Lecture 3

Deterministic and Non-Blocking Hybrid Automata

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Continuous Dynamical Systems Revisited

Consider the ordinary differential equation

$$\dot{x}(t) = F\big(x(t)\big), \qquad x(0) = x_0. \tag{1}$$

A solution[1] on $[0, T]$, $T > 0$, to (1) is a continuously differentiable function $x : [0, T] \to \mathbb{R}^n$ satisfying

$$x(t) = x_0 + \int_0^t F\big(x(s)\big)\, ds.$$

We may ask for which functions F there exist a solution to (1) and, if so, if the solution is unique. Is it, for instance, sufficient that F is a continuous function? The answer is no concerning the uniqueness issue, as illustrated by the following example:

Example 1. Let $F(z) = \sqrt{z}$ and consider the differential equation (1) for $x_0 = 0$. Then both $x(t) = t^2/4$ and $x(t) = 0$ are solutions.[2]

If we, however, not only assume continuity but also restrict the rate of change of F, we can show some nice properties. The following definition is needed, where $\|\cdot\|$ denotes the Euclidean norm ($\|z\|^2 = z_1^2 + \cdots + z_n^2$): a function F is locally Lipschitz continuous if there exist $r, L > 0$ such that

$$\|F(z) - F(y)\| \le L\|z - y\|$$

for all $\|z\|, \|y\| < r$. If there exists $L > 0$ (independent of r) such that the condition holds for any $r > 0$, then F is globally Lipschitz continuous.

Theorem 1 (Existence and Uniqueness). *If F is locally Lipschitz continuous, then there exists $\delta > 0$ such that (1) has a unique solution on $[0, \delta]$. Moreover, if F is globally Lipschitz continuous, then $\delta < \infty$ can be chosen arbitrarily large.*

[1] This solution is called a solution in the sense of Caratheodory.

[2] Actually, for any $t_0 > 0$, $x(t) = 0$ for $t \in (0, t_0)$ and $x(t) = (t - t_0)^2/4$ for $t \in (t_0, \infty)$ is a solution.

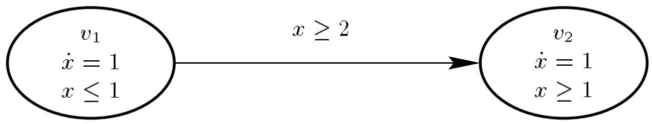

Fig. 1. Blocking hybrid automaton if $\{(v_1, z) : z < 2\} \cap \text{Init} \neq \emptyset$.

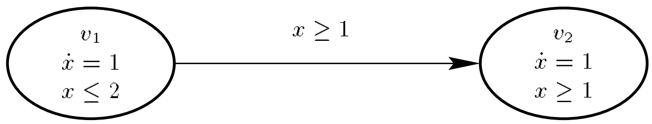

Fig. 2. Non-deterministic hybrid automaton if $\{(v_1, z) : z < 2\} \cap \text{Init} \neq \emptyset$.

Theorem 2 (Continuous Dependence on Initial State). *Assume F is globally Lipschitz continuous. If $x(\cdot)$ and $y(\cdot)$ are two solutions of (1) with initial conditions x_0 and y_0, respectively, then for a given $\epsilon > 0$ there exists $\delta(\epsilon, T) > 0$ such that*

$$\|x_0 - y_0\| \leq \delta \quad \Rightarrow \quad \|x(t) - y(t)\| \leq \epsilon$$

for all $t \in [0, T]$.

Next we will study extensions of these properties to hybrid automata.

Non-Blocking and Deterministic Hybrid Automata

Recall from Lecture 2 that $\mathcal{E}_H(v_0, x_0)$ denotes the set of all executions of H with initial condition $(v_0, x_0) \in \text{Init}$, $\mathcal{E}_H^M(v_0, x_0)$ the set of all maximal executions, and $\mathcal{E}_H^\infty(v_0, x_0)$ the set of all infinite executions.

Definition 1 (Non-Blocking Hybrid Automaton). *A hybrid automaton H is non-blocking if $\mathcal{E}_H^\infty(v_0, x_0)$ is non-empty for all $(v_0, x_0) \in \text{Init}$.*

Figure 1 shows an example of a hybrid automaton, which is blocking if $\{(v_1, z) : z < 2\} \cap \text{Init} \neq \emptyset$.

