Lecture 8

Hybrid Control Design

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Control of hybrid systems is discussed in this lecture. So far we have studied only the analysis problem and, for instance, given conditions on when a hybrid automata is stable or not. However, we have not explicitly discussed control design, i.e., how control actions should be chosen. In Lecture 5 on open hybrid automata, we introduced continuous and discrete control variables. Here we will focus on discrete control for purely continuous systems. Most of the lecture is dedicated to stabilization. In future lectures we will also study extensions of optimal control to hybrid systems.

A Motivation for Hybrid Control

We start with an example motivating why it may be good to introduce discrete control for a continuous dynamical system. Consider the mobile robot in Figure 1, where the angular velocity of the rear wheels are individually controlled while the front wheel turns freely. We may regard the forward velocity v and the angular velocity of the heading angle ω as the two control signals. Then the dynamics of the vehicle is described by the equations

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega, \end{aligned} \tag{1}$$

where (x, y) is the coordinate of the center of the rear axis and θ the heading angle. The vehicle is said to have *non-holonomic constraints*, i.e., the instantaneous velocities $(\dot{x}, \dot{y}, \dot{\theta})$ are constrained to a manifold of lower dimension (two) than the dimension of the state space (three). Non-holonomic systems form an important class of nonlinear control systems.

With a smooth state transformation, it is possible to transform (1) into the following form

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1u_2 - x_2u_1, \end{aligned} \tag{2}$$

which is so-called feedback equivalent to (1). This dynamical system is called the Brockett Integrator. An important property of (2) (and thus of (1)) is that

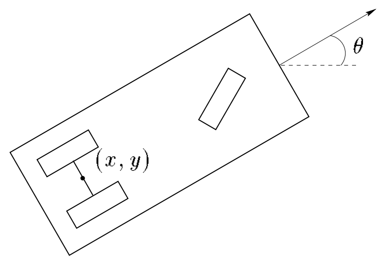

Fig. 1. Mobile robot.

the origin cannot be asymptotically stabilized with smooth state feedback. For a general nonlinear control system, we namely have the following theorem.

Theorem 1. *Consider the control system*

$$\dot{x} = F(x, u), \quad x(t_0) = x_0, \tag{3}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is a* C^1 *function and* $F(0,0) = 0$*. If there exists a* C^0 *function* $g : \mathbb{R}^n \to \mathbb{R}^m$ *such that* $x^* = 0$ *is an asymptotically stable equilibrium point of* $\dot{x} = F\big(x, g(x)\big)$*, then the image under* F *of every open neighborhood of* $(x, u) = (0, 0)$ *contains an open neighborhood of* $x = 0$*.*

For the Brockett Integrator, we have

$$F(x, u) = \begin{bmatrix} u_1 \\ u_2 \\ x_1 u_2 - x_2 u_1 \end{bmatrix}.$$

The image of this map does not contain $(0, 0, \epsilon)$ for any $\epsilon \neq 0$. Hence, it follows from Theorem 1 that (2) cannot be asymptotically stabilized with a continuous feedback law. The system is, however, asymptotically stabilizable by a hybrid control law. An example of such a controller will be given next.

Stabilization Through Hybrid Control

Consider the continuous system (3) and assume that the feedback is given by a function $u = g(v, x) : V \times \mathbb{R}^n \to \mathbb{R}^m$, where V is a finite set of discrete modes. The continuous state feedback controller considered in Theorem 1 is thus a special case of such a feedback. The discrete mode is given by a switching function $\sigma : V \times \mathbb{R}^n \to V$. The closed-loop system is illustrated in Figure 2, where hence the upper block is a continuous dynamical system and the lower block is an automaton. The described control system (and variations thereof) have many names, including switching control, supervisory control, and logic-based switching. It is also a special class of hybrid automata, which we call switching control automata and define as follows.

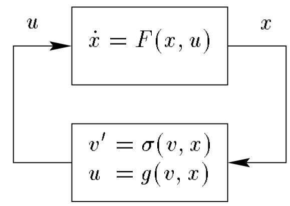

Fig. 2. Continuous dynamical system and discrete controller.

Definition 1 (Switching Control Automata). *Consider functions* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : V \times \mathbb{R}^n \to \mathbb{R}^m$ *and* $\sigma : V \times \mathbb{R}^n \to V$. *The hybrid automaton* $H = (V, X, f, \text{Init}, \text{Inv}, \text{Jump})$ *is a* switching control automaton *if for all* $(v, z) \in V \times \mathbb{R}^n$,

- $f(v, z) = F\big(z, g(v, z)\big)$
- $\text{Init} = V \times \mathbb{R}^n$,
- $\text{Inv} = \{(v, z) \in V \times \mathbb{R}^n : \ v = \sigma(v, z)\}$,
-

$$\text{Jump}(v, z) = \begin{cases} \{(\sigma(v, z), z)\}, & \textit{if } \sigma(v, z) \neq v \\ \emptyset, & \textit{otherwise.} \end{cases}$$

The switching function σ determines the evolution of the discrete part of the execution. In general, σ may force a discrete transition to any discrete mode. Hence, (V, E), where E denotes the edges as defined in Lecture 2, forms a connected graph.

