Lecture 4

Zeno Hybrid Automata

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Zeno Hybrid Automata

Zeno hybrid automata[1] accept executions with infinitely many discrete transitions within a finite time interval. Real physical systems are, of course, not Zeno, but hybrid automata modeling real systems may be Zeno. The Zeno phenomena is often due to a too high level of abstraction. In this lecture, we discuss some properties of Zeno hybrid automata.

Recall from Lecture 2 that an infinite execution $\chi = (\tau, v, x) \in \mathcal{E}_H^\infty$ is Zeno if the execution time $\tau_\infty(\chi) = \sum_{i=0}^\infty (\tau_i' - \tau_i)$ is bounded. If $\tau_\infty(\chi) < \infty$, this time is called the Zeno time.

Definition 1 (Zeno Hybrid Automaton). *A hybrid automaton H is Zeno if there exists $(v_0, x_0) \in \mathrm{Init}$ such that all executions in $\mathcal{E}_H^\infty(v_0, x_0)$ are Zeno.*[2]

Example 1. The non-analytic function $z \mapsto z \sin(1/z)$ has infinitely many zeros in the interval $(-1, 0)$. Hence, the hybrid automaton in Figure 1 with $\mathrm{Init} = \{(v, z) \in V \times \mathbb{R} : z \in (-1, 0)\}$ is Zeno.

Example 2. Consider the bouncing ball automaton in Figure 2, where $\mathrm{Init} = \{(v, z) : z_1 > 0\}$, $c \in (0, 1)$, and $g > 0$. The simulated execution indicates that the hybrid automaton is Zeno. In fact, this is easily proved by considering a generic execution with initial state $(v, x_0) \in \mathrm{Init}$ and $x_0 = (x_{01}, x_{02})$. Recall from Lecture 3 that the hybrid automaton accepts a unique infinite execution for every initial state. The first jump (bounce) for the execution occurs at

$$\tau_1 = \tau_0' = \frac{x_{20} + \sqrt{x_{20}^2 + 2g x_{10}}}{g}.$$

[1] The name Zeno refers to the philosopher Zeno of Elea (ca. 500–400 B.C.), whose major work consisted of a number of paradoxes, designed to support his view that the concepts of motion and evolution lead to contradictions. An example is Zeno's Second Paradox of Motion, in which Achilles is racing against a tortoise.

[2] An alternative definition is to say that a hybrid automaton is Zeno if there is *at least one* Zeno execution in $\mathcal{E}_H^\infty(v_0, x_0)$. In that case, a non-deterministic Zeno hybrid automaton may accept both Zeno and non-Zeno executions, which may be an undesirable feature for instance in Reach set calculations. For deterministic hybrid automata the two definitions coincide.

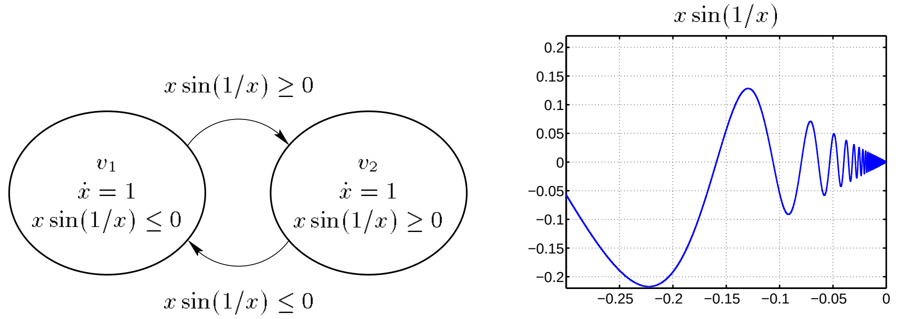

Fig. 1. A hybrid automaton that is Zeno for Init $= \{(v, z) \in V \times \mathrm{R} : z \in (-1, 0)\}$ due to that the invariant and jump conditions are defined by a non-analytic function $x \mapsto x\sin(1/x)$, which has infinitely many zeros in the interval $(-1, 0)$.