Definition 2 (Deterministic Hybrid Automaton). *A hybrid automaton H is deterministic if $\mathcal{E}_H^M(v_0, x_0)$ contains at most one element for all $(v_0, x_0) \in \text{Init}$.*

Figure 2 shows an example of a hybrid automaton, which is non-deterministic if $\{(v_1, z) : z < 2\} \cap \text{Init} \neq \emptyset$.

Motivated by previous section on continuous dynamical system, we impose the following standing assumption on the hybrid automaton.

Assumption 1 *The vector field $f(v, \cdot)$ is globally Lipschitz continuous for all $v \in V$.*

The examples in Figures 1 and 2 illustrate that a similar result to Theorem 1 does not hold. Next we will impose conditions that ensure a hybrid automaton to be non-blocking and deterministic. To do so, it is convenient to introduce the following two subsets of the state space.

The set of states reachable by H is denoted

$$\begin{aligned}\mathrm{Reach}_H = \{&(v,z) \in V \times \mathbb{R}^n : \\ &\exists \chi = (\tau, v, x) \in \mathcal{E}_H, \big(v(\tau_N'), x(\tau_N')\big) = (v,z), N < \infty\},\end{aligned}$$

where sometimes the subscript will be dropped. Note that $\mathrm{Reach}_H \supset \mathrm{Init}$, since in the definition we may choose $\tau_N' = \tau_N$ and $N = 0$.

Let $\phi(t,a)$ denote the solution to $\dot{x} = f(v,x)$ for $x(0) = a$. The set of states from which continuous evolution is impossible is then given by

$$\mathrm{Out}_H = \{(v,z) \in V \times \mathbb{R}^n : \forall \epsilon > 0,\ \exists t \in [0,\epsilon),\ \big(v, \phi(t,z)\big) \notin \mathrm{Inv}\}.$$

As usual, we will use $\mathrm{Out}_H(v)$ to denote the projection of Out to discrete state $v \in V$, and drop the subscript H whenever the automaton is clear from the context.

Note that if Inv is an open set, then Out is simply Inv^c. If Inv is closed, then Out may also contain parts of the boundary of Inv as in the following example.

Example 2. Consider the hybrid automaton in Figure 2 with $\mathrm{Init} = \{v_1\} \times \mathbb{R}^n$. Then,

$$\mathrm{Reach} = \{(v_1,z) :\ z \in \mathbb{R}\} \cup \{(v_2,z) :\ z \geq 1\}$$

and

$$\mathrm{Out} = \{(v_1,z) :\ z \geq 2\} \cup \{(v_2,z) :\ z < 1\}.$$

Note that since

$$\mathrm{Inv} = \{(v_1,z) :\ z \leq 2\} \cup \{(v_2,z) :\ z \geq 1\},$$

it holds that

$$\mathrm{Inv}^c = \{(v_1,z) :\ z > 2\} \cup \{(v_2,z) :\ z < 1\} \neq \mathrm{Out}.$$

Theorem 3 (Non-Blocking Hybrid Automaton). *A hybrid automaton H is non-blocking if for all $(v,z) \in \mathrm{Out}_H \cap \mathrm{Reach}_H$, $\mathrm{Jump}(v,z) \neq \emptyset$.*

Proof. Consider an initial state $(v_0,x_0) \in \mathrm{Init}$ and assume, for the sake of contradiction, that there does not exist an infinite execution starting at (v_0,x_0). Let $\chi = (\tau,v,x) \in \mathcal{E}_H^M(v_0,x_0)$ denote a maximal execution starting at (v_0,x_0), and note that τ is a finite sequence. We consider two cases: when τ ends with a right-open interval and when τ ends with a closed interval.

First consider $\tau = \{[\tau_i,\tau_i']\}_{i=0}^{N-1}[\tau_N,\tau_N')$. Let $(v_N,x_N) = \lim_{t\to\tau_N'}(v(t),x(t))$. Note that, by the definition of execution and by Theorem 1, the limit exists and χ can be extended to $\widehat{\chi} = (\widehat{\tau},\widehat{v},\widehat{x})$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N}$, $\widehat{v}(\tau_N') = v_N$, and $\widehat{x}(\tau_N') = x_N$. This contradicts the maximality of χ.

