_j(t)\|^2 \leq 2\|z_i(t) - \bar{z}(t)\|^2 + 2\|\bar{z}(t) - z_j(t)\|^2 \leq \frac{4}{\rho_2(P)}V_d(z(t)) < k_V^d e^{-2\beta_d t},
\end{aligned} \tag{5.57}
$$

where

$$k_V^d = \frac{4V_d(z(0))}{\rho_2(P)} + \frac{\rho(P)n\alpha_d^2}{2\rho_2(P)\beta_1[(2-k_4)-\beta_d\rho(P)]}. \tag{5.58}$$

Hence

$$\lim_{t\to\infty}(x_i(t)-x_j(t)) = \lim_{t\to\infty}(y_i(t)-\tau_i-(y_j(t)-\tau_j)) = d_{ij}, \tag{5.59}$$

and

$$\lim_{t\to\infty}(r_i(t)-r_j(t)) = 0, \tag{5.60}$$

exponentially.

(iii) From

$$\dot r_i(t) = u_i^d(t) = E_i(t) - k_1\sum_{l\in N_i}\omega_{il}(t)(y_i(t)-y_l(t)) - k_2\sum_{l\in N_i}\omega_{il}(t)(r_i(t)-r_l(t)) - k_3 r_i(t),$$

we have

$$\frac{de^{k_3t}r_i(t)}{dt} = \Big[E_i(t) - k_1\sum_{l\in N_i}\omega_{il}(t)(y_i(t)-y_l(t)) - k_2\sum_{l\in N_i}\omega_{il}(t)(r_i(t)-r_l(t))\Big]e^{k_3t}.$$

Then, similar to (5.19), we have

$$\begin{aligned}\frac{d\|e^{k_3t}r_i(t)\|}{dt} &\le \left\|\frac{de^{k_3t}r_i(t)}{dt}\right\| = \Big\|E_i(t) - k_1\sum_{l\in N_i}\omega_{il}(t)(y_i(t)-y_l(t)) - k_2\sum_{l\in N_i}\omega_{il}(t)(r_i(t)-r_l(t))\Big\|e^{k_3t}\\ &\le c_i^r e^{(k_3-\beta_d)t},\end{aligned}$$

where

$$c_i^r = \alpha_d + (k_1+k_2)\sum_{l\in N_i} f_{il}(k_{il}^d)\sqrt{k_V^d}.$$

From

$$e^{k_3t}\frac{d\|r_i(t)\|}{dt} \le e^{k_3t}\frac{d\|r_i(t)\|}{dt} + k_3e^{k_3t}\|r_i(t)\| = \frac{de^{k_3t}\|r_i(t)\|}{dt} = \frac{d\|e^{k_3t}r_i(t)\|}{dt},$$

we have

$$\frac{d\|r_i(t)\|}{dt} \le c_i^r e^{-\beta_d t},\ \forall t\ge 0. \tag{5.61}$$

Thus

$$\|r_i(t)\| \le \|r_i(0)\| + \frac{c_i^r}{\beta_d}, \tag{5.62}$$

and

$$\begin{aligned}\|u_i^d(t)\| =&\|\dot{r}_i(t)\| = \Big\|E_i(t) - k_1\sum_{l\in N_i}\omega_{il}(t)(y_i(t)-y_l(t)) - k_2\sum_{l\in N_i}\omega_{il}(t)(r_i(t)-r_l(t)) - k_3 r_i(t)\Big\| \\ \le& c_i^r e^{-\beta_d t} + k_3\Big(\|r_i(0)\| + \frac{c_i^r}{\beta_d}\Big). \end{aligned} \tag{5.63}$$

Again, similar to (5.19), we have

$$\begin{aligned}\|\dot{r}_i(t) - \dot{r}_j(t)\| =& \|E_i(t) - E_j(t) - k_1\sum_{l\in N_i}\omega_{il}(t)(y_i(t)-y_l(t)) \\ &- k_2\sum_{l\in N_i}\omega_{il}(t)(r_i(t)-r_l(t)) + k_1\sum_{l\in N_j}\omega_{jl}(t)(y_j(t)-y_l(t))\| \\ &+ k_2\sum_{l\in N_j}\omega_{jl}(t)(r_j(t)-r_l(t)) - k_3(r_i(t)-r_j(t))\| \qquad (5.64) \\ <& c_{ij}^r e^{-\beta_d t}, \qquad (5.65)\end{aligned}$$

where

$$c_{ij}^r = 2\alpha_d + \Big\{(k_1+k_2)\Big[\sum_{l\in N_i} f_{il}(k_{il}^d) + \sum_{l\in N_j} f_{jl}(k_{jl}^d)\Big] + k_3\Big\}\sqrt{k_V^d}. \tag{5.66}$$

Similar to (5.20), we have

$$\begin{aligned}\dot{e}_i(t) =& \sum_{j\in N_i}[\dot{\omega}_{ij}(t)(y_i(t)-y_j(t)) + \omega_{ij}(t)(\dot{y}_i(t)-\dot{y}_j(t))] \\ =& \sum_{j\in N_i}\Big\{h_{ij}(\|y_i(t)-y_j(t)\|)\frac{(y_i(t)-y_j(t))^\top}{\|y_i(t)-y_j(t)\|}(r_i(t)-r_j(t))(y_i(t)-y_j(t)) \\ &+ \omega_{ij}(t)(r_i(t)-r_j(t))\Big\}, \end{aligned} \tag{5.67}$$

and

$$\begin{aligned}\dot{e}_i^r(t) =& \sum_{j\in N_i}[\dot{\omega}_{ij}(t)(r_i(t)-r_j(t)) + \omega_{ij}(t)(\dot{r}_i(t)-\dot{r}_j(t))] \\ =& \sum_{j\in N_i}\Big\{h_{ij}(\|y_i(t)-y_j(t)\|)\frac{(y_i(t)-y_j(t))^\top}{\|y_i(t)-y_j(t)\|}(r_i(t)-r_j(t))(r_i(t)-r_j(t)) \\ &+ \omega_{ij}(t)(\dot{r}_i(t)-\dot{r}_j(t))\Big\}. \end{aligned} \tag{5.68}$$

