Lecture 6

Lyapunov Stability

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This lecture introduces stability for hybrid automata. By generalizing some classical concepts from continuous dynamical systems, we are able to derive a Lyapunov stability theorem for hybrid automata.

Continuous Dynamical Systems Revisited

Consider the ordinary differential equation

$$\dot{x}(t) = F\big(x(t)\big), \qquad x(0) = x_0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous (see Lecture 3).

Definition 1 (Equilibrium Point). *The point $x^* = 0$ is an* equilibrium point *of (1) if $F(0) = 0$.*

An equilibrium point is trivially an invariant set, since $x_0 = 0$ implies that $x(t) = 0$ for all $t \geq 0$.

Definition 2 (Stability). *The equilibrium point $x^* = 0$ of (1) is* stable[1] *if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \;\; \forall t \geq 0,$$

where $x : [0, \infty) \to \mathbb{R}^n$ is the solution of (1).

The equilibrium point $x^* = 0$ is asymptotically stable *if it is stable and $\delta > 0$ can be chosen such that*

$$\|x_0\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} \|x(t)\| = 0.$$

It is no loss of generality to assume $x^* = 0$, because to consider other points we can simply make a change of variables.

If an equilibrium point is not stable, it is called unstable. If the equilibrium point is (asymptotically) stable for all $x_0 \in \mathbb{R}^n$, it is called globally (asymptotically) stable. Note that for a stable equilibrium point x^*, there is no implication that the solution will converge to x^*. Furthermore, for a system (1) with a single unstable equilibrium point, the solution can, of course, be bounded.

[1] This stability concept is often called stable in the sense of Lyapunov.

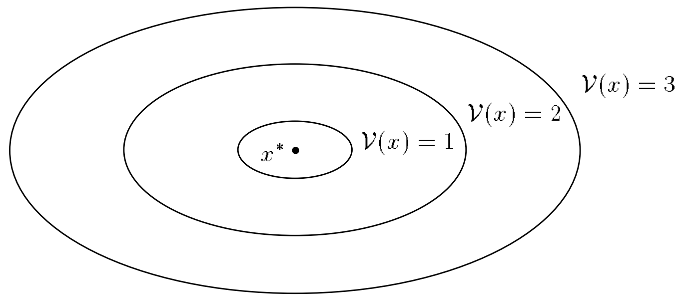

Fig. 1. Levels sets for a Lyapunov function. If a solution enters one of these sets, it has to stay there.

Consider a continuously differentiable (C^1) function $\mathcal{V} : \Omega \to \mathbb{R}$ where Ω is a subset of $\mathbb{R}^n$. The rate of change of $\mathcal{V}$ along solutions of (1) is denoted

$$\dot{\mathcal{V}}(x) = \sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial x_i} F_i(x) = \frac{\partial \mathcal{V}}{\partial x} F(x).$$

Theorem 1 (Lyapunov's Stability Theorem). *Let $x^* = 0$ be an equilibrium point of (1) and $\Omega \subset \mathbb{R}^n$ a set containing x^*. If $\mathcal{V} : \Omega \to \mathbb{R}$ is a C^1 function such that*

$$\begin{aligned} &\mathcal{V}(0) = 0 \\ &\mathcal{V}(x) > 0, \ \ \forall x \in \Omega \setminus \{0\} \\ &\dot{\mathcal{V}}(x) \le 0, \ \ \forall x \in \Omega, \end{aligned}$$

then x^* is stable. Furthermore, if $x^* = 0$ is stable and

$$\dot{\mathcal{V}}(x) < 0, \ \ \forall x \in \Omega \setminus \{0\},$$

then x^* is asymptotically stable.