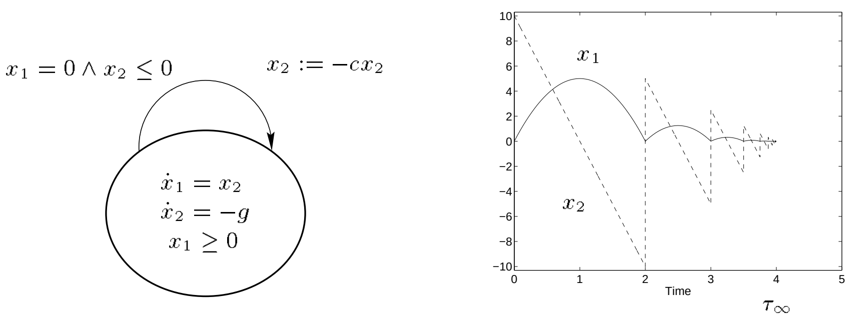

Fig. 2. Hybrid automaton for bouncing ball, where Init $= \{(v, z) : z_1 > 0\}$.

The next occurs at

$$\tau_2 = \tau_1' = \tau_1 + \frac{2x_2(\tau_1)}{g}.$$

More generally, we have

$$\tau_N = \tau_1 + \frac{2x_2(\tau_1)}{g} \sum_{k=1}^{N} c^{k-1},$$

where $x_2(\tau_1) = -cx_2(\tau_0') = c\sqrt{x_{20}^2 + 2gx_{10}}$ and $\sum_{k=1}^{N} c^{k-1} \to 1/(1-c)$ as $N \to \infty$. Hence,

$$\tau_\infty(\chi) = \frac{x_{20}}{g} + \frac{(1+c)\sqrt{x_{20}^2 + 2gx_{10}}}{g(1-c)} < \infty$$

for every execution $\chi \in \mathcal{E}_H^\infty(v, x_0)$, so H is Zeno. For $c = 1/2$, $g = 10$, and $x_0 = (0, 10)$, we get $\tau_\infty(\chi) = 4$. This corresponds to the execution shown in Figure 2.

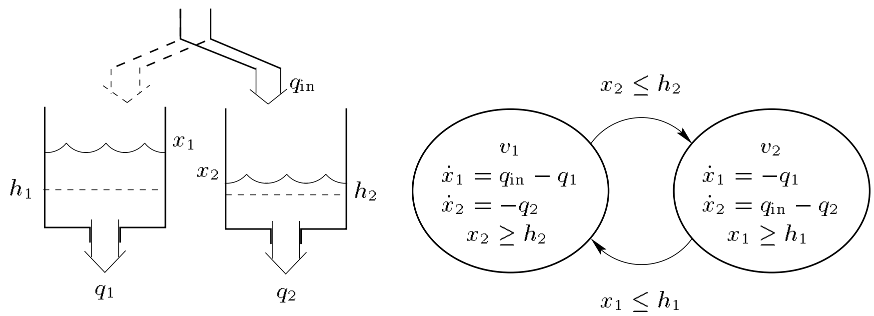

Fig. 3. Water tank system and corresponding hybrid automaton.

Example 3. Consider the water tank system in Figure 3 introduced in Lecture 2, where $\text{Init} = V \times \{z \in \mathbb{R}^2 : z_1 > h_1, z_2 > h_2\}$. The hybrid automaton illustrates the dynamics of the system interacting with a switching control strategy, which tries to keep the volumes in the two tanks above h_1 and h_2, respectively. Assume that the inflow is larger than any of the outflows but that the sum of the outflows are larger than the inflow, i.e., $\max\{q_1, q_2\} < q_{\text{in}} < q_1 + q_2$. This means of course that the tanks in reality eventually will drain. The hybrid automaton, however, does not, capture this, because it is Zeno. The executions of the hybrid automaton does not exceed the Zeno time, which is the approximate time when the sum of the tank volumes $x_1 + x_2$ for a physical implementation would go below $h_1 + h_2$. It is straightforward to prove that the hybrid automaton is Zeno, because (similar to the bouncing ball example) it is easy to explicitly derive the hybrid time trajectory τ. By doing this, we get the Zeno time

$$\tau_\infty(\chi) = \frac{x_{10} + x_{20} - h_1 - h_2}{q_1 + q_2 - q_{\text{in}}} < \infty$$

for all executions $\chi \in \mathcal{E}_H^\infty(v, x_0)$, where $v \in \{v_1, v_2\}$ and $x_0 = (x_{01}, x_{02}) \in \text{Init}$. For $h_1 = h_2 = 1$, $(q_1, q_2, q_{\text{in}}) = (2, 3, 4)$, and $(v_0, x_0) = \big(v_1, (2, 2)\big)$, the Zeno time is $\tau_\infty(\chi) = 2$. A simulation illustrating this case is shown in Figure 4.