Similar to (5.23), we have

$$\begin{aligned}\frac{d\|E_i(t)\|}{dt} =& \frac{d\|k_1 e_i(t) + k_2 e_i^r(t) + k_3(r_i(t) - r_i(t_{k_i(t)}^i))\|}{dt} \\ \le& \|k_1\dot{e}_i(t) + k_2\dot{e}_i^r(t) + k_3\dot{r}_i(t)\| \end{aligned} \tag{5.69}$$

$$
\begin{aligned}
&\leq k_1\|\dot{e}_i(t)\| + k_2\|\dot{e}_i^r(t)\| + k_3\|\dot{r}_i(t)\| \\
&\leq \sum_{j\in N_i}\Big\{k_1 g_{ij}(\|y_i(t)-y_j(t)\|)\|r_i(t)-r_j(t)\| \\
&\quad + k_2 h_{ij}(\|y_i(t)-y_j(t)\|)\|r_i(t)-r_j(t)\|^2 + k_2\omega_{ij}(t)(\|\dot{r}_i(t)-\dot{r}_j(t)\|)\Big\} + k_3\|u_i^d(t)\| \\
&\leq \sum_{j\in N_i} k_1 g_{ij}(k_{ij}^d)\|r_i(t)-r_j(t)\| + k_2 h_{ij}(k_{ij}^d)\|r_i(t)-r_j(t)\|^2 \\
&\quad + k_2 f_{ij}(k_{ij}^d)(\|\dot{r}_i(t)-\dot{r}_j(t)\|) + k_3\Big[c_i^q e^{-\beta_d t} + k_3\Big(\|r_i(0)\| + \frac{c_i^r}{\beta_d}\Big)\Big] \\
&< c_i^d e^{-\beta_d t} + k_3\Big[c_i^r e^{-\beta_d t} + k_3\Big(\|r_i(0)\| + \frac{c_i^r}{\beta_d}\Big)\Big],
\end{aligned}
\tag{5.70}
$$

where

$$
c_i^d = \sum_{j\in N_i}\Big\{k_1 g_{ij}(k_{ij}^d)\sqrt{k_V^d} + k_2 h_{ij}(k_{ij}^d)k_V^d + k_2 f_{ij}(k_{ij}^d)c_{ij}^r\Big\}.
$$

Thus

$$
\frac{d\|E_i(t)\|}{dt} < c_i^e, \tag{5.71}
$$

where

$$
c_i^e = c_i^d + k_3\Big[c_i^r + k_3\Big(\|q_i(0)\| + \frac{c_i^r}{\beta_d}\Big)\Big].
$$

From (5.71), similar to the way to exclude Zeno behavior in the proof of Theorem 3.1 or 4.2, we can prove that there is no Zeno behavior by contradiction. □

Similar to the analysis after Theorem 5.1, in order to monitor the inequality in the triggering law (5.47), each agent needs to continuously sense its absolute speed, the relative positions and speeds to its neighbors. In the following we will give an event-triggered algorithm to implement Theorem 5.3 and at the same time to avoid continuous sensing by using the similar idea as Algorithm 5.1.

Since it is assumed that every agent knows its initial speed, every agent $i \in \mathcal{I}$, can know $\{r_i(t_k^i)\}_{k=1}^{\infty}$ by iterative computation as follows

$$
r_i(t_{k+1}^i) = r_i(t_k^i) + (t_{k+1}^i - t_k^i)u_i^d(t_k^i). \tag{5.72}
$$

Thus, at any time $s \geq 0$, agent i can predict

$$
r_i(t) = r_i(t_{k_i(s)}^i) + (t - t_{k_i(s)}^i)u_i^d(t_{k_i(s)}^i), \ \forall t \geq s. \tag{5.73}
$$

This means that no agent needs to sense absolute speed.

Algorithm 5.4

1: Choose $0 < \beta_1 \leq \beta_0$ and determine P by (5.43);
2: Choose $0 < k_3 < \frac{4}{k_2 + \sqrt{k_1^2 + k_2^2}}$, $\alpha_d > 0$ and $0 < \beta_d < \frac{2 - k_4}{\rho(P)}$;
3: Initialize $t_1^i = 0$ and $k = 1$;
4: Agent $i \in \mathcal{I}$ sends $\{d_{ij}, (v_i, v_j) \in \mathcal{E}(\mathcal{G})\}$ to its neighbors;
5: Agent i continuously listens to whether there is broadcasting from its neighbors and receives the broadcasted information if it occurs;
6: At time $s = t_k^i$, agent i senses the relative position $x_i(s) - x_j(s)$ and relative speed $r_i(s) - r_j(s)$, and predicts future relative position $x_i(t) - x_j(t)$, the relative speed $r_i(t) - r_j(t)$, $\forall j \in \mathcal{N}_i$, and its future speed $r_i(t)$, $t \geq s$ by (5.74), (5.75) and (5.73), respectively;
7: Agent i substitutes these into $E_i(t)$ and finds out τ_{k+1}^i which is the smallest solution of equation $\|E_i(t)\| = \alpha_d e^{-\beta_d t}$;
8: **if** there is broadcasting from its neighbors at $t_0 \in (s, \tau_{k+1}^i)$, i.e., there exists $j \in \mathcal{N}_i$ such that agent j broadcasts its triggering information at $t_0 \in (s, \tau_{k+1}^i)$ **then**
9:     agent i receives information at t_0, and updates $s = t_0$, and goes back to Step 6;
10: **else**
11:     agent i determines $t_{k+1}^i = \tau_{k+1}^i$, and gets $r_i(t_{k+1}^i)$ by (5.72), and updates $u_i^d(t_{k+1}^i)$ by sensing the relative positions and speeds to its neighbors, and broadcasts its triggering information $\{t_{k_i(t)}^i, u_i^d(t_{k_i(t)}^i)\}$ to its neighbors, and resets $k = k + 1$, and goes back to Step 6;
12: **end if**