A function $\mathcal{V}$ that satisfies the three assumptions above for stability is called a Lyapunov function. A Lyapunov function defines level sets $\{x \in \mathbb{R}^n : \ \mathcal{V}(x) \le c\}$ for $c > 0$. The boundaries of these are illustrated in Figure 1 for $c = 1, 2, 3$. The condition $\dot{\mathcal{V}}(x) \le 0$ in Theorem 1 gives that when a solution crosses a boundary $\{x \in \mathbb{R}^n : \ \mathcal{V}(x) = c\}$ at $t = t_0 \ge 0$, it can never leave the corresponding level set again because

$$\dot{\mathcal{V}}\big(x(t)\big) \le 0 \quad \Rightarrow \quad \mathcal{V}\big(x(t)\big) \le \mathcal{V}\big(x(0)\big) \le c, \ \ \forall t \ge t_0.$$

We apply Theorem 1 to the pendulum of Example 4 in Lecture 1.

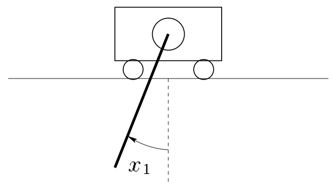

Fig. 2. Pendulum.

Example 1. Consider the pendulum in Figure 2 with unit mass and unit length (and the cart fixed to the ground), whose dynamics is governed by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -g\sin x_1.\end{aligned}$$

To show that the downright position $x^* = 0$ of the pendulum is a stable equilibrium point, consider the function

$$\mathcal{V}(x) = g(1-\cos x_1) + \frac{x_2^2}{2},$$

defined over the set $\Omega = \{x \in \mathbb{R}^2 : \ -\pi < x_1 < \pi\}$. Then,

$$\begin{aligned}&\mathcal{V}(0) = 0\\ &\mathcal{V}(x) > 0, \ \ \forall x \in \Omega \setminus \{0\},\end{aligned}$$

and

$$\dot{\mathcal{V}}(x) = \frac{\partial \mathcal{V}}{\partial x}F(x) = \begin{bmatrix} g\sin x_1 \ x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -g\sin x_1 \end{bmatrix} \equiv 0,$$

so the equilibrium point $x^* = 0$ is stable. Note, however, that we cannot prove that x^* is asymptotically stable using Theorem 1, because $\dot{\mathcal{V}}(x) \not< 0$. In fact, it is easy to show (and intuitively clear) that x^* is not asymptotically stable.

Note that Theorem 1 is not constructive, in the sense that no hints are given on how to find Lyapunov functions. In fact, this may be a difficult problem in general. In the example, $\mathcal{V}$ corresponds to the energy of the pendulum, which is often a solution to the problem in practice. For a stable linear system $\dot{x} = Ax$ a Lyapunov function is given by $\mathcal{V}(x) = x^T P x$, where P is a symmetric positive definite matrix[2] such that $A^T P + PA = -I$.

The strength of the Lyapunov result is that we can draw conclusion about all solutions with initial conditions in a certain set without having to integrate the vector field.

[2] A matrix M is called positive definite if $x^T M x > 0$ for all $x \neq 0$. It is called positive semidefinite if $x^T M x \geq 0$ for all x.

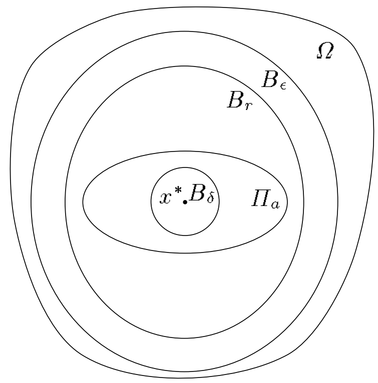

Fig. 3. Sets used in the proof of Theorem 1.

