Lecture 7

Computational Lyapunov Methods

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Theorem 2 in Lecture 6 presents a sufficient condition for stability of hybrid automata. If there exists a Lyapunov-like function $\mathcal{V}$,[1] which satisfies a certain property at the jump instances, stability of the hybrid automaton follows. As was pointed out, it is a non-trivial task to find such a $\mathcal{V}$. This lecture is devoted to that problem. It will be shown that for certain classes of hybrid automata, which have vector fields linear in x, computationally attractive methods exist to derive $\mathcal{V}$.

Lyapunov Equation

For a matrix $M \in \mathbb{R}^{n\times n}$, let $\lambda(M) = \{\lambda_i\}_{i=1}^n$, $\lambda_i \in \mathbb{C}$, denote its eigenvalues. The trace of M is denoted $\operatorname{tr} M = \sum_{i=1}^n M_{ii}$, where M_{ii} is the diagonal elements of M. Recall that $\operatorname{tr} M = \sum_{i=1}^n \lambda_i$. If $\operatorname{Re}\lambda_i < 0$ for all i, the matrix M is called Hurwitz. A symmetric matrix $M = M^T$ is called positive definite (write $M \succ 0$) if $x^T M x > 0$ for all non-zero $x \in \mathbb{R}^n$. It is called positive semidefinite ($M \succeq 0$) if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$. Similarly, M is negative definite ($M \prec 0$) if $-M$ is positive definite and it is negative semidefinite ($M \preceq 0$) if $-M$ is positive semidefinite.

Recall that a linear differential equation $\dot{x} = Ax$ is asymptotically stable if and only if A is a Hurwitz matrix. A check if asymptotically stability holds can be done by solving a linear algebraic equation, as stated next by the following classical result.

Theorem 1. *The equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is asymptotically stable if and only if for all matrices $Q = Q^T \succ 0$ there exists $P = P^T \succ 0$ such that*

$$A^T P + PA = -Q. \tag{1}$$

Moreover, if $\dot{x} = Ax$ is asymptotically stable, then P is unique.

Proof. The "if" part follows from Theorem 1 in Lecture 6 together with $V(x) = x^T P x$, because then $V(0) = 0$, and for all $x \neq 0$, $V(x) > 0$ and

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (A^T P + PA) x = -x^T Q x < 0.$$

[1] Sometimes we drop "-like" in "Lyapunov-like" even if $\mathcal{V}$ is not strictly a Lyapunov function in the sense in continuous dynamical systems.

For the "only if" part, consider

$$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt,$$

which is well-defined since A is a Hurwitz matrix. It is obvious that P is symmetric. It remains to show that P is positive definite, satisfies (1), and is unique. First, recall that $\langle f, g\rangle = \int_0^\infty f^T(t) g(t)\, dt$ defines an inner product on the space of vector valued L_2 functions ($\int_0^\infty \|f(t)\|^2\, dt < \infty$). With $f(t) = Q^{1/2} e^{At} x$ (where a factorization $Q = Q^{1/2}(Q^{1/2})^T$ exists because $Q \succ 0$), it follows that $x^T P x = \langle f, f\rangle > 0$ for all $x \neq 0$ and hence $P \succ 0$. Second, a straightforward calculation gives

$$\begin{aligned} PA + A^T P &= \int_0^\infty e^{A^T t} Q e^{At} A\, dt + \int_0^\infty A^T e^{A^T t} Q e^{At}\, dt \\ &= \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{At}\, dt = -Q. \end{aligned}$$

Third, assume there exists another solution $\hat{P} \neq P$. Then,

$$\begin{aligned} 0 &= e^{A^T t}(Q - Q) e^{At} \\ &= e^{A^T t}[A^T(P - \hat{P}) + (P - \hat{P})] e^{At} \\ &= \frac{d}{dt} e^{A^T t}(P - \hat{P}) e^{At}, \end{aligned}$$

so that $e^{A^T t}(P - \hat{P}) e^{At}$ is constant for all $t \geq 0$. Evaluating at $t = 0$ and $t = \infty$ gives

$$e^{A^T \cdot 0}(P - \hat{P}) e^{A \cdot 0} = \lim_{t\to\infty} e^{A^T t}(P - \hat{P}) e^{At},$$

that is,

$$P - \hat{P} = 0.$$

This concludes the proof.