Each agent $i \in \mathcal{I}$, at any time $s \geq 0$, knows its last triggering time $t_{k_i(s)}^i$ and control input $u_i^d(t_{k_i(s)}^i)$ which is a constant until it determines its next triggering time. If agent i also knows the relative position $x_i(s) - x_j(s)$, relative speed $r_i(s) - r_j(s)$ and $u_j^d(s) = u_j^d(t_{k_j(s)}^j)$ which is a constant until agent j determines its next triggering time, for $j \in \mathcal{N}_i$, then agent i can predict

$$x_i(t) - x_j(t) = x_i(s) - x_j(s) + (t - s)(r_i(s) - r_j(s)) + \frac{1}{2}(t - s)^2(u_i^d(t_{k_i(s)}^i) - u_j^d(t_{k_j(s)}^j)), \quad (5.74)$$

$$r_i(t) - r_j(t) = r_i(s) - r_j(s) + (t - s)(u_i^d(t_{k_i(s)}^i) - u_j^d(t_{k_j(s)}^j)), \ t \geq s, \quad (5.75)$$

until $t \leq \min\left\{t_{k_i(s)+1}^i, t_{k_j(s)+1}^j\right\}$. This means that continuous sensing, broadcasting and receiving are not needed any more.

The above implement idea is summarized in Algorithm 5.4.

5.3.2 Self-triggered algorithms

As noted earlier, each agent still needs to continuously listen to incoming information from its neighbors. In order to avoid this, in the following we will first give a self-triggered algorithm which is similar to Algorithm 5.2 such that each agent only needs to listen at its neighbors' triggering times. Then, we will give another self-triggered algorithm which is similar to Algorithm 5.3 such that broadcasting, receiving, and listening only occur at the beginning.

From (5.53) and (5.57), we have

$$\|y_i(t) - y_j(t)\| < \hat{k}^y_{ij}(t),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \forall t \geq 0, \tag{5.76}$$

where

$$\hat{k}^y_{ij}(t) = \min\left\{k^d_{ij},\ \sqrt{k^d_V} e^{-\beta_d t}\right\}.$$

From (5.57) and (5.62), we have

$$\|r_i(t) - r_j(t)\| < \hat{k}^r_{ij}(t),\ \forall i, j \in \mathcal{I}, \forall t \geq 0. \tag{5.77}$$

where

$$\hat{k}^r_{ij}(t) = \min\left\{\sqrt{k^d_V} e^{-\beta_d t}, \|r_i(0)\| + \|r_j(0)\| + \frac{c^r_i + c^r_j}{\beta_d}\right\}.$$

Then, similar to (5.65), we have

$$\|\dot{r}_i(t) - \dot{r}_j(t)\| = \|u^d_i(t) - u^d_j(t)\| < \theta^d_{ij}(t),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \forall t \geq 0, \tag{5.78}$$

where

$$\theta^d_{ij}(t) = 2\alpha_d e^{-\beta_d t} + \sum_{l \in N_i} f_{il}(\hat{k}^y_{il}(t))\Big(k_1 \hat{k}^y_{il}(t) + k_2 \hat{k}^r_{il}(t)\Big) + \sum_{l \in N_j} f_{jl}(\hat{k}^y_{jl}(t))\Big(k_1 \hat{k}^y_{jl}(t) + k_2 \hat{k}^r_{jl}(t)\Big) + k_3 \hat{k}^r_{ij}(t).$$

Then, similar to (5.31), we have

$$r_i(t) - r_j(t) = z^r_{ij}(t^i_k, t) + \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}(t)} (u^d_i(s) - u^d_j(s))ds,\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t^i_k, t^i_{k+1}), \tag{5.79}$$

where $t^1_{ij}(t)$ and $t^2_{ij}(t)$ defined in (5.30), and

$$z^r_{ij}(t^i_k, t) = r_i(t^i_k) - r_j(t^i_k) + (t^1_{ij}(t) - t^i_k)(u^d_i(t^i_k) - u^d_j(t^j_{k_j(t^i_k)})).$$

Thus

$$\begin{aligned}\|r_i(t) - r_j(t)\| \leq& \|z^r_{ij}(t^i_k, t)\| + \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}(t)} \|u^d_i(s) - u^d_j(s)\| ds \\ \leq& \check{k}^r_{ij}(t),\ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}),\ t \in [t^i_k, t^i_{k+1}),\end{aligned}$$

where

$$\check{k}^r_{ij}(t) = \|z^r_{ij}(t^i_k, t)\| + \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}(t)} \theta^d_{ij}(s) ds,\ t \in [t^i_k, t^i_{k+1}). \tag{5.80}$$

Hence, then, from (5.77) and (5.80), we have

$$\|r_i(t) - r_j(t)\| \leq \tilde{k}^r_{ij}(t), \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), t \in [t^i_k, t^i_{k+1}), \tag{5.81}$$

where

$$\tilde{k}^r_{ij}(t) = \min\{\hat{k}^r_{ij}(t), \check{k}^r_{ij}(t)\}, t \in [t^i_k, t^i_{k+1}). \tag{5.82}$$

From $\dot{y}_i(t) = r_i(t)$ and (5.79), we have

$$\begin{aligned} y_i(t) - y_j(t) =& y_i(t^i_k) - y_j(t^i_k) + \int_{t^i_k}^t [r_i(s) - r_j(s)]ds \\ =& z^y_{ij}(t^i_k, t) + \int_{t^i_k}^t \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}(r)} (u^d_i(s) - u^d_j(s))dsdr, \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \ t \in [t^i_k, t^i_{k+1}), \end{aligned} \tag{5.83}$$

where

$$z^y_{ij}(t^i_k, t) = y_i(t^i_k) - y_j(t^i_k) + (t^1_{ij}(t) - t^i_k)(r_i(t^i_k) - r_j(t^i_k)) + \frac{1}{2}(t^1_{ij}(t) - t^i_k)^2(u^d_i(t^i_k) - u^d_j(t^j_{k_j(t^i_k)})).$$

Thus

$$\begin{aligned} \|y_i(t) - y_j(t)\| \leq& \|z^y_{ij}(t^i_k, t)\| + \int_{t^i_k}^t \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}(r)} \|u^d_i(s) - u^d_j(s)\|dsdr \\ \leq& \check{k}^y_{ij}(t), \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \ t \in [t^i_k, t^i_{k+1}), \end{aligned} \tag{5.84}$$

where

$$\check{k}^y_{ij}(t) = \|z^y_{ij}(t^i_k, t)\| + \int_{t^i_k}^t \int_{t^j_{k_j(t^i_k)+1}}^{t^2_{ij}(r)} \theta^d_{ij}(s)dsdr. \tag{5.85}$$