Proof (Theorem 1). We give a sketch for the proof of stability, i.e., that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \ \forall t \geq 0.$$

In Figure 3, the balls are defined as $B_r = \{x \in \mathbb{R}^n : \ \|x\| \leq r\}$, with $r \in (0, \epsilon]$, and and the level set Π_a is defined as

$$\Pi_a = \{x \in B_r : \ \mathcal{V}(x) \leq a\},$$

where $a \in (0, b)$ and $b = \min_{x \in \partial B_r} \mathcal{V}(x)$. A solution $x(t)$ that starts in Π_a stays there, because $\dot{\mathcal{V}}\big(x(t)\big) \leq 0$ implies that

$$\mathcal{V}\big(x(t)\big) \leq \mathcal{V}\big(x(0)\big) \leq a, \ \forall t \geq 0.$$

We may choose $\delta > 0$ such that $\mathcal{V}(x) < a$ for all $x \in B_\delta$, since $\mathcal{V}$ is continuous and $\mathcal{V}(0) = 0$. Then, $B_\delta \subset \Pi_a \subset B_r$, so that

$$x(0) \in B_\delta \quad \Rightarrow \quad x(0) \in \Pi_a \quad \Rightarrow \quad x(t) \in \Pi_a, \ \forall t \geq 0 \quad \Rightarrow \quad x(t) \in B_r, \ \forall t \geq 0.$$

Since $r \in (0, \epsilon)$, this proves the result. Asymptotic stability is shown by proving that $\mathcal{V}\big(x(t)\big) \to 0$ as $t \to \infty$, see the references.

Stability of Hybrid Automata

We now generalize concepts from previous section to hybrid automata.

Definition 3 (Equilibrium Point of Hybrid Automata). *The continuous state $x^* = 0 \in \mathbb{R}^n$ is an* equilibrium point *of a hybrid automaton H if there exists a non-empty set $\bar{V} \subset V$ such that for all $v \in \bar{V}$,*

– $(v', z') \in \mathrm{Jump}(v, 0)$ *implies that $z' = 0$ and $v' \in \bar{V}$,*

– $f(v, 0) = 0$.

If $\text{Jump}(v, 0) = \emptyset$, Definition 3 is the trivial generalization of Definition 1. If $\text{Jump}(v, 0) \neq \emptyset$, then we require that the vector field should vanish in the origin also for all reachable discrete states.

An equilibrium point $x^* = 0$ together with $\bar{V}$ define an invariant set in the following sense.

Definition 4 (Invariant Set). *A set $M \subset \text{Init}$ is called invariant if for all $(v_0, x_0) \in M$, $(\tau, v, x) \in \mathcal{E}_H(v_0, x_0)$, and $t \in \tau$, it holds that $\big(v(t), x(t)\big) \in M$.*

Lemma 1. *The set $W = \{x^*\} \times \bar{V}$ in Definition 3 is invariant.*

Proof. Consider an execution $\chi = (\tau, v, x) \in \mathcal{E}_H(v_0, x_0)$ with $(v_0, x_0) \in W$. If $\text{Jump}(v_0, 0) = \emptyset$, the invariance follows from that $f(v_0, 0) = 0$. If $\text{Jump}(v_0, 0) \neq \emptyset$, by the definition of equilibrium point, $(v_1, z') \in \text{Jump}(v_0, 0)$ implies that $z' = 0$ and $v_1 \in \bar{V}$. The result now follows by induction.

To simplify the presentation we assume that $x^* = 0$, as in the previous section for continuous systems. The state transformation required to discuss other points may however be more involved for hybrid automata.

Definition 5 (Stability of Hybrid Automata). *Assume $x^* = 0$ is an equilibrium point of a hybrid automaton H. It is called stable, if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all executions $\chi = (\tau, x, v) \in \mathcal{E}_H(v_0, x_0)$ with $\|x_0\| < \delta$ it holds that $\|x(t)\| < \epsilon$ for all $t \in \tau$.*

The equilibrium point $x^* = 0$ is asymptotically stable *if it is stable and $\delta > 0$ can be chosen such that for all executions $\chi = (\tau, x, v) \in \mathcal{E}_H^\infty(v_0, x_0)$ with $\|x_0\| < \delta$ it holds that $\lim_{t \to \tau_\infty} \|x(t)\| = 0$.*

Note that in the definition of stability, the discrete states of the hybrid automata are not taken into account. This means that an equilibrium point of a hybrid automaton is considered asymptotically stable, although the discrete evolution does not converge. An execution χ in the definition may be blocking, non-deterministic, as well as Zeno.