In the previous three examples, the Zeno executions satisfy the inequality $\tau_i' > \tau_i$ for all $i \geq 0$. This is not always the case for Zeno hybrid automata, as illustrated by the following example.

Example 4. Consider the hybrid automaton in Figure 5. Every execution $\chi \in \mathcal{E}_H^\infty(v_0, x_0)$ with $(v_0, x_0) \in \text{Init} = V \times \mathbb{R}$ is Zeno. In fact, $\tau_0' = \tau_0 + \|x_0\|$ and $\tau_i = \tau_i' = \tau_0'$ for all $i > 0$, so $\tau_\infty(\chi) = \|x_0\| < \infty$.

Motivated by these examples, we first show a couple of existence results for Zeno hybrid automata and then characterize the limiting behavior of Zeno executions.

Proposition 1. *A hybrid automaton is Zeno only if the graph (V, E) has a cycle.*

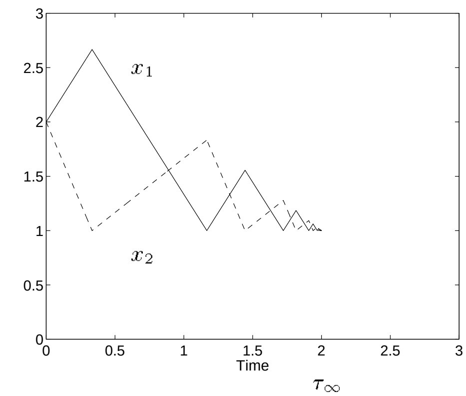

Fig. 4. Zeno execution for water tank hybrid automaton.

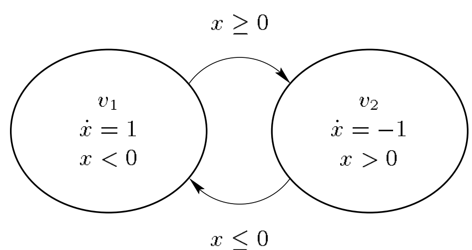

Fig. 5. A Zeno hybrid automaton with Zeno executions satisfying $\tau_i = \tau_i' = \tau_0' = \tau_0 + \|x_0\|$ for all $i > 0$.

Proof. If (V, E) has no cycle, then H accept executions only with a finite number of jumps. Such an execution cannot be Zeno.

Proposition 2. *If there exists a finite collection of states* $\{(v_i, z_i)\}_{i=1}^K$ *in* $V \times \mathbb{R}^n$ *such that*

- $(v_1, z_1) = (v_K, z_K)$,
- *there exists* $i \in \{1, \ldots, K\}$, $(v_i, z_i) \in \text{Reach}_H$,
- *for all* $i \in \{1, \ldots, K-1\}$, $(v_{i+1}, z_{i+1}) \in \text{Jump}(v_i, z_i)$,

then the hybrid automaton H accepts a Zeno execution.

Proof. Consider a finite execution $\chi = (\tau, v, x) \in \mathcal{E}_H$ with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$, such that $\big(v(\tau_N), x(\tau_N)\big) = (v_k, x_k)$ for some $k \in \{1, \ldots, K\}$. Such χ exists by the definition of Reach_H. Let $\hat{\tau} = \{[\hat{\tau}_i, \hat{\tau}_i']\}_{i=0}^\infty$, where $\hat{\tau}_i = \tau_i$ and $\hat{\tau}_i' = \tau_i'$ if $i \in \{0, \ldots, N\}$, and $\hat{\tau}_i' = \hat{\tau}_i = \tau_N$ if $i > N$. Then, H accepts the Zeno execution