Equation (1) is called a Lyapunov equation. Note that it is linear in the elements of P. Hence, it is easy to find a Lyapunov function $V(x) = x^T P x$ for a linear differential equation by solving its corresponding Lyapunov equation for some $Q \succ 0$.

We may also write the matrix condition in Theorem 1 as: if there exists a matrix $P = P^T \succ 0$ satisfying the inequality

$$A^T P + PA \prec 0.$$

Such an expression is called a linear matrix inequality (LMI), since the left-hand side is linear in the unknown P. As we will see next, generalizations of Theorem 1 to hybrid automata lead to LMI's of similar nature. Today there exists efficient softwares to solve LMI problems. They will be further discussed in the end of the lecture.

Globally Quadratic Lyapunov Function

Now we generalize Theorem 1 to a class of hybrid automata.

Theorem 2. *Consider a hybrid automaton $H = (V, X, f, \text{Init}, \text{Inv}, \text{Jump})$ and let $V = \{v_1, \dots, v_K\}$, $K \geq 1$. Assume that for all $k \in \{1, \dots, K\}$,*

- $f(v_k, x) = A_k x$, $A_k \in \mathbb{R}^{n \times n}$,
- $\text{Init} \subset \text{Inv}$,
- *for all* $x \in \mathbb{R}^n$,

$$|\text{Jump}(v_k, x)| = \begin{cases} 1, & \text{if } (v_k, x) \in \partial\text{Inv} \\ 0, & \text{otherwise} \end{cases}$$

and

$$(v', x') \in \text{Jump}(v_k, x) \quad \Rightarrow \quad (v', x') \in \text{Inv},\ x' = x.$$

Furthermore, assume that for all $\chi \in \mathcal{E}_H^\infty$, $\tau_\infty(\chi) = \infty$ and that $x^* = 0$ is an equilibrium point of H. Then, if there exists $P = P^T \succ 0$ such that

$$A_k^T P + P A_k \prec 0, \qquad \forall k \in \{1, \dots, K\},$$

$x^* = 0$ is asymptotically stable.

Proof. A sketch of the proof is as follows. First note that there exists $\gamma > 0$ such that

$$A_k^T P + P A_k + \gamma I \preceq 0, \qquad \forall k \in \{1, \dots, K\}.$$

Also, note that with the given assumptions there exists a unique, infinite, and non-Zeno execution $\chi = (\tau, v, x)$ for every $(v_0, x_0) \in \text{Init}$ (see Lecture 3). For all $i \geq 0$, the continuous evolution $x : \tau \to \mathbb{R}^n$ of such an execution fulfills the time-varying linear differential equation

$$\dot{x}(t) = \sum_{k=1}^{K} \mu_k(t) A_k x(t), \quad t \in [\tau_i, \tau_i'],$$

where $\mu_k : \tau \to [0,1]$ is a function such that for $t \in [\tau_i, \tau_i']$, $\sum_{i=1}^{K} \mu_i(t) = 1$. Let $V(v, x) = x^T P x$. This leads to that for all $t \in [\tau_i, \tau_i']$,

$$\begin{aligned} \dot{V}\big(v(t), x(t)\big) &= \sum_{i=1}^{K} \left[\mu_i(t) x(t)^T (A_i^T P + P A_i) x(t)\right] \\ &\leq -\gamma \|x(t)\|^2 \sum_{i=1}^{K} \mu_i(t) \\ &= -\gamma \|x(t)\|^2. \end{aligned}$$

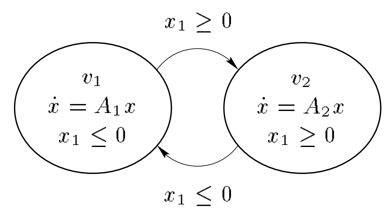

Fig. 1. Hybrid automaton for Examples 1 and 2.