The following example illustrates that an equilibrium point of a hybrid automaton may be unstable, even if the corresponding equilibrium points in each individual discrete mode are stable.

Example 2. Consider the hybrid automaton in Figure 4, where

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}.$$

Both matrices have eigenvalues $-1 \pm i10\sqrt{10}$, so the differential equations $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ are stable. The eigenvectors are different for A_1 and A_2, which is reflected in the phase portraits in Figure 5. The equilibrium point $x^* = 0$ of the hybrid automaton in Figure 4 is, however, unstable. An example of the continuous evolution for an execution is shown to the left in Figure 6.

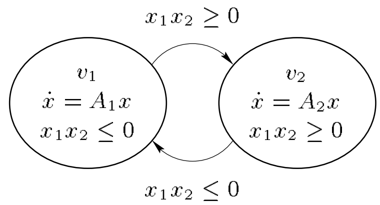

Fig. 4. Hybrid automaton with stable dynamics in each discrete mode, but still the equilibrium point $x^* = 0$ is unstable.

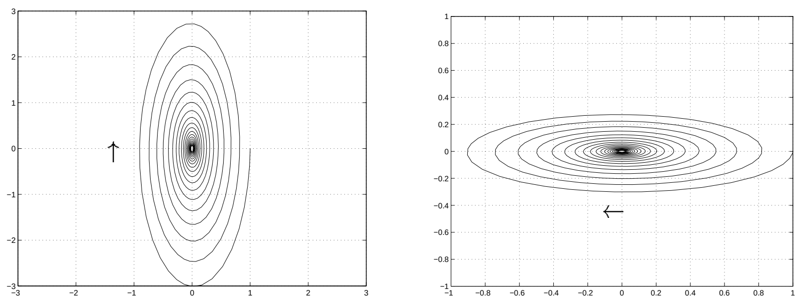

Fig. 5. Phase portraits for $\dot{x} = A_1x$ (left) and $\dot{x} = A_2x$ (right).

To the right, the continuous evolution is shown for an execution of the hybrid automaton where A_1 and A_2 have changed places such that $f(v_1, x) = A_2x$ and $f(v_2, x) = A_1x$. For this hybrid automaton, $x^* = 0$ is stable.

The example illustrates that it is possible to destabilize two stable systems by introducing switching. It is easy to construct a similar example showing that it is possible to stabilize two unstable systems.

Lyapunov's Stability Theorem for Hybrid Automata

Theorem 2 (Lyapunov's Stability Theorem). *Consider a hybrid automaton H, such that $(v, z) \in V \times \mathbb{R}^n$ and $(v', z') \in \mathrm{Jump}(v, z)$ imply that $z' = z$. Assume there exists an open set $\Omega \subset V \times \mathbb{R}^n$, such that $(v, 0) \in \Omega$ for some $v \in V$ and that $x^* = 0$ is an equilibrium point of H. Let $\mathcal{V} : \Omega \to \mathbb{R}$ be a C^1 function in its second argument such that for all $v \in V$,*

$$\begin{aligned} \mathcal{V}(v, 0) &= 0 \\ \mathcal{V}(v, x) &> 0, \ \ \forall x, \ (v, x) \in \Omega \setminus 0 \\ \frac{\partial \mathcal{V}}{\partial x} f(v, x) &\leq 0, \ \ \forall x, \ (v, x) \in \Omega. \end{aligned}$$

If for all $\chi = (\tau, v, x) \in \mathcal{E}_H(v_0, x_0)$ with $(v_0, x_0) \in \mathrm{Init} \cap \Omega$ and for all $\hat{v} \in V$, the sequence $\{\mathcal{V}(v(\tau_i), x(\tau_i)) : \ v(\tau_i) = \hat{v}\}$ is non-increasing (or empty), then $x^* = 0$ is stable.