Hence, then, from (5.76) and (5.84), we have

$$\|y_i(t) - y_j(t)\| \leq \tilde{k}^y_{ij}(t), \ \forall (v_i, v_j) \in \mathcal{E}(\mathcal{G}), \ t \in [t^i_k, t^i_{k+1}), \tag{5.86}$$

where

$$\tilde{k}^y_{ij}(t) = \min\left\{\hat{k}^y_{ij}(t), \check{k}^y_{ij}(t)\right\}, t \in [t^i_k, t^i_{k+1}). \tag{5.87}$$

Then from (5.67)–(5.70), (5.79), (5.81), (5.83), and (5.86), we have

$$\|E_i(t)\| \leq \varphi^d_i(t), t \in [t^i_k, t^i_{k+1}), \tag{5.88}$$

where

$$\varphi^d_i(t) = \Big\| \sum_{j \in N_i} \int_{t^i_k}^{t^1_{ij}(t)} \Big\{ k_1 h_{ij}(\|z^y_{ij}(t^i_k, s)\|) \frac{(z^y_{ij}(t^i_k, s))^\top}{\|z^y_{ij}(t^i_k, s)\|} z^r_{ij}(t^i_k, s) z^y_{ij}(t^i_k, s)$$

$$
\begin{aligned}
&+k_1 f_{ij}(\|z^y_{ij}(t^i_k,s)\|)(z^r_{ij}(t^i_k,s)) + k_2 h_{ij}(\|z^y_{ij}(t^i_k,s)\|)\frac{(z^y_{ij}(t^i_k,s))^\top}{\|z^y_{ij}(t^i_k,s)\|} z^r_{ij}(t^i_k,s) z^r_{ij}(t^i_k,s)\\
&+k_2 f_{ij}(\|z^y_{ij}(t^i_k,s)\|)(u^d_i(t^i_k) - u^d_j(t^j_{k_j(t^i_k)}))\Big\}ds + k_3(t-t^i_k)u^d_i(t^i_k)\Big\|\\
&+\sum_{j\in N_i}\int_{t^1_{ij}(t)}^{t}\Big\{k_1 g_{ij}(\tilde{k}^y_{ij}(s))\tilde{k}^r_{ij}(s) + k_2 h_{ij}(\tilde{k}^y_{ij}(s))(\tilde{k}^r_{ij}(s))^2 + k_2 f_{ij}(\tilde{k}^y_{ij}(s))\theta^d_{ij}(s)\Big\}ds\\
=&\Big\|\sum_{j\in N_i}\Big\{f_{ij}(\|z^y_{ij}(t^i_k,t^1_{ij}(t))\|)(k_1 z^y_{ij}(t^i_k,t^1_{ij}(t)) + k_2 z^r_{ij}(t^i_k,t^1_{ij}(t)))\\
&- f_{ij}(\|z^y_{ij}(t^i_k,t^i_k)\|)(k_1 z^y_{ij}(t^i_k,t^i_k) + k_2 z^r_{ij}(t^i_k,t^i_k))\Big\} + k_3(t-t^i_k)u^d_i(t^i_k)\Big\|\\
&+\sum_{j\in N_i}\int_{t^1_{ij}(t)}^{t}\Big\{k_1 g_{ij}(\tilde{k}^y_{ij}(s))\tilde{k}^r_{ij}(s) + k_2 h_{ij}(\tilde{k}^y_{ij}(s))(\tilde{k}^r_{ij}(s))^2\\
&+k_2 f_{ij}(\tilde{k}^y_{ij}(s))\theta^d_{ij}(s)\Big\}ds,\ t\in[t^i_k,t^i_{k+1}).
\end{aligned}
\tag{5.89}
$$

Hence, a necessary condition to guarantee the inequality in (5.47), i.e.,

$$\alpha_d e^{-\beta_d t} \le \|E_i(t)\|,\ \forall t\in[t^i_k,t^i_{k+1}),$$

is

$$\alpha_d e^{-\beta_d t} = \varphi^d_i(t),\ \forall t\in[t^i_k,t^i_{k+1}).$$

Since $\alpha_d e^{-\beta_d t}$ decreases with respect to t, $\varphi^d_i(t)$ increases with respect to t during $[t^i_k,t^i_{k+1})$ and $\varphi^d_i(t^i_k)=0$, then given t^i_k, agent i can estimate t^i_{k+1} by the solution to

$$\alpha_d e^{-\beta_d t} = \varphi^d_i(t),\ t\ge t^i_k. \tag{5.90}$$

In other words, if at time t^i_k agent i knows $t^j_{k_j(t^i_k)}$, $t^j_{k_j(t^i_k)+1}$, $u^d_j(t^j_{k_j(t^i_k)})$, $\forall j\in\mathcal{N}_i$, then it can estimate its next triggering time t^i_{k+1} by solving (5.90). The above implement idea is summarized in Algorithm 5.5.

Similar to the single integrators case, broadcasting, receiving and listening can be ruled out except at the beginning, and each agent only needs to sense the relative positions to its neighbors and to update its control input at its triggering times. The idea is illustrated as follows.