Note that

$$\lambda_{\min}\|x\|^2 \leq \mathcal{V}(v,x) \leq \lambda_{\max}\|x\|^2, \tag{2}$$

where $0 < \lambda_{\min} \leq \lambda_{\max}$ are the smallest and largest eigenvalues of P, respectively. It follows that

$$\dot{\mathcal{V}}\big(v(t), x(t)\big) \leq -\frac{\gamma}{\lambda_{\max}}\mathcal{V}\big(x(t)\big), \quad t \in [\tau_i, \tau_i'],$$

and, hence,

$$\mathcal{V}\big(v(t), x(t)\big) \leq \mathcal{V}\big(v(\tau_i), x(\tau_i)\big)e^{-\gamma(t-\tau_i)/\lambda_{\max}}, \quad t \in [\tau_i, \tau_i'].$$

Using (2) again gives

$$\lambda_{\min}\|x(t)\|^2 \leq \lambda_{\max}\|x(\tau_i)\|^2 e^{-\gamma(t-\tau_i)/\lambda_{\max}}, \quad t \in [\tau_i, \tau_i'].$$

Since the execution χ by assumption is non-Zeno, we have that $\tau_i \to \infty$ as $i \to \infty$. Hence, $\|x(t)\|$ goes to zero exponentially as $t \to \tau_\infty$, which implies that the equilibrium point $x^* = 0$ is asymptotically stable.[2]

The second and third assumptions imply that H is non-blocking and deterministic. The non-Zenoness assumption $\tau_\infty(\chi) = \infty$ is in general hard to check. However, it holds, for instance, if the vector field is pointing (strictly) out of Inv for all states in the domain of Jump and (strictly) into Inv for all states in the range of Jump. The non-Zenoness assumption, could (probably[3]) in some cases be removed by introducing relaxed executions (or Filippov solutions). This will be further discussed in Lecture 8.

Example 1. Consider the hybrid automaton H in Figure 1 with

$$A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}.$$

[2] In fact, the equilibrium point is *exponentially stable*, a notion that is stronger than asymptotically stable, but that we did not introduce here.

[3] This has been pointed out by several researchers, but there seems to be no formal results, yet.

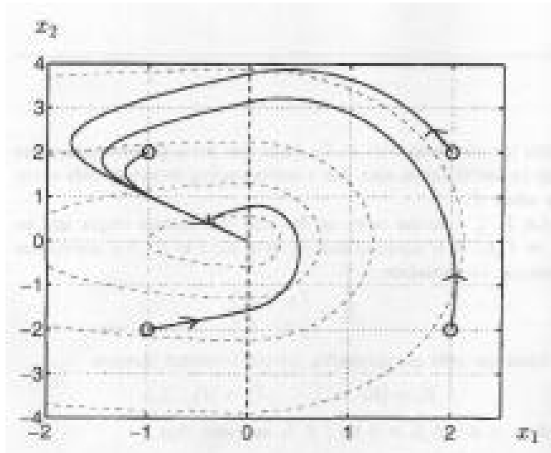

Fig. 2. Continuous evolution for a hybrid automaton that does not have a globally quadratic Lyapunov function. Still, the origin is an asymptotically stable equilibrium point, which can be proved by using a Lyapunov function quadratic in each discrete mode.

The eigenvalues are $\lambda(A_1) = \{-1 \pm i\}$ and $\lambda(A_2) = \{-2 \pm i\}$, so both $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ have an asymptotically stable focus (i.e., the solutions are spiraling toward the origin). It is straightforward to see that H satisfies the assumptions of Theorem 2. Moreover, $A_1^T + A_1 < 0$ and $A_2^T + A_2 < 0$, so the inequalities in the theorem are fulfilled for $P = I$. Hence, the origin is an asymptotically stable equilibrium point for H.

Theorem 2 provides a candidate for a globally quadratic Lyapunov function $\mathcal{V}(x) = x^T P x$. A hybrid automaton may, however, be stable even if there is no such Lyapunov function. This is illustrated by the following example.