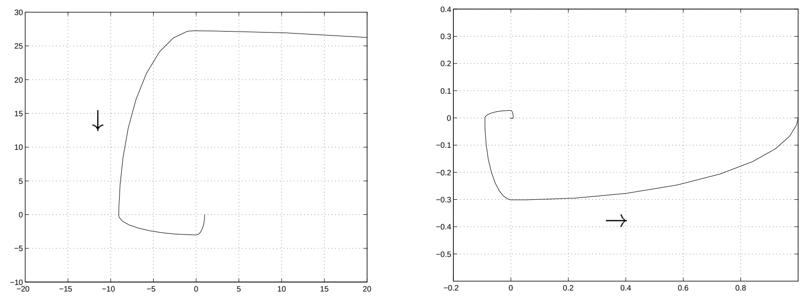

Fig. 6. Execution of hybrid automaton in Figure 4 (left) and execution of a similar hybrid automaton but with $f(v_1, x) = A_2 x$ and $f(v_2, x) = A_1 x$ (right).

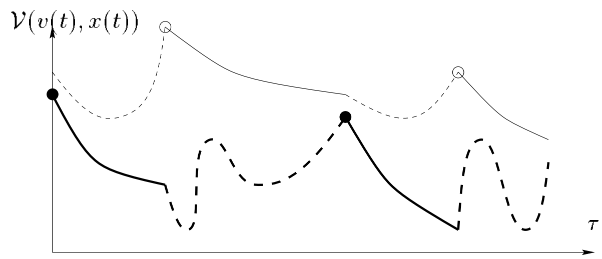

Fig. 7. Evolution of $\mathcal{V}$ for a stable case. The thick line correspond to v_1 and the thin line to v_2. The "active" part of $\mathcal{V}$ is solid.

The theorem is illustrated in Figure 1 for the case $V = \{v_1, v_2\}$ and $(v_i, \cdot) \notin \text{Jump}(v_i, \cdot)$. The thick line represent $\mathcal{V}\big(v_1, x(t)\big)$ and the thin line represent $\mathcal{V}\big(v_2, x(t)\big)$. The solid lines show when a discrete mode is "active," in the sense that v_1 is actice for $t \in [\tau_i, \tau_i']$ if $v(t) = v_1$. In Figure 1, both v_1 and v_2 are active twice. The two dots illustrate the sequence $\{\mathcal{V}\big(v(\tau_i), x(\tau_i)\big) \,:\, v(\tau_i) = v_1\}$ and the two circles the sequence $\{\mathcal{V}\big(v(\tau_i), x(\tau_i)\big) \,:\, v(\tau_i) = v_2\}$. The last assumption in Theorem 2 says that all such sequences formed by executions in $\mathcal{E}_H(v_0, x_0)$ should be non-increasing.

Proof. We sketch the proof for $V = \{v_1, v_2\}$ and $(v_i, \cdot) \notin \text{Jump}(v_i, \cdot)$. Define the sets in Figure 8 similar to the proof of Theorem 1), for example,