From (5.76), (5.77), (5.64) and (5.70), we have

$$\frac{d\|E_i(t)\|}{dt} < \hat{c}^d_i(t), \tag{5.91}$$

where

$$\hat{c}^d_i(t) = \sum_{j\in N_i}\Big\{k_1 g_{ij}(\hat{k}^y_{ij}(t))\hat{k}^r_{ij}(t) + k_2 h_{ij}(\hat{k}^y_{ij}(t))(\hat{k}^r_{ij}(t))^2 + k_2 f_{ij}(\hat{k}^y_{ij}(t))\theta^d_{ij}(t)\Big\} + k_3\|u^d_i(t)\|.$$

Algorithm 5.5

1: Choose $0 < \beta_1 \leq \beta_0$ and determine P by (5.43);
2: Choose $0 < k_3 < \frac{4}{k_2 + \sqrt{k_1^2 + k_2^2}}$, $\alpha_d > 0$ and $0 < \beta_d < \frac{2-k_4}{\rho(P)}$;
3: Agent $i \in \mathcal{I}$ sends $\{d_{ij}, (v_i, v_j) \in \mathcal{E}(\mathcal{G}), r_i(0)\}$ to its neighbors;
4: Initialize $t_1^i = 0$ and $k = 1$;
5: At time $s = t_k^i$, agent i gets $r_i(t_k^i)$ by (5.72), and updates $u_i^d(t_k^i)$ by sensing the relative positions and speeds to its neighbors, and determines t_{k+1}^i by (5.90)[1], and broadcasts its triggering information $\{t_{k+1}^i, u_i^d(t_k^i)\}$ to its neighbors;
6: At agent i's neighbors' triggering times which are between $[t_k^i, t_{k+1}^i]$, agent i receives triggering information for its neighbors[2];
7: resets $k = k + 1$, and goes back to Step 5.

Algorithm 5.6

1: Choose $0 < \beta_1 \leq \beta_0$ and determine P by (5.43);
2: Choose $0 < k_3 < \frac{4}{k_2 + \sqrt{k_1^2 + k_2^2}}$, $\alpha_d > 0$ and $0 < \beta_d < \frac{2-k_4}{\rho(P)}$;
3: Agent $i \in \mathcal{I}$ sends $\{d_{ij}, (v_i, v_j) \in \mathcal{E}(\mathcal{G}), r_i(0)\}$ to its neighbors;
4: Initialize $t_1^i = 0$ and $k = 1$;
5: At time $s = t_k^i$, agent i gets $r_i(t_k^i)$ by (5.72), and updates $u_i^d(t_k^i)$ by sensing the relative positions and speeds to its neighbors, and determines t_{k+1}^i by (5.92), and resets $k = k+1$, and repeats this step.

If t_k^i is known, then agent i can estimate t_{k+1}^i by

$$\int_{t_k^i}^{t_{k+1}^i} \hat{c}_i^d(t)dt = \alpha_d e^{-\beta_d t_{k+1}^i}. \tag{5.92}$$

The above implement idea is summarized in Algorithm 5.6.

The following theorem shows that the formation with connectivity preservation can be established and Zeno behavior can be excluded.

Theorem 5.4. *Under the same settings as Theorem 5.3. All agents perform Algorithm 5.5 or Algorithm 5.6, then the multi-agent system (5.40) with event-triggered control input (5.46) converges to the formation exponentially with connectivity preservation, and there is no Zeno behavior.*

Proof. Under both Algorithm 5.5 and Algorithm 5.6, $\|E_i(t)\| \leq \alpha_d e^{-\beta_d t}$ holds for all $i \in \mathcal{I}$ and $t \geq 0$. Then from Theorem 5.3, we know that the formation is achieved exponentially and the connectivity is preserved. The method of the exclusion of Zeno behavior is similar to the way in the proof of Theorem 5.3. □

[1] Agent i uses $t^j_{k_j(t_k^i)}$ to replace $t^j_{k_j(t_k^i)+1}$ to determine t_{k+1}^i by (5.90) when $t_k^i = t^j_{k_j(t_k^i)}$.

[2] In other words, agent i only listen to incoming information at its neighbors' triggering times. Thus continuous listening is avoided.

Remark 5.5. *In real applications, it is reasonable to assume the initial speed of each agent is zero. By this assumption and Remark 5.3, we know that the only global parameter that is needed to perform Algorithm 5.5 and Algorithm 5.6 is n the number of agents.*

Remark 5.6. *The absolute measurements of positions and speeds are not needed when performing Algorithms 5.1–5.6.*

Similar to Table 5.1, we can summarizes what and when information should be exchanged by each agent when Algorithms 5.4–5.6 are performed. Since it is similar to Table 5.1, we omit it here. Moreover, the comparison of the inter-event times determined by Algorithms 5.4–5.6 is similar to Property 5.1.

5.4 Simulations

In this section, two numerical examples are given to demonstrate the effectiveness of the presented results.

Consider a network of $n = 3$ agents in $\mathbb{R}^2$ whose Laplacian matrix is given by

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The three agents are trying to establish a right triangle formation with

$$d_{12} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \; d_{13} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \; d_{23} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

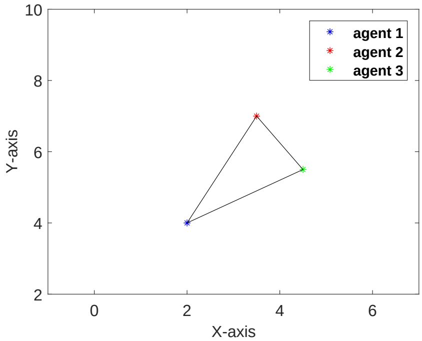
The communication radius is $\Delta = 20$. We have $\beta_0 = 0.1765$.

Firstly, we consider the situation that the three agents are modeled as single integrators. The initial positions of agents can be randomly selected as long as the initial condition (5.5) is satisfied. Here, the initial positions of agents are chosen by

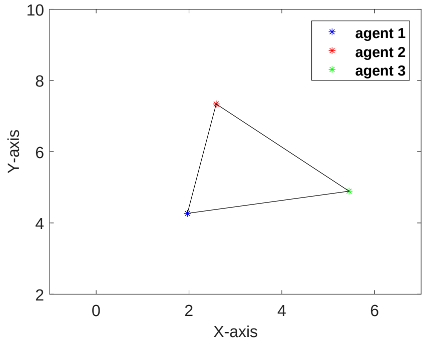
$$x_1(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \; x_2(0) = \begin{pmatrix} 3.5 \\ 7 \end{pmatrix}, \; x_3(0) = \begin{pmatrix} 4.5 \\ 5.5 \end{pmatrix}.$$

One can easily check that both Assumption 5.1 and initial condition (5.5) hold. Choose $\alpha = 100$ and $\beta = \frac{\beta_0}{50}$, by applying the Algorithm 5.2, we get the evolutions of the formation shown in Figure 5.2.