$\hat{\chi} = (\hat{\tau}, \hat{v}, \hat{x})$, where

$$\hat{v}(t) = \begin{cases} v(t), & \text{if } t \in \tau \\ v_{k+1}, & \text{if } t \in [\hat{\tau}_{N+1}, \hat{\tau}_{N+1}] \\ \vdots & \vdots \\ v_K, & \text{if } t \in [\hat{\tau}_{N+K-k}, \hat{\tau}_{N+K-k}] \\ v_2, & \text{if } t \in [\hat{\tau}_{N+K-k+1}, \hat{\tau}_{N+K-k+1}] \\ \vdots & \vdots \end{cases}$$

and $\hat{x}(t)$ is defined similarly.

Zeno States

The asymptotic behavior of an infinite execution is captured in terms of its ω limit set. This is a generalization of the concept from continuous dynamical systems.

Definition 2 (ω Limit Point). *An ω limit point $(\hat{v}, \hat{z}) \in V \times \mathbb{R}^n$ of an execution $\chi = (\tau, v, x) \in \mathcal{E}_H^\infty$ is a point for which there exists a sequence $\{\theta_n\}_{n=0}^\infty$, $\theta_n \in \tau$, such that as $n \to \infty$, $\theta_n \to \tau_\infty$ and $\big(v(\theta_n), x(\theta_n)\big) \to (\hat{v}, \hat{z})$. The ω limit set of an execution χ is the set of all ω limit points.*

If the continuous part of a Zeno execution is bounded, then it has an ω limit point. We introduce the term Zeno state for such a point.

Definition 3 (Zeno State). *An ω limit point of a Zeno execution is called a Zeno state. The ω limit set of a Zeno execution is called the Zeno set.*

We use $Z_\infty \subset V \times \mathbb{R}^n$ to denote the Zeno set. We write $V_\infty \subset V$ for the discrete part of Z_∞ and $X_\infty \subset \mathbb{R}^n$ for the continuous part, i.e.,

$$\begin{aligned} V_\infty &= \{v \in V : \exists z \in \mathbb{R}^n, (v, z) \in Z_\infty\} \\ X_\infty &= \{z \in \mathbb{R}^n : \exists v \in V, (v, z) \in Z_\infty\}. \end{aligned}$$

For the bouncing ball example, we have $V_\infty = V$ and $X_\infty = \{(0,0)\}$, and for the water tank example, $V_\infty = V$ and $X_\infty = \{(1,1)\}$.

A hybrid automaton may accept a Zeno execution, but the execution must not have a Zeno state. This is the case if the continuous part of the execution become unbounded as $t \to \tau_\infty$. It is straightforward to derive examples where the Zeno set have any finite number of elements, as well as an infinite but countable or an uncountable number of elements. In Examples 1–3 we saw that the discrete parts of the Zeno execution χ were periodic in the limit as $t \to \tau_\infty(\chi)$. This is not the case in general as illustrated by the following example.

Example 5. Consider the Zeno hybrid automaton in Figure 6 where $c \in (0,1)$ (cf. the bouncing ball hybrid automaton). This system does not accept Zeno

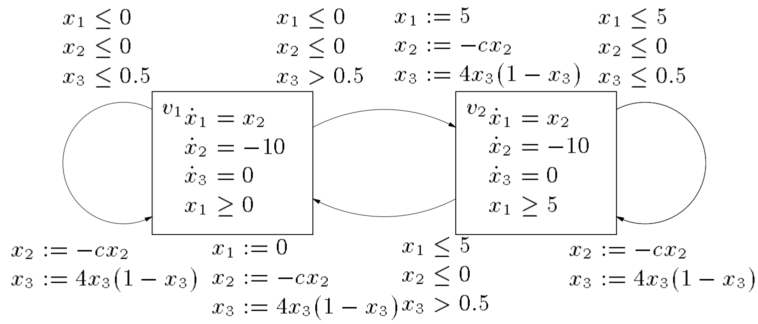

Fig. 6. A hybrid automaton that accepts Zeno executions that do not periodically jump between the discrete states as $t \to \tau_\infty$.

executions that periodically jump between the two discrete modes in the limit. A simulation is presented in Figure 7, where x_1 and x_2 are shown. The third continuous state x_3 is initialized at $x_3(\tau_0) = 0.9$. The reason for the quasi-periodic behavior is that the jump map for x_3 is defined by the so called logistic map, which shows this kind of quasi-periodic behavior when iterated.