$$\varPi_1 = \{x \in B_{r_1} \,:\, \mathcal{V}(v_1, x) \leq a_1\},$$

where $a_1 \in (0, b_1)$ and $b_1 = \min_{x \in \partial B_{r_1}} \mathcal{V}(v_1, x)$. The left sets correspond v_1 and the right sets to v_2. Now let $r = \min\{\bar{r}_1, \bar{r}_2\}$ and define the sets in Figure 9, where again the left sets correspond v_1 and the right sets to v_2. Let $\delta = \min\{\tilde{r}_1, \tilde{r}_2\}$. Consider $\chi = (\tau, v, x) \in \mathcal{E}_H(v_0, x_0)$ with $\|x_0\| < \delta$ and assume $v_0 = v_1$. By a continuous Lyapunov argument, $x(t) \in \bar{\varPi}_1 \subset B_r$ for $t \in [\tau_0, \tau_0']$. Therefore, $x(\tau_1) = x(\tau_0') \in \varPi_2$ (where equality holds because of identity jump condition). By a continuous Lyapunov argument again, $x(t) \in \bar{\varPi}_2 \subset B_\epsilon$ for $t \in [\tau_1, \tau_1']$. By

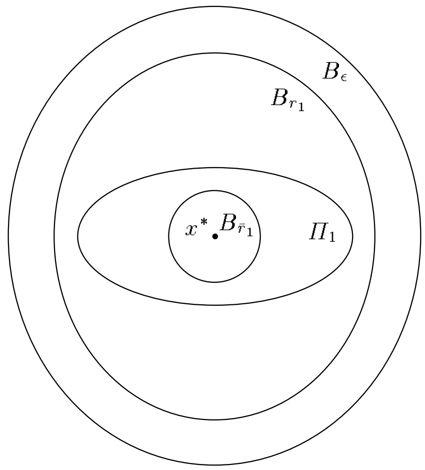

Fig. 8. Sets used in the proof of Theorem 2. The left ones correspond to $v = v_1$ and the right ones to $v = v_2$.

the assumption of the non-increasing sequence, $x(\tau_1') = x(\tau_2) \in \bar{\Pi}_1$. The result now follows from induction.

A severe drawback with this result compared to its counterpart for continuous systems is that in Theorem 1 we have to verify that a certain sequence taken over the execution is non-increasing for all exeuctions. It practice we might hence have to integrate the vector fields for all possible executions. Hence, the strength of the classic Lyapunov result is thus not completely carried over to hybrid automata.

The next problem is how to find the Lyapunov function $\mathcal{V}$ in Theorem 2. This will be discussed in next lecture.

Background

The introductory section on continuous dynamical systems are based on [3, 2]. The Lyapunov theorem for hybrid automata presented as Theorem 2 is a slight extension of Theorem 2.3 in[1]. Variations of that Lyapunov theorem are given in [4]. Example 2 is taken from [1].

References

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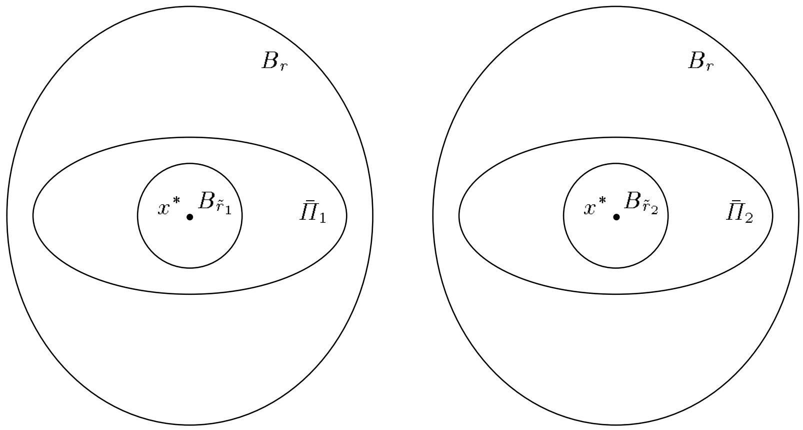

Fig. 9. Sets used in the proof of Theorem 2. The left ones correspond to $v = v_1$ and the right ones to $v = v_2$. The ball B_r corresponds to the smaller of $B_{\bar{r}_1}$ and $B_{\bar{r}_1}$ in Figure 8.

4. H. Ye, A. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 43(4):461–474, April 1998.