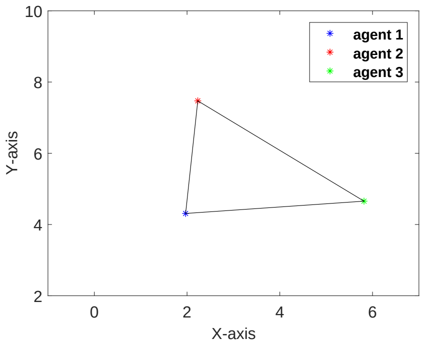
Figure 5.3 (a) shows the position evolutions of the multi-agent system (5.1) with event-triggered control input (5.4) when performing Algorithm 5.2, where "circles" denote the initial positions and "triangle" denotes the desired formation, and the triggering times for each agent shown in Figure 5.3 (b), respectively. When every agent performs Algorithm 5.3, Figure 5.4 (a) and Figure 5.4 (b) show the position evolutions and the triggering times, respectively.



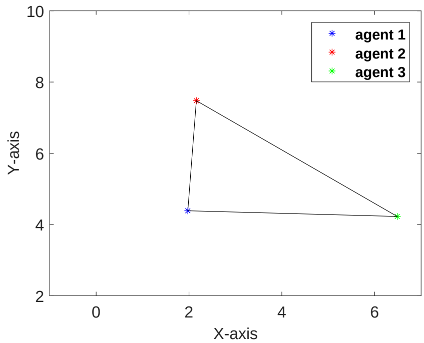
(a) At $t = 0$.



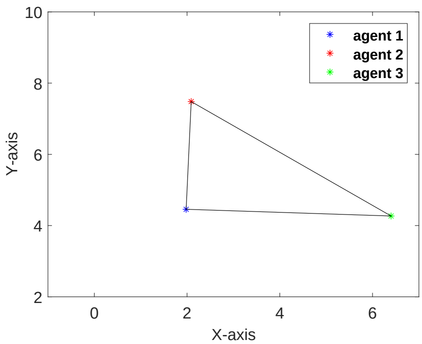
(b) At $t = 1.2$.



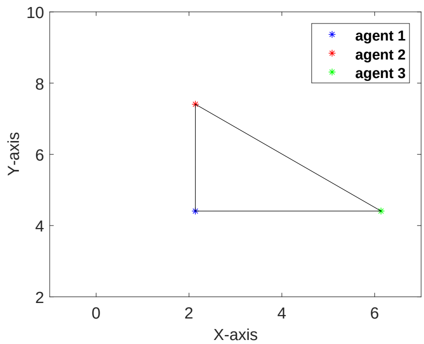
(c) At $t = 1.6$.



(d) At $t = 2.4$.

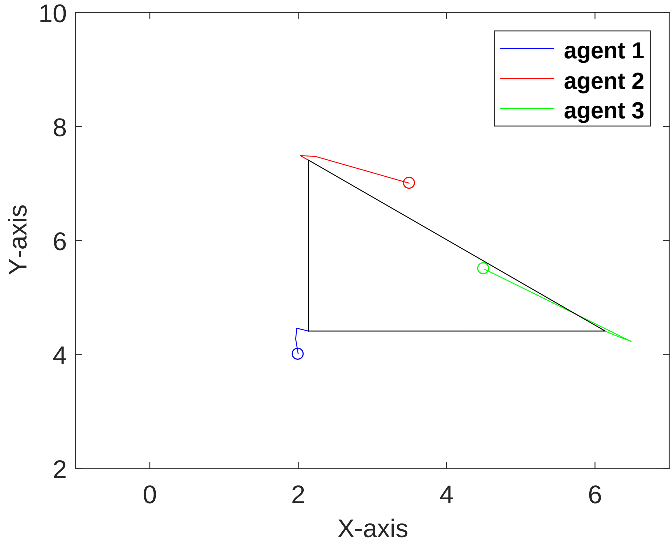


(e) At $t = 3.2$.

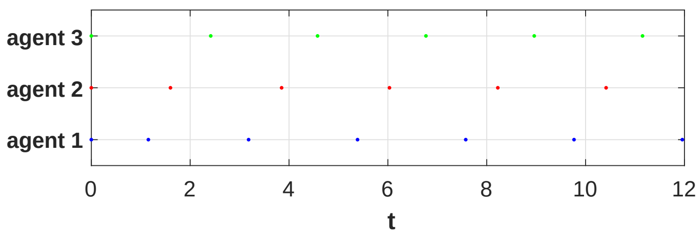


(f) At $t = 10.4$.

Figure 5.2: Evolutions of the formation process of the multi-agent system (5.1) with event-triggered control input (5.4) when performing Algorithm 5.2.

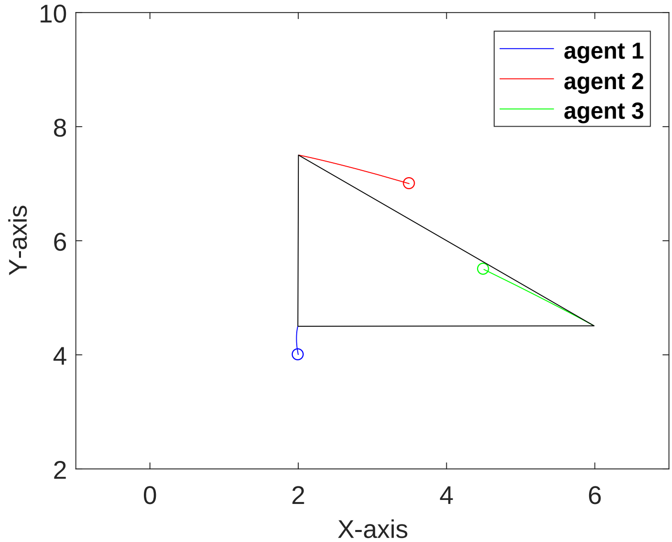


(a)

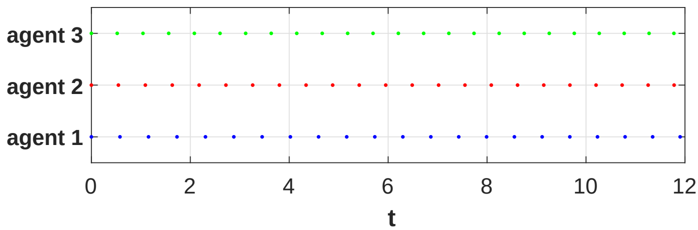


(b)

Figure 5.3: (a) The position evolutions of the multi-agent system (5.1) with event-triggered control input (5.4) when performing Algorithm 5.2. (b) The triggering times for each agent.