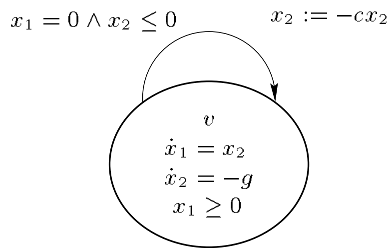

Fig. 3. Hybrid automaton for bouncing ball, where $\text{Init} = \{(v, z) : z_1 \ge 0\}$.

Now consider the case $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$, and let $(v_N, x_N) = (v(\tau_N'), x(\tau_N'))$. Clearly, $(v_N, x_N) \in \text{Reach}_H$. If $(v_N, x_N) \notin \text{Out}_H$, then there exists $\epsilon > 0$ such that χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{v}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N' + \epsilon)$ by continuous evolution. If, on the other hand, $(v_N, x_N) \in \text{Out}_H$, then there exists $(\tilde{v}, \tilde{z}) \in V \times \mathbb{R}^n$ such that $(\tilde{v}, \tilde{z}) \in \text{Jump}(v_N, x_N)$ (by the assumption in the theorem). Therefore, χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{v}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N+1}$, $\tau_{N+1} = \tau_{N+1}' = \tau_N'$, $v(\tau_{N+1}) = \tilde{v}$, $x(\tau_{N+1}) = \tilde{z}$ by a discrete transition. Both when the execution is extended with continuous evolution and with a discrete transition, the maximality of χ is contradicted. This completes the proof.

Example 3. Consider the bouncing ball hybrid automaton again shown in Figure 3. Here

$$\begin{aligned} \text{Out} &= \{(v, z) : z_1 < 0,\} \cup \{(v, z) : z_1 = 0, z_2 \le 0\} \\ \text{Reach} &= \{(v, z) : z_1 \ge 0\}, \end{aligned}$$

so that $\text{Out} \cap \text{Reach} = \{(v, z) : z_1 = 0, z_2 \le 0\}$. By the definition of Jump, we see that for all $(v, z) \in \text{Out} \cap \text{Reach}$, $\text{Jump}(v, z) \neq \emptyset$. Hence, the hybrid automaton is non-blocking.

Proposition 1. *A deterministic hybrid automaton H is non-blocking if and only if for all $(v, z) \in \text{Out}_H \cap \text{Reach}_H$, $\text{Jump}(v, z) \neq \emptyset$.*

Proof. The "if" part follows from Theorem 3. For the "only if" part, consider a deterministic hybrid automaton H and assume there exists $(\tilde{v}, \tilde{z}) \in \text{Out} \cap \text{Reach}$ such that $\text{Jump}(\tilde{v}, \tilde{z}) = \emptyset$. We will show that H then is blocking.

Since $(\tilde{v}, \tilde{z}) \in \text{Reach}$, there exists $(v_0, x_0) \in \text{Init}$ and a finite execution $\chi = (\tau, v, x) \in \mathcal{E}_H(v_0, x_0)$, such that $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ and $(\tilde{v}, \tilde{z}) = (v(\tau_N'), x(\tau_N'))$. Let us show that $\chi \in \mathcal{E}_H^M(v_0, x_0)$, i.e., χ is maximal.

Assume first that there exists $\widehat{\chi} = (\widehat{\tau}, \widehat{v}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N + \epsilon)$ for some $\epsilon > 0$. This would, however, violate the assumption that $(\tilde{v}, \tilde{z}) \in \text{Out}$. Next assume that there exists $\widehat{\chi} = (\widehat{\tau}, \widehat{v}, \widehat{x})$ with $\widehat{\tau} = \tau[\tau_{N+1}, \tau_{N+1}']$ with $\tau_{N+1} = \tau_N'$. This requires that the execution can be extended beyond $(\tilde{v}, \tilde{x})$ by a discrete transition, i.e., that there exists $(\bar{v}, \bar{z}) \in V \times \mathbb{R}^n$ such that $(\bar{v}, \bar{z}) \in \text{Jump}(\tilde{v}, \tilde{z})$.

This would, however, violate the assumption that $\text{Jump}(\tilde{v}, \tilde{z}) = \emptyset$. Hence, we have shown that $\chi \in \mathcal{E}_H^M(v_0, x_0)$.