A switching control automaton may also be viewed as a continuous control system with a particular type of discontinuous feedback.

The Brockett Integrator

Next we present an asymptotically stable switching control hybrid automaton H, which presents a stabilizing control law for the Brockett Integrator. Consider the switching control automaton H with $V = \{1, \ldots, 4\}$ and

$$g(1, x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad g(2, x) = \begin{bmatrix} x_1 + \frac{x_2 x_3}{x_1^2 + x_2^2} \\ x_2 - \frac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}$$

$$g(3, x) = \begin{bmatrix} -x_1 + \frac{x_2 x_3}{x_1^2 + x_2^2} \\ -x_2 - \frac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \qquad g(4, x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

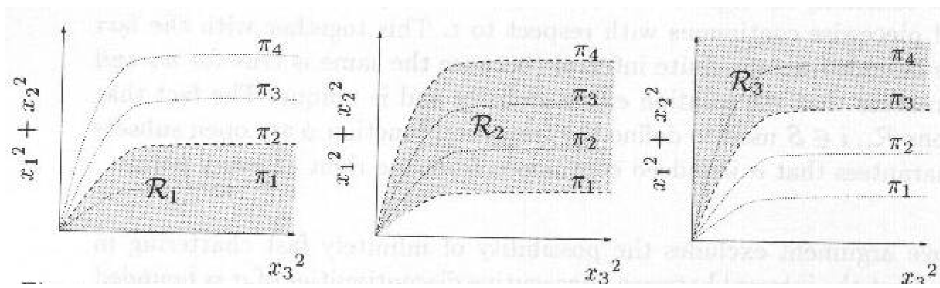

Fig. 3. Functions π_i defining the jump condition together with the projections of the regions $\mathcal{R}_i$ into the $(x_3^2, x_1^2 + x_2^2)$ space.

To define the switching function σ introduce the functions $\pi_i : [0, \infty) \to \mathbb{R}$, $i \in \{1, \dots, 4\}$, illustrated in Figure 3 and define regions

$$\begin{aligned}
\mathcal{R}_1 &= \{x \in \mathbb{R}^3 :\ 0 \le x_1^2 + x_2^2 < \pi_2(x_3^2)\} \\
\mathcal{R}_2 &= \{x \in \mathbb{R}^3 :\ \pi_1(x_3^2) < x_1^2 + x_2^2 < \pi_4(x_3^2)\} \\
\mathcal{R}_3 &= \{x \in \mathbb{R}^3 :\ \pi_3(x_3^2) < x_1^2 + x_2^2\} \\
\mathcal{R}_4 &= \{(0,0,0)\}.
\end{aligned}$$

Then,for all $(v, z) \in V \times \mathbb{R}^3$, let

$$\sigma(v, z) = \begin{cases} v, & \text{if } z \in \mathcal{R}_v \\ \max\{v \in V :\ z \in \mathcal{R}_v\}, & \text{otherwise.} \end{cases}$$

It is possible to show that H accepts a unique, infinite, and non-Zeno execution for all initial states. It is also possible to show that the equilibrium point $x^* = 0$ of H is (globally) asymptotically stable. An example of an execution accepted by H is shown in Figure 4. Note particularly in the right-most diagram how the dynamics of the continuous evolution changes due to the regions $\mathcal{R}_i$ and the state feedback maps g.

Stabilization of Linear Dynamics

Assume that $K > 1$ matrices A_v, $v \in \{1, \dots, K\}$, are given. They form a switched linear system $\dot{x} = A_\sigma x$, where σ defines the current mode of the system. We may ask if it is possible to choose σ, for instance, as a feedback from the continuous state, $\sigma : \mathbb{R}^n \to V$, such that the overall system is asymptotically stable. This problem can be posed as a switched control automaton in Definition 1 with $g(v, z) = A_v z$ and $F(z, u) = u$.

In Lecture 6 we saw that even if A_v is stable for all $v \in V$, a switching function may give an overall unstable system. Next we give a sufficient condition

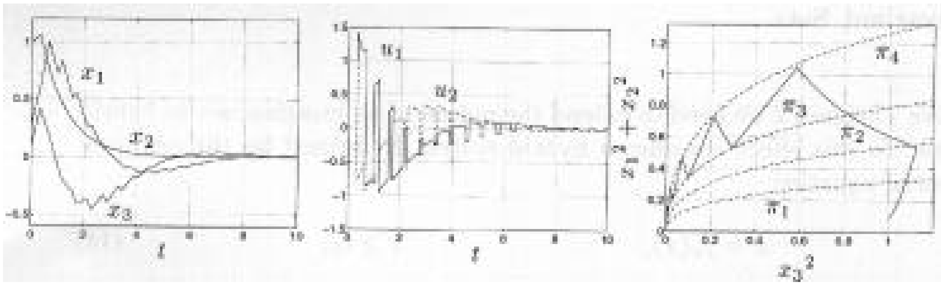

Fig. 4. An execution of the switching control automaton for the Brockett integrator.

that guarantees the existence of a stabilizing switching function σ. We also give a specific example of such a function. First, however, note that if A_k is a Hurwitz matrix for some k, then the stabilization problem is trivial because we can simply choose $\sigma(x) \equiv k$.