Example 2. Consider the hybrid automaton in Figure 1 again, but let

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}.$$

The eigenvalues are $\lambda(A_1) = \{-6, -1\}$ and $\lambda(A_2) = \{-2 \pm 4\sqrt{5}i\}$, so $\dot{x} = A_1 x$ has an asymptotically stable node and $\dot{x} = A_2 x$ has an asymptotically stable focus. The evolution of the continuous state is shown in Figure 2 for four different initial states. Origin seems to be a stable equilibrium point. Indeed, the Lyapunov function indicated by dashed level sets proves asymptotic stability of the origin. Note that the Lyapunov function is not globally quadratic, but piecewise quadratic in the sense that it is quadratic in each discrete mode.

From the following result it follows that for the previous example, it is indeed not possible to find a quadratic Lyapunov function.

Proposition 1. *If there exists $R_i = R_i^T \succ 0$, $i \in \{1, \dots, K\}$, such that*

$$\sum_{i=1}^{K} (R_i A_i^T + A_i R_i) \succ 0,$$

then there does not exist $P = P^T \succ 0$ such that

$$A_i^T P + P A_i \prec 0, \qquad \forall i \in \{1, \dots, K\}.$$

Proof. Recall that for two matrices $M \succ 0$ and $N \succ 0$, it holds that $\mathrm{tr}(MN) = \mathrm{tr}(NM) \succ 0$. Therefore, if $P = P^T \succ 0$,

$$\begin{aligned}\sum_{i=1}^{K} \mathrm{tr}[R_i(A_i^T P + P A_i)] &= \sum_{i=1}^{K} [\mathrm{tr}(R_i A_i^T P) + \mathrm{tr}(R_i P A_i)] \\ &= \sum_{i=1}^{K} [\mathrm{tr}(P R_i A_i^T) + \mathrm{tr}(P A_i R_i)] \\ &= \mathrm{tr}\left[P \sum_{i=1}^{K} (R_i A_i^T + A_i R_i)\right] \succ 0.\end{aligned}$$

But $R_i = R_i^T \succ 0$ by assumption, so there exists i such that $A_i^T P + P A_i \succ 0$. This proves the result.

Piecewise Quadratic Lyapunov Function

In this section we restrict the class of hybrid automata considered in Theorem 2 further, by assuming that the the invariant condition is given by polyhedra. Consider a hybrid automaton H with

$$\mathrm{Inv} = \bigcup_{k=1}^{K} \{v_k\} \times \{x \in \mathbb{R}^n : \ E_{k1} x \geq 0, \dots, E_{kn} x \geq 0\},$$

where

$$E_k = \begin{bmatrix} E_{k1} \\ \vdots \\ E_{kn} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

It then follows that $(v_k, 0) \in \mathrm{Inv}$ for all k. Let Jump : $V \times \mathbb{R}^n \to P(V \times \mathbb{R}^n)$ for all $(v_k, x) \in V \times \mathbb{R}^n$ satisfy

$$|\mathrm{Jump}(v_k, x)| = \begin{cases} 1, & \text{if } (v_k, x) \in\in \partial \mathrm{Inv} \\ 0, & \text{otherwise} \end{cases}$$

such that

$$(v_\ell, x') \in \mathrm{Jump}(v_k, x) \quad \Rightarrow \quad F_\ell x = F_k x, \ v_\ell \neq v_k, \ x' = x,$$

where $F_k, F_\ell \in \mathbb{R}^{n\times n}$ are given matrices (which hence define the boundaries of $\text{Inv}(v)$).

The LMI condition in Theorem 2 implies that

$$x^T(A_k^T P + PA_k)x < 0, \quad \forall x \neq 0, (v_k, x) \in V \times \mathbb{R}^n.$$

It is, however, sufficient to require that

$$x^T(A_k^T P + PA_k)x < 0, \quad \forall x \neq 0, (v_k, x) \in \text{Inv}.$$

This can be done by specifying a matrix S_k such that $x^T S_k x \geq 0$ for all x with $(v_k, x) \in \text{Inv}$ (and, thus, not necessarily positive semidefinite). Then,

$$A_k^T P + PA_k + S_k \prec 0$$

still implies that

$$x^T(A_k^T P + PA_k)x < 0, \quad \forall x \neq 0, (v_k, x) \in \text{Inv},$$

but $x^T(A_k^T P + PA_k)x < 0$ must not hold for $x \neq 0$ with $(v_\ell, x) \in \text{Inv}$ and $\ell \neq k$. The matrix S_k can be chosen as $S_k = E_k^T U_k E_k$, where E_k is given by the representation of H and $U_k = U_k^T \in \mathbb{R}^{n\times n}$ is to be chosen and should have non-negative elements.