A jump condition Jump is *non-expanding*, if there exists $\delta \in [0,1]$ such that $(v', z') \in \text{Jump}(v, z)$ implies $\|z'\| \le \delta \|z\|$. It is *contracting*, if there exists $\delta \in [0,1)$ such that $(v', z') \in \text{Jump}(v, z)$ and $(v', y') \in \text{Jump}(v, y)$ imply $\|z' - y'\| \le \delta \|y - x\|$. Note that the jump map defines a single-valued function in the second case.

For continuous dynamical systems, an assumption of global Lipschitz continuity on the vector field excludes the possibility for finite escape time.[3] A hybrid automaton may accept executions with finite execution times for which the continuous state is unbounded. To avoid this, it is not sufficient to assume that $f(v, \cdot)$ is globally Lipschitz continuous for all $v \in V$. However, if also the jump condition is non-expanding, then the continuous state is bounded along executions.

Lemma 1. *Consider a hybrid automaton with non-expanding jump condition. Then, there exists $c > 0$ such that for all executions $\chi = (\tau, q, x) \in \mathcal{E}_H$ and $t \in \tau$,*

$$\|x(t)\| \le \big(\|x(\tau_0)\| + 1\big) e^{c(t-\tau_0)} - 1.$$

Note that the right-hand side of this expression tends to infinity as $t \to \infty$. When $x(\cdot)$ is bounded, the Bolzano–Weierstrass Property implies that there exists at least one Zeno state for each Zeno execution. If the continuous part of the jump condition is the identity map, then the continuous part of the Zeno set X_∞ is a singleton ($|X_\infty| = 1$), as proved next.

[3] *Finite escape time* is that a trajectory of a continuous dynamical system goes to infinity in finite time. For example, $\dot{x} = -x^2$, $x(0) = -1$ has the (unique) solution $x(t) = 1/(t-1)$, which tends to $-\infty$ as $t \to 1$.

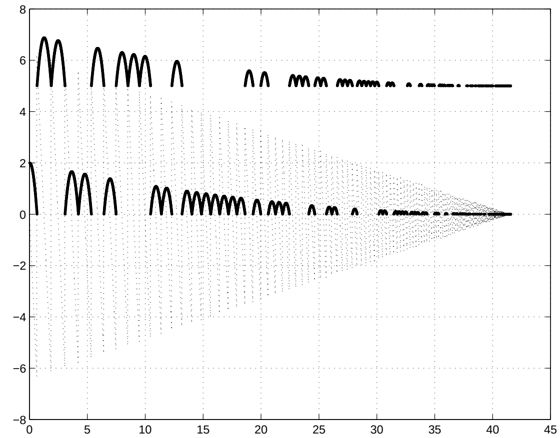

Fig. 7. An example of an execution for the hybrid automaton in Example 5. The continuous variables x_1 (solid) and x_2 (dotted) are shown. Note how x_1 seems to randomly jump either to 0 or 5.

Theorem 1. *Consider a hybrid automaton such that* $(v', z') \in \text{Jump}(v, z)$ *implies* $z' = z$. *Then, for every Zeno execution* $\chi = (\tau, v, x)$, *it holds that* $|X_\infty| = 1$.

Proof. For all sequences $\{\theta_i\}_{i=0}^{\infty}$, $\theta_i \in \tau$, such that $\theta_i \to \tau_\infty$, suppose $\theta_i \in [\tau_{n_i}, \tau'_{n_i}]$, where $n_i \to \infty$ as $i \to \infty$. We have

$$\begin{aligned} x(\theta_i) &= x(\tau_{n_i}) + \int_{\tau_{n_i}}^{\theta_i} f\big(v(\tau_{n_i}), x(\tau)\big)\, d\tau \\ &= x(\tau_{n_i}) + (\theta_i - \tau_{n_i}) f\big(v(\tau_{n_i}), (x_1(\xi^1_{n_i}), \dots, x_n(\xi^n_{n_i}))^T\big), \end{aligned}$$