Now, assume, for the sake of contradiction, that H is non-blocking. Then, there exists $\bar{\chi} \in \mathcal{E}_H^\infty(v_0, x_0)$. But $\mathcal{E}_H^\infty(v_0, x_0) \subset \mathcal{E}_H^M(v_0, x_0)$ and $\chi \neq \chi'$ (as the former is finite and the latter infinite), therefore $\mathcal{E}_H^M(v_0, x_0) \supset \{\chi, \chi'\}$. This contradicts the assumption that H is deterministic. Therefore, H is blocking and the proof is complete.

In the following necessary and sufficient condition for a hybrid automaton to be deterministic, we let $|\cdot|$ denote cardinality.

Theorem 4 (Deterministic Hybrid Automaton). *A hybrid automaton H is deterministic if and only if for all $(v, z) \in \text{Reach}_H$, $|\text{Jump}(v, z)| \leq 1$ and, if $\text{Jump}(v, z) \neq \emptyset$, $(v, z) \in \text{Out}_H$.*

Proof. For the "if" part, assume, for the sake of contradiction, that there exists an initial state $(v_0, x_0) \in \text{Init}$ and two maximal executions $\chi = (\tau, v, x)$ and $\widehat{\chi} = (\widehat{\tau}, \widehat{v}, \widehat{x})$ starting at (v_0, x_0) with $\chi \neq \widehat{\chi}$. Let $\tilde{\chi} = (\tilde{\tau}, \tilde{v}, \tilde{x}) \in \mathcal{E}_H(v_0, x_0)$ denote the maximal common prefix of χ and $\widehat{\chi}$ (see Lecture 2 for the definition of prefix). Such a prefix exists as the executions start at the same initial state. Moreover, $\tilde{\chi}$ is not infinite, as $\chi \neq \widehat{\chi}$. Therefore, as in the proof of Theorem 3, $\tilde{\tau}$ can be assumed to be of the form $\tilde{\tau} = \{[\tilde{\tau}_i, \tilde{\tau}_i']\}_{i=0}^N$. Let $(v_N, x_N) = (v(\tilde{\tau}_N'), x(\tilde{\tau}_N')) = (\widehat{v}(\tilde{\tau}_N'), \widehat{x}(\tilde{\tau}_N'))$. Clearly, $(v_N, x_N) \in \text{Reach}$. We distinguish the following four cases:

Case 1: $\tilde{\tau}_N' \notin \{\tau_i'\}$ and $\tilde{\tau}_N' \notin \{\widehat{\tau}_i'\}$, i.e., $\tilde{\tau}_N'$ is not a time when a discrete transition takes place in either χ or $\widehat{\chi}$. Then, by a standard existence and uniqueness argument for continuous systems, there exists $\epsilon > 0$ such that the prefixes of χ and $\widehat{\chi}$ are defined over $\widehat{\tilde{\tau}} = \{[\tilde{\tau}_i, \tilde{\tau}_i']\}_{i=0}^{N-1}[\tilde{\tau}_N, \tilde{\tau}_N' + \epsilon)$ and are identical. This contradicts the maximality of $\tilde{\chi}$.

Case 2: $\tilde{\tau}_N' \in \{\tau_i'\}$ and $\tilde{\tau}_N' \notin \{\widehat{\tau}_i'\}$, i.e., $\tilde{\tau}_N'$ is a time when a discrete transition takes place in χ but not in $\widehat{\chi}$. The fact that a discrete transition takes place from (v_N, x_N) in χ indicates that there exists $(\bar{v}, \bar{z}) \in V \times \mathbb{R}^n$, such that $(\bar{v}, \bar{z}) \in \text{Jump}(v_N, x_N)$. The fact that no discrete transition takes place from (v_N, x_N) in $\widehat{\chi}$ indicates that there exists $\epsilon > 0$ such that $\widehat{\chi}$ is defined over $\widehat{\tilde{\tau}} = \{[\tilde{\tau}_i, \tilde{\tau}_i']\}_{i=0}^{N-1}[\tilde{\tau}_N, \tilde{\tau}_N' + \epsilon)$. Therefore $(v_N, x_N) \notin \text{Out}$. This contradicts the second condition of the theorem.

Case 3: $\tilde{\tau}_N' \notin \{\tau_i'\}$ and $\tilde{\tau}_N' \in \{\widehat{\tau}_i'\}$, symmetric to 2.