We may also let $\mathcal{V}$ depend on the discrete mode. Hence, let $\mathcal{V}(v_k, x) = x^T P_k x$ for $(v_k, x) \in \text{Inv}$. We choose $P_k = F_k^T M F_k$, where F_k is given by the representation of H and $M = M^T \in \mathbb{R}^{n\times n}$ is to be chosen. The dependence of P_k on F_k leads to that $\mathcal{V}$ is C^0 in x. We summarize in the following theorem, which can be proved similar to Theorem 2.

Theorem 3. *Consider a hybrid automaton* $H = (V,\ X,\ f,\ \text{Init},\ \text{Inv},\ \text{Jump})$ *and let* $V = \{v_1, \ldots, v_K\}$, $K \geq 1$. *Assume that for all* $k \in \{1, \ldots, K\}$,

– $f(v_k, x) = A_k x$, $A_k \in \mathbb{R}^{n\times n}$,
–
$$\text{Inv} = \bigcup_{k=1}^{K} \{v_k\} \times \{x \in \mathbb{R}^n : \ E_{k1}x \geq 0, \ldots, E_{kn}x \geq 0\},$$
  $E_k = (E_{k1}^T, \ldots, E_{kn}^T)^T \in \mathbb{R}^{n\times n}$,
– $\text{Init} \subset \text{Inv}$,
–
$$|\text{Jump}(v_k, x)| = \begin{cases} 1, & \textit{if } (v_k, x) \in \partial\text{Inv} \\ 0, & \textit{otherwise} \end{cases}$$
  such that
$$(v_\ell, x') \in \text{Jump}(v_k, x) \quad \Rightarrow \quad F_\ell x = F_k x,\ v_\ell \neq v_k,\ x' = x,$$
  where $F_k, F_\ell \in \mathbb{R}^{n\times n}$

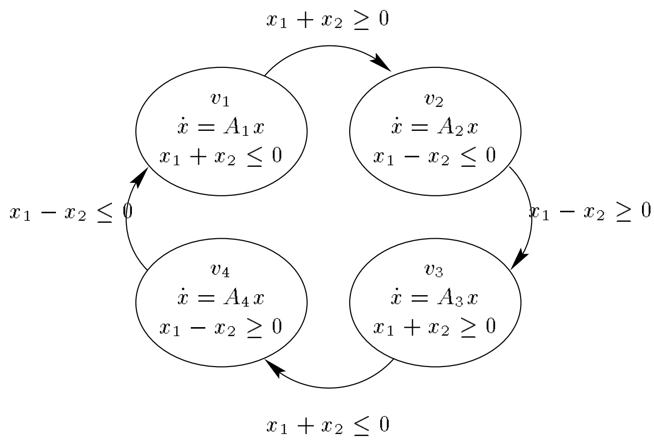

Fig. 3. Hybrid automaton for Example 3.

Furthermore, assume that for all $\chi \in \mathcal{E}_H^\infty$, $\tau_\infty(\chi) = \infty$ *and that* $x^* = 0$ *is an equilibrium point of* H. *Then, if for all* $k \in \{1, \ldots, K\}$ *there exists* $U_k = U_k^T$, $W_k = W_k^T$, *and* $M = M^T$, *where* U_k *and* W_k *have non-negative elements, such that* $P_k = F_k^T M F_k$ *satisfies*

$$\begin{aligned} A_k^T P_k + P_k A_k + E_k^T U_k E_k &\prec 0 \\ P_k - E_k^T W_k E_k &\succ 0, \end{aligned}$$

$x^* = 0$ *is asymptotically stable.*

Note that the matrix inequalities are linear in the unknown matrices U_k, W_k, and M. The second inequality assures that $\mathcal{V}(v_k, x) = x^T P_k x > 0$ for all $x \neq 0$, $(v_k, x) \in \text{Inv}$. The assumptions on H impose that $(v_k, 0) \in \text{Inv}$ for all k. This assumption can be removed by introducing a bit more involved notation, see the references.