for some $\xi^1_{n_i}, \dots, \xi^n_{n_i} \in [\tau_{n_i}, \tau'_{n_i}]$. Hence, for all $k > \ell \geq 0$,

$$\begin{aligned} x(\theta_k) = x(\theta_\ell) &+ (\tau'_{n_\ell} - \theta_\ell) f\big(v(\tau_{n_\ell}), (x_1(\xi^1_{n_\ell}), \dots, x_n(\xi^n_{n_\ell}))^T\big) \\ &+ \sum_{i=n_\ell+1}^{n_k-1} (\tau'_i - \tau_i) f\big(v(\tau_i), (x_1(\xi^1_i), \dots, x_n(\xi^n_i))^T\big) \\ &+ (\theta_k - \tau_{n_k}) f\big(v(\tau_{n_k}), (x_1(\xi^1_{n_k}), \dots, x_n(\xi^n_{n_k}))^T\big), \end{aligned}$$

which gives that

$$\|x(\theta_k) - x(\theta_\ell)\| \leq K \sum_{i=n_\ell}^{n_k} (\tau'_i - \tau_i),$$

where $K > 0$ is a constant such that $\|f(v, z)\| \leq K$ for all $(v, z) \in V \times \mathbb{R}^n$. Such constant exists due to Assumption 1 in Lecture 3 (global Lipschitz continuity) and Lemma 1. By the fact that $\sum_{i=0}^{\infty} (\tau'_i - \tau_i) < \infty$, we know that $\{x(\theta_i)\}_{i=0}^{\infty}$ is a Cauchy sequence. The space $\mathbb{R}^n$ is complete, so the sequence has a limit

$\hat{x} = \lim_{i\to\infty} x(\theta_i)$. This limit is independent of the choice of sequence $\{\theta_i\}_{i=0}^{\infty}$, as follows from the following argument. Consider two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$, $\alpha_i, \beta_i \in \tau$, such that $\alpha_i \to \tau_\infty$ and $\beta_i \to \tau_\infty$. Suppose $\alpha_i \in [\tau_{m_i}, \tau'_{m_i}]$ and $\beta_i \in [\tau_{n_i}, \tau'_{n_i}]$, where $m_i \to \infty$ and $n_i \to \infty$ as $i \to \infty$, and $m_i \geq n_i$. Then,

$$\begin{aligned} x(\alpha_i) = x(\beta_i) &+ (\tau'_{n_i} - \beta_i) f\big(v(\tau_{n_i}), (x_1(\xi^1_{n_i}), \ldots, x_n(\xi^n_{n_i}))^T\big) \\ &+ \sum_{j=n_i+1}^{m_i-1} (\tau'_j - \tau_j) f\big(v(\tau_j), (x_1(\xi^1_j), \ldots, x_n(\xi^n_j))^T\big) \\ &+ (\alpha_i - \tau_{m_i}) f\big(v(\tau_{m_i}), (x_1(\xi^1_{m_i}), \ldots, x_n(\xi^n_{m_i}))^T\big). \end{aligned}$$

This gives that $\|x(\alpha_i) - x(\beta_i)\| \leq K \sum_{j=n_i}^{m_i} (\tau'_j - \tau_j)$. Hence, $\|x(\alpha_i) - x(\beta_i)\| \to 0$ as $i \to \infty$, which shows that both sequences have the same limit. This completes the proof.

Note that Theorem 1 gives the structure of the Zeno set for the large class of hybrid systems called switched systems, since these systems can be modeled as hybrid automata with identity reset relation.

If the reset relation is contracting and $(v', x') \in \text{Reset}(v, 0)$ implies that x' is the origin, then the continuous part of the Zeno state is also the origin.

Theorem 2. *Consider a Zeno hybrid automaton with contracting jump condition such that* $(v', z') \in \text{Jump}(v, 0)$ *implies* $z' = 0$. *Then, for every Zeno execution* $\chi = (\tau, v, x)$, *it holds that* $X_\infty = \{0\}$.