(a)



(b)

Figure 5.4: (a) The position evolutions of the multi-agent system (5.1) with event-triggered control input (5.4) when performing Algorithm 5.3. (b) The triggering times for each agent.

Secondly, we consider the situation that the three agents are modeled as double integrators. The initial positions of agents are chosen as before. The initial speeds of agents can be randomly selected and here we choose

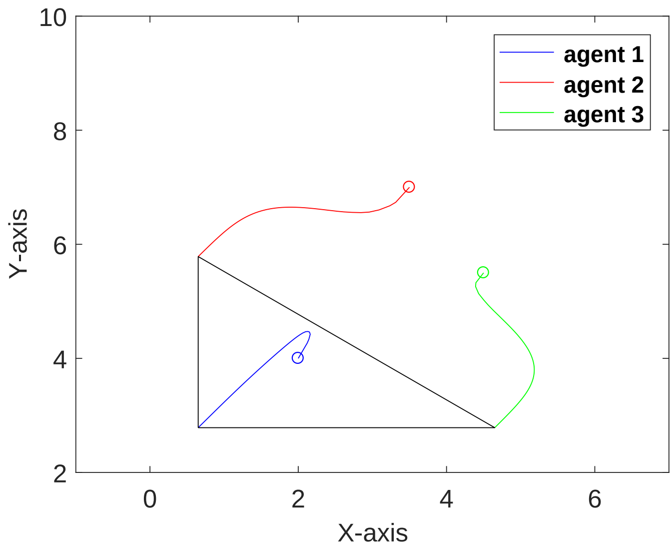
$$r_1(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; r_2(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \; r_3(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

We have $P = \begin{bmatrix} 5.0237 & 1.1547 \\ 1.1547 & 1.4502 \end{bmatrix}$, $k_1 = 1.1547$, $k_2 = 1.4502$, and $\rho(P) = 5.3643$. Choose $k_3 = \frac{2}{k_2 + \sqrt{k_1^2 + k_2^2}} = 0.6053$, $\alpha_d = 10$, and $\beta_d = \frac{(2-k_4)}{10\rho(P)}$, by applying the Algorithm 5.5, we get the evolutions of the position shown in Figure 5.5 (a), where "circles" denote the initial positions and "triangle" denotes the desired formation, and the triggering times for each agent shown in Figure 5.5 (b), respectively. When every agent performs Algorithm 5.6, Figure 5.6 (a) and Figure 5.6 (b) show the position evolutions and the triggering times, respectively.

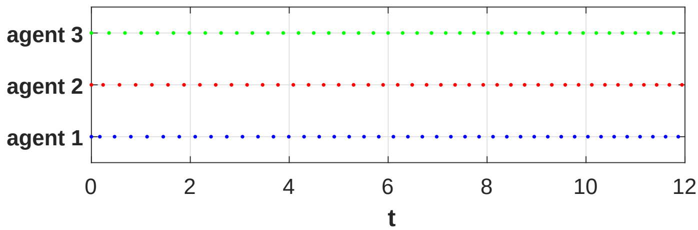
It can be seen that the formation is achieved when any one of the four self-triggered algorithms is performed, but the formation could be achieved in different positions. It can also be seen that the average inter-event time determined by Algorithm 5.2 is greater than that determined by Algorithm 5.3. However, just as Table 5.1 summarized, the only communication requirement that each agent needs to perform in Algorithm 5.3 is to sense the relative positions to its neighbors. Similar comparison can be made between Algorithms 5.5 and 5.6. Moreover, we can see that double integrators have more smooth trajectories compared with single integrators.

5.5 Summary

In this chapter, formation control for multi-agent systems with limited communication, including sensing, broadcasting, receiving and listening, was addressed. We first considered the situation that agents are modeled as single integrators. An event-triggered algorithm and two self-triggered algorithms, to avoid continuous communication and using absolute measurements of positions, were proposed. It was shown that each agent only updates its control input by sensing the relative state to its neighbors and broadcasts its triggering information at its triggering times, and listens to and receives its neighbors' triggering information at their triggering times. Moreover, the desired formation was established exponentially with connectivity preservation and exclusion of Zeno behavior. Then, these results were extended to double integrators. Future research directions of this work include taking input saturation into account since the proposed event-triggered control input could be very large, which is unrealistic.

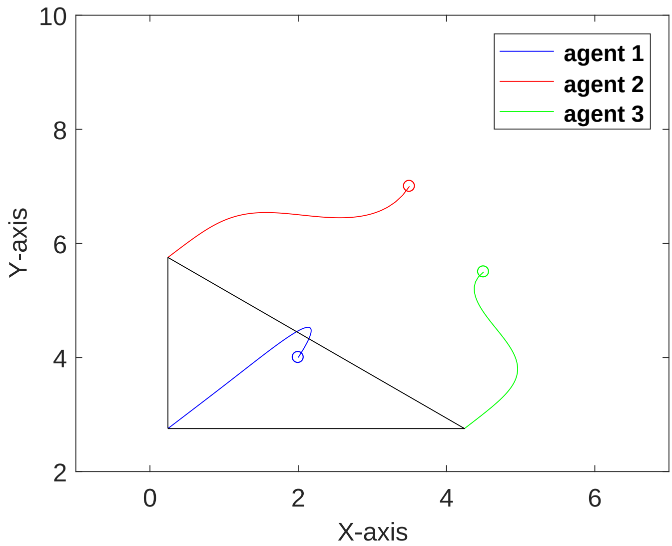


(a)

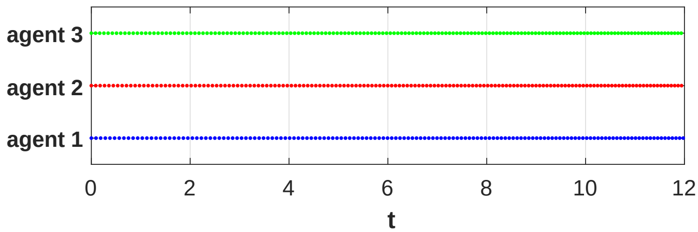


(b)

Figure 5.5: (a) The position evolutions of the multi-agent system (5.40) with event-triggered control input (5.46) when performing Algorithm 5.5. (b) The triggering times for each agent.