Example 3. Consider the hybrid automaton in Figure 3 with

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}, \qquad A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}.$$

Here we may choose

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$F_k = \begin{bmatrix} E_k \\ I \end{bmatrix}, \quad \forall k \in \{1, \ldots, 4\}.$$

The eigenvalues of A_k are $\lambda(A_k) = \{-1/10 \pm \sqrt{5}i\}$. The evolution of the continuous state is shown in Figure 4 for the initial state $(v_0, x_0) = \big(v_1, (-2, 0)\big)$. Origin

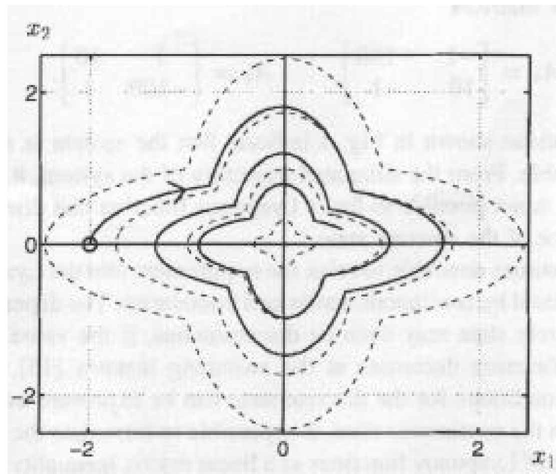

Fig. 4. Continuous evolution for hybrid automaton in Example 3. The level sets for the Lyapunov function is indicated by dashed lines. Note that function is not globally quadratic.

seems to be a stable equilibrium point. Indeed, the Lyapunov function indicated by dashed level sets proves asymptotic stability for the hybrid automaton. Note that it is not quadratic, but only quadratic in each discrete mode. The Lyapunov function is given by $\mathcal{V}(v_k, x) = x^T P_k x$ with

$$P_1 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Linear Matrix Inequalities

Many problems in control and system theory results in LMI's. For example, in Theorem 3 we have in the unknown M, U_k, and W_k, the LMI

$$\begin{aligned} E_k^T M E_k &\succ 0 \\ A_k^T P_k + P_k A_k + E_k^T U_k E_k &\prec 0 \\ P_k - E_k^T W_k E_k &\succ 0. \end{aligned}$$

The problem of finding M, U_k, and W_k, such as the LMI is satisfied can be cast as an optimization problem. It turns out that it is a convex optimization problem, so it can be efficiently solved. Recently, softwares for solving LMI problems have been developed. For example, there is a MATLAB toolbox called LMI Control Toolbox for solving LMI problems. In MATLAB, you may type

```
>> help lmilab
```

for more information. Try the demo

```
>> lmidem
```

and you will see that there is a graphical user interface to specify LMI's. It is entered by

```
>> lmiedit
```

After specifying an LMI, for instance,

$$P = P^T \succ 0$$
$$A^T P + PA \prec 0,$$

where A is a matrix that should be stored in MATLAB's workspace, and storing it in `lmisys`, a feasible solution `p` is found (if it exists) by running the commands

```
>> [tmin,pfeas]=feasp(lmisys)
>> p=dec2mat(lmisys,pfeas,p)
```

Note that the feasibility problem is solved as a convex optimization problem, which hence has a global minimum. This means that if the feasibility problem has a solution, the optimization algorithm will (at least theoretically) always find it.

Background

Most of this lecture is based on [4, 3]. More general cases are covered there, for instance, when the origin does not necessarily belong to the invariant condition of every discrete mode. The introduction on the Lyapunov equation is inspired by [5]. The book [1] is devoted to LMI's.

References

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