Proof. For all sequences $\{\theta_i\}_{i=0}^{\infty}$, $\theta_i \in \tau$, such that $\theta_i \to \tau_\infty$, suppose $\theta_i \in [\tau_{n_i}, \tau'_{n_i}]$, where $n_i \to \infty$ as $i \to \infty$. We have

$$\begin{aligned} \|x(\theta_i)\| &\leq \|x(\tau_{n_i})\| + \left\| \int_{\tau_{n_i}}^{\theta_i} f\big(v(\tau_{n_i}), x(\tau)\big)\, d\tau \right\| \\ &\leq \|x(\tau_{n_i})\| + K(\tau'_{n_i} - \tau_{n_i}), \end{aligned}$$

where $K > 0$ is the same constant as in the proof of Theorem 1. Using the fact that $\|x(\tau_{n_i})\| \leq \delta \|x(\tau'_{n_i-1})\|$, it follows that

$$\begin{aligned} \|x(\theta_i)\| &\leq \delta \|x(\tau'_{n_i-1})\| + K(\tau'_{n_i} - \tau_{n_i}) \\ &= \delta \left\| x(\tau_{n_i-1}) + \int_{\tau_{n_i-1}}^{\tau'_{n_i-1}} f\big(v(\tau_{n_i-1}), x(\tau)\big)\, d\tau \right\| + K(\tau'_{n_i} - \tau_{n_i}) \\ &\leq \delta \|x(\tau_{n_i-1})\| + K\delta(\tau'_{n_i-1} - \tau_{n_i-1}) + K(\tau'_{n_i} - \tau_{n_i}). \end{aligned}$$

By induction,

$$\|x(\theta_i)\| \leq \delta^{n_i} \|x(\tau_0)\| + K \sum_{m=0}^{n_i} \delta^{n_i - m} (\tau'_m - \tau_m).$$

Since

$$\sum_{n_i=0}^{\infty} K \sum_{m=0}^{n_i} \delta^{n_i-m}(\tau'_m - \tau_m) = K \sum_{m=0}^{\infty}(\tau'_m - \tau_m) \sum_{n_i=0}^{\infty} \delta^{n_i} = \frac{K\tau_\infty}{1-\delta} < \infty,$$

it holds that $K\sum_{m=0}^{n_i} \delta^{n_i-m}(\tau'_m - \tau_m) \to 0$ as $n_i \to \infty$. This yields that $\|x(\theta_i)\| \to 0$ as $i \to \infty$, which, hence, completes the proof.

For a large class of Zeno hybrid automata, the continuous part of the Zeno set X_∞ is located on the intersection of the boundaries of $\text{Inv}(v, \cdot)$ for $v \in V_\infty$. Next this result is stated for hybrid automata with non-expanding jump condition. Recall that Ω^o denotes the interior of a set Ω and $\partial\Omega$ the boundary.

Proposition 3. *Consider a hybrid automaton H with non-expanding jump condition. Assume it accepts a Zeno execution $\chi = (\tau, v, x) \in \mathcal{E}_H^\infty$ with Zeno set $Z_\infty = \{(v_i, z_i\}_{i=1}^N$, $N \geq 1$. If, for all $i \in \{1, \ldots, N\}$ and $y \in \text{Inv}(v_i)^o$, $\text{Jump}(v_i, y) = \emptyset$, then $z_i \in \partial\text{Inv}(v_i)$ for all $i \in \{1, \ldots, N\}$. Furthermore, if there exists $\hat{z} \in \mathbb{R}^n$ such that for all $i \in \{1, \ldots, N\}$, $z_i = \hat{z}$, then $\hat{z} \in \bigcap_{i=1}^N \partial\text{Inv}(v_i)$.*

It follows from Proposition 3 that if the boundaries of $\text{Inv}(\cdot)$ are not intersecting, then there exist no Zeno executions with non-empty Zeno state and $N > 1$. Proposition 3 is thus a refinement of the condition given in Proposition 1, which states that a hybrid automaton is non-Zeno if the graph (V, E) has no cycle.

Background

This lecture is mainly based on Section 4 in [4]. Zeno hybrid automata are also discussed in [1, 2]. Lemma 1 and Proposition 3 are proved in [3].

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