Theorem 2. *Consider the set of switching control automata $\mathcal{H} = \{H_\sigma\}$, parameterized in the switching function $\sigma : \mathbb{R}^n \to V$, such that $V = \{1, \ldots, K\}$, for all $(v, z) \in V \times \mathbb{R}^n$, $f(v, z) = A_v z$, and H_σ is non-Zeno. If there exists constants $\mu_k \geq 0$, $k \in \{1, \ldots, K\}$, with $\sum_{k=1}^K \mu_k = 1$, such that $A = \sum_{k=1}^K \mu_k A_k$ is a Hurwitz matrix, then there exists $H_\sigma \in \mathcal{H}$ such that the equilibrium point $x^* = 0$ is asymptotically stable. Furthermore,*

$$\sigma(z) = \arg\min_{k \in V} z^T(A_k^T P + PA_k)z, \tag{4}$$

defines one such H_σ, where $P = P^T \succ 0$ is the solution to $A^T P + PA = -Q$ for any given $Q = Q^T \succ 0$.

Proof. The main idea of the proof is as follows. By the choice of P in the theorem, we have $z^T(A^T P + PA)z < 0$ for an arbitrary $z \neq 0$. Therefore,

$$\sum_{k=1}^K \mu_k z^T(A_k^T P + PA_k)z < 0.$$

Because all μ_k are non-negative, there exists k such that $z^T(A_k^T P + PA_k)z < 0$. Hence, by choosing $\sigma(z) = k$, the Lyapunov function $\mathcal{V}(v, z) = z^T P z$ is decreasing for that particular z. Since H_σ is non-Zeno by assumption, asymptotic stability follows similar to the proof of Theorem 2 in Lecture 7.

The idea of choosing switching function as in (4) may be interpreted as choosing k such that $\mathcal{V}\big(v(t), z(t)\big)$ is decreasing as much as possible. Another suggestion is to choose

$$\sigma(x) = \arg\min_{k \in V}\{\mathcal{V}(1, x), \ldots, \mathcal{V}(K, x)\}.$$

Stability for Arbitrary Switching

In previous section the switching function σ was a feedback signal depending on the continuous state of the hybrid automaton. A related problem is if the hybrid automaton is stable when the switching function σ defines an arbitrary (unknown) sequence of discrete transitions. These could be considered as discrete disturbances acting on a continuous system. Next we define a hybrid automaton with such a switching function.

Definition 2 (Switching Disturbance Automata). *Consider a piecewise constant switching function $\sigma : \mathbb{R} \to V$. The hybrid automaton $H = (V, X, f,$ Init, Inv, Jump$)$ is a* switching disturbance automaton *if for all $(v, z) \in V \times \mathbb{R}^n$,*

–
$$f(v,z) = \begin{bmatrix} \bar{f}(v,\bar{z}) \\ 1 \end{bmatrix}, \qquad \bar{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \end{bmatrix},$$

– $\text{Init} = V \times \mathbb{R}^n$,
– $\text{Inv} = \{(v,z) \in V \times \mathbb{R}^n : \ v = \sigma(z_n)\}$,
–
$$\text{Jump}(v,z) = \begin{cases} \{(\sigma(z_n), z)\}, & \textit{if } \sigma(z_n) \neq v \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Note that x_n act as a time variable and that σ thus, in a sense, defines a piecewise constant function on the time axis.

Proposition 1. *Consider a switching disturbance automaton H such that $V = \{1, \ldots, K\}$ and for all $(v, \bar{z}) \in V \times \mathbb{R}^{n-1}$, $\bar{f}(v,\bar{z}) = A_v \bar{z}$, where A_v is a Hurwitz matrix. If $A_k A_\ell = A_\ell A_k$ for all $k, \ell \in V$, then $x^* = 0$ is an asymptotically stable equilibrium point of H.*

Proof. We present a sketch for $K = 2$. Let $t \in [\tau_i, \tau_i']$ and assume that $v(t) = v_0 = 1$. Then

$$x(t) = \exp[A_1(t - \tau_i)] \exp[A_2(\tau_{i-1}' - \tau_{i-1})] \cdots \exp[A_1(\tau_0' - \tau_0)] x_0.$$

Since A_1 and A_2 commute, we have

$$x(t) = \exp\left[A_1[(t-\tau_i) + \cdots + (\tau_0' - \tau_0)]\right] \exp\left[A_2[(\tau_{i-1}' - \tau_{i-1}) + \cdots + (\tau_1' - \tau_1)]\right].$$

By construction, H is non-Zeno. Hence, the Hurwitz assumption gives that $\lim_{t \to \tau_\infty} x(t) = 0$ where $\tau_\infty = \infty$.

Background

Brockett's condition for smooth stabilizability is given in [1]. The stabilizing hybrid control law presented here is taken from [3]. See [6] for a discussion on non-holonomic constraints. Proposition 1 is from the survey [5], which inspired some of the discussion in this lecture.

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