Case 4: $\tilde{\tau}_N' \in \{\tau_i'\}$ and $\tilde{\tau}_N' \in \{\widehat{\tau}_i'\}$, i.e., $\tilde{\tau}_N'$ is a time when a discrete transition takes place in both χ and $\widehat{\chi}$. The fact that a discrete transition takes place from (v_N, x_N) in both χ and $\widehat{\chi}$ indicates that there exist (v, z) and $(\widehat{v}, \widehat{z})$, such that $(v, z) \in \text{Jump}(v_N, x_N)$ and $(\widehat{v}, \widehat{z}) \in \text{Jump}(v_N, x_N)$. By the first condition of the theorem, $(v, z) = (\widehat{v}, \widehat{z})$. Therefore, χ and $\widehat{\chi}$ have identical prefixes defined over $\widehat{\tilde{\tau}} = \{[\tilde{\tau}_i, \tilde{\tau}_i']\}_{i=0}^N[\tilde{\tau}_{N+1}, \tilde{\tau}_{N+1}']$, with $\tilde{\tau}_{N+1} = \tilde{\tau}_{N+1}' = \tilde{\tau}_N'$. This contradicts the maximality of $\tilde{\chi}$.

This concludes the "if" part of the proof, since the assumption $\chi \neq \widehat{\chi}$ in the beginning of the proof thus does not hold.

For the "only if" part, assume that there exists $(\bar{v}, \bar{z}) \in$ Reach such that at least one of the two conditions of the theorem is violated. Since $(\bar{v}, \bar{z}) \in$ Reach, there exists $(v_0, x_0) \in$ Init and a finite execution $\chi = (\tau, v, x) \in \mathcal{E}_H(v_0, x_0)$ such that $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ and $(\bar{v}, \bar{z}) = (v(\tau_N'), x(\tau_N'))$. If the first condition is violated, then there exist $\widehat{\chi} = (\widehat{\tau}, \widehat{v}, \widehat{x})$ and $\tilde{\chi} = (\tilde{\tau}, \tilde{v}, \tilde{x})$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1}, \tau_{N+1}']$, $\tau_{N+1} = \tau_N'$, and $(\widehat{v}(\tau_{N+1}), \widehat{x}(\tau_{N+1})) \neq (\tilde{v}(\tau_{N+1}), \tilde{x}(\tau_{N+1}))$, such that $\chi < \widehat{\chi}$ and $\chi < \tilde{\chi}$. If the second condition is violated, then there exists $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N + \epsilon)$, $\epsilon > 0$, and $\tilde{\tau} = \tau[\tau_{N+1}, \tau_{N+1}']$, $\tau_{N+1} = \tau_N'$, such that $\chi < \widehat{\chi}$ and $\chi < \tilde{\chi}$. In both these cases, let $\overline{\widehat{\chi}} \in \mathcal{E}_H^M(v_0, x_0)$ and $\overline{\tilde{\chi}} \in \mathcal{E}_H^M(v_0, x_0)$ denote maximal executions of which $\widehat{\chi}$ and $\tilde{\chi}$ are prefixes, respectively. Since $\widehat{\chi} \neq \tilde{\chi}$, it follows that $\overline{\widehat{\chi}} \neq \overline{\tilde{\chi}}$. Therefore $|\mathcal{E}_H^M(v_0, x_0)| \geq 2$ and thus H is non-deterministic. This concludes the "only if" part.

Example 4. Consider the bouncing ball hybrid automaton in Figure 3 again. Since $|\text{Jump}(v, z)| \leq 1$ for all $z \in \mathbb{R}^2$, the first condition of Theorem 4 holds. From

$$
\begin{aligned}
&\{(v, z) \in V \times \mathbb{R}^n : \text{Jump}(v, z) = \emptyset\} \\
&\quad = \{(v, z) : z_1 = 0, z_2 \leq 0\} \\
&\quad \subset \{(v, z) : z_1 < 0, \} \cup \{(v, z) : z_1 = 0, z_2 \leq 0\} \\
&\quad = \text{Out},
\end{aligned}
$$

we see that also the second condition holds. Hence, the hybrid automaton is deterministic.

Summarizing Theorems 3 and 4, we get the following result.

Corollary 1 (Existence and Uniqueness). *If a hybrid automaton H satisfies the conditions of Theorems 3 and 4, then it accepts a unique infinite execution for all $(v_0, x_0) \in$ Init.*

Proof. If H satisfies the condition in Theorem 3, then $|\mathcal{E}_H^\infty(v_0, x_0)| \geq 1$. If it satisfies the conditions in Theorem 4, then $|\mathcal{E}_H^M(v_0, x_0)| \leq 1$. But $\mathcal{E}_H^\infty(v_0, x_0) \subset \mathcal{E}_H^M(v_0, x_0)$, so therefore, $1 \leq |\mathcal{E}_H^\infty(v_0, x_0)| \leq |\mathcal{E}_H^M(v_0, x_0)| \leq 1$, which proves the result.