(a)



(b)

Figure 5.6: (a) The position evolutions of the multi-agent system (5.40) with event-triggered control input (5.46) when performing Algorithm 5.5. (b) The triggering times for each agent.

Chapter 6

Conclusions and future research

In this chapter, we summarize the main results presented in Chapters 3–5 and discuss possible directions for future research.

6.1 Conclusions

This thesis proposed distributed dynamic event-triggered control strategies for multi-agent systems to reduce energy consumption, the amount of information exchanged, and system update in general. In particular, the three problems of average consensus for single-integrator agents, agents systems with input saturation, and formation control for single- and double-integrator agents with connectivity preservation were solved.

Dynamic event-triggered control for multi-agent systems

We first proposed two dynamic event-triggered control strategies for first-order continuous-time multi-agent systems to solve average consensus problem. Compared with existing event-triggered control strategies, our dynamic event-triggered control strategies involve internal dynamic variables which play an essential role to guarantee that the triggering time sequence does not exhibit Zeno behavior. Some of the existing event-triggered control strategies are special cases of our strategies. We proved that average consensus is achieved exponentially if and only if the communication graph is connected, and Zeno behavior was excluded by proving that the triggering time sequence of each agent is divergent. Then, we proposed a self-triggered control strategy to avoid continuous listening over the network. As a result, each agent only needs to sense and broadcast at its triggering times, and to listen to and receive incoming information from its neighbors at their triggering times. Thus continuous listening is avoided. With some modifications, the results in this part can be extended to the cases that the underlying graph is directed and has a directed spanning tree. Furthermore, the results also can most likely be extended to general linear and nonlinear multi-agent systems with standard controllability assumptions for linear dynamics and standard continuity assumptions for the nonlinear dynamics.

Multi-agent systems with input saturation

We extended the results above to multi-agent systems with input saturation constraints over digraphs. We showed that consensus is achieved if and only if the underlying directed communication topology has a directed spanning tree. We considered event-triggered control and presented a distributed triggering law to reduce the overall need of communication and system updates. The triggering law was a special kind of dynamic triggering and was inspired by a Lyapunov function we used in the proof of the first result. We showed that consensus is achieved for the event-triggered control under the same connectivity condition, and the triggering law was proven to be free of Zeno behavior. Moreover, we presented a self-triggered algorithm to avoid continuous listening. With some modifications, we believe that the results in this part can be extended to multi-agent systems with output saturation constraints and even nonlinear multi-agent systems with standard continuity assumptions.

Event-triggered formation control with connectivity preservation

Formation control for multi-agent systems with limited communication was addressed. We first considered the situation that agents are modeled as single integrators and designed distributed event-triggered control. An event-triggered algorithm and two self-triggered algorithms were proposed. It was shown that each agent only updates its control input by sensing the relative state to its neighbors and broadcasts its triggering information at its triggering times, and listens to and receives its neighbors' triggering information at their triggering times. The desired formation was shown to be established exponentially with connectivity preservation and exclusion of Zeno behavior. Then, these results were extended to double integrators. With some modifications, we think the results in this part can be extended to position- and distance-based formation control, and can most likely be extended to systems with input saturation.

Summary

We summarize some aspects of the thesis results in Table 6.1 and compare them with the literature. The rows list some specific properties of the considered multi-agent control protocols and the implication of the developed analysis. None of the listed work assume continuous broadcasting of the agents' state to its neighbors, but it is common in the literature to assume continuous listening. None of results in Chapters 3–5 require that agents has to continuously listen to their neighbors. The table specifies if the considered control laws are based on absolute agent state information or relative state information. Finally, as shown in the thesis it is important to exclude Zeno behavior. In the literature, this issue has not always been carefully investigated. In particular, references [16,17,67–69, 71,73,75] do not strictly show that Zeno behavior is excluded, while [18,64–66,70,72,74] do.

Table 6.1: Summary of the thesis results and comparison with the literature.

	[16–18, 64–68]	[69–75]	Chapter 3	Chapter 4	Chapter 5
Continuous broadcasting?	No	No	No	No	No
Continuous listening?	Yes	Yes	No	No	No
State information?	Absolute	Relative	Absolute	Absolute	Relative
Avoiding Zeno?	?	?	Yes	Yes	Yes

6.2 Future research directions

There are several interesting research directions based on the work of this thesis. Some of the immediate ones were mentioned above. Other extensions are discussed in this section.

Optimal target tracking

For the autonomous target tracking example in Section 1.1, the target is not static. It would be interesting to consider how to control a multi-agent system to track a moving target. It is reasonable to study optimal formation control to track a moving target such that the resource consumed (for motion and communication) to reach the desired formation is minimized.

Number of triggering times

In Chapters 3–5, we showed when agents perform the dynamic event-triggered strategies not only the desired properties are achieved but also the overall need of communication and system updates are reduced. It would be interesting to quantify this reduction systematically and compare it with other event- and time-triggered strategies. One specific problem is to determine the number of triggering times that are needed to guarantee that all agents reach a ball of given radius centered at the average of all agents' states.

Time delays and noise

There are time delays and noise in sensing and communication in most real applications. The loss of perfect information due to such issues can degrade the performance of the multi-agent systems and even destabilize it. It would be interesting to extend the results in Chapters 3–5 to models with time delays and noise.

Bit rate

It would be interesting to find the minimum communication rate between agents to guarantee that desired properties still can be achieved. Such minimum rate question is well studied for single-agent systems [105], the data rate theorem [106]. However, it is not well studied for multi-agent systems [107]. It would be interesting to quantify bit rate conditions to guarantee desired properties are achieved for multi-agent systems based on event-triggered control approaches.

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