Note that even if a hybrid automaton accepts a unique infinite execution for all initial states, it does not necessarily accept executions with execution time $\tau_\infty = \sum_{i=0}^N (\tau_i' - \tau_i) = \infty$, i.e., the execution is not necessary defined for all times. An example of such an hybrid automaton is the bouncing ball hybrid automaton. This issue will be further discussed in next lecture, where Zeno hybrid automata are introduced.

Continuous Dependence on Initial State

There are other properties than existence and uniqueness that are important for hybrid automata. For example, it is often desirable that a model shows robustness to modeling errors and initial conditions. Unfortunately, there exist few such results. In this section we will only illustrate through a couple of examples that executions of hybrid automata in general do not show continuous dependence on initial states. Continuity is interpreted in the metric

$$d\big((v,z),(\hat v,\hat z)\big) = d_D(v,\hat v) + \|z-\hat z\|,$$

where d_D denotes the discrete metric given by

$$d_D(v,\hat v) = \begin{cases} 0, & \text{if } v=\hat v \\ 1, & \text{if } v\neq \hat v, \end{cases}$$

and $\|\cdot\|$ denotes the Euclidean norm.

Example 5. Consider the hybrid automaton

- $\mathbf{V}=\{v_1,v_2\}$ and $\mathbf{X}=\mathbb{R}^2$;
- $\text{Init} = \{v_1\}\times\mathbb{R}^2$;
- $f(\cdot,\cdot)\equiv(1,0)^T$;
- $\text{Inv} = \{(v_1,x):\ x_1\le 0\}\cup\{(v_2,x):\ x_1\ge 0\}$;
-
$$\text{Jump}\big(v,(x_1,x_2)\big) = \begin{cases} \big(v_2,(x_1,1)\big), & \text{if } v=v_1,\ x_2>0 \\ \big(v_2,(x_1,0)\big), & \text{if } v=v_1,\ x_2\le 0 \\ \emptyset, & \text{otherwise.} \end{cases}$$

The hybrid automaton has a unique infinite execution for every initial state. It shows, however, in general not continuous dependence on the initial state as illustrated next. Consider two executions $\chi=(\tau,v,x)$ and $\hat\chi=(\hat\tau,\hat v,\hat x)$ with initial states $\big(v_1,(0,0)\big)$ and $\big(v_1,(0,\epsilon)\big)$, respectively. For every $\epsilon>0$, $\|x(t)-\hat x(\hat t)\|=1$ for all $t\in[\tau_i,\tau_i']$ and $\hat t\in[\hat\tau_i,\hat\tau_i']$, $i>0$.

The reason for the absence of continuous dependence in the example is due to the discontinuous jump condition.

Example 6. Consider the hybrid automaton

- $\mathbf{V}=\{v_1,v_2,v_3\}$ and $\mathbf{X}=\mathbb{R}^2$;
- $\text{Init} = \{v_1\}\times\mathbb{R}^2$;
-
$$f(v,x) = \begin{cases} (1,0)^T, & \text{if } v=v_1 \\ (1,1)^T, & \text{if } v=v_2 \\ (1,-1)^T, & \text{if } v=v_3; \end{cases}$$
- $\text{Inv} = \{(v_1,x):\ x_1\le 0\}\cup\{(v_2,x):\ x_2\ge 0\}\cup\{(v_3,x):\ x_2\le 0\}$;
-
$$\text{Jump}(v,x) = \begin{cases} (v_2,x), & \text{if } v=v_1,\ x_1\ge 0,\ x_2\ge 0 \\ (v_2,x), & \text{if } v=v_1,\ x_1\ge 0,\ x_2<0\ . \\ \emptyset, & \text{otherwise.} \end{cases}$$

To see that this hybrid automaton in general does not show continuous dependence on the initial state, consider initial states in a neighborhood of $\big(v_1,(0,0)\big)$.

Background

The lecture is mainly based on [2]. For more results on continuous dynamical systems, see [3]. For some work on continuous dependence on initial state, see [5,1,4].

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