Lecture 1

Introduction

Karl Henrik Johansson

The importance, ubiquity, and complexity of embedded systems are growing tremendously thanks to the revolution in digital technology. This has created a need for design techniques that can guarantee safety and performance specifications. Hybrid system theory addresses this problem by providing a mathematical framework for analyzing systems with interacting discrete and continuous dynamics. A hybrid system captures the coupling between the digital computations and the analog physical environment inherent in many of today's real-time systems. The theory of hybrid systems has a large number of applications in areas such as real-time software, embedded systems, robotics, mechatronics, aeronautics, and process control. As an introduction, we present a number of examples from these areas.

Hybrid systems are modeled as hybrid automata, which may be represented as a directed graph with continuous dynamics in each node. The continuous flow evolve according to the differential equation specified in the current node of the graph. When certain conditions are fulfilled, a discrete transition may take place from one node to another if the nodes are connected by an edge. The continuous flow is then forced to satisfy the differential equation in the new node. Depending on the number of discrete states (nodes) and the differential equation in each state, the hybrid automaton may show a more or less complex behavior. The ultimate cases are on one side a hybrid automaton with only one discrete state and no edges and on the other side a hybrid automaton with trivial continuous dynamics ($\dot{x} = 0$) in each discrete state. The first case corresponds to a continuous-time dynamical system and the second to a purely discrete system. The rest of this introduction is devoted to a few examples of hybrid systems.

Examples

Example 1: Bouncing Ball

A simple non-trivial hybrid automaton is the model of a bouncing ball shown to the left in Figure 1. The vertical position of the ball is denoted x_1 and the velocity x_2. The acceleration of gravity is denoted g and $c \in [0, 1]$ is a coefficient of restitution. To the right in the figure, the evolution of the continuous state of the system (x_1, x_2) is shown. As long as the ball is above the ground ($x_1 > 0$), the continuous state flow according to the differential equation specified in the single discrete state of the hybrid automaton. When the transition condition is fulfilled,

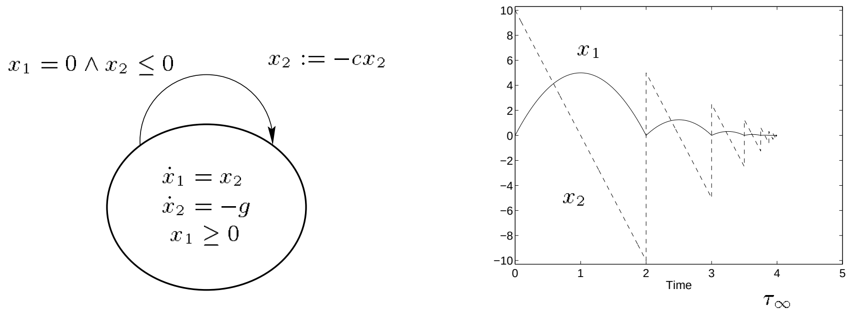

Fig. 1. A hybrid automaton modeling a bouncing ball and an example of an execution.

a discrete jump takes place. This corresponds to that the ball bounces. The speed of the ball is reset at each discrete jump ($x_2 := -cx_2$). The reset models the physics of the bouncing ball as if the ball loses a proportional amount of its energy at each bounce. As is indicated by the simulation in Figure 1, this model leads to that the ball bounces infinitely many times in a finite time interval.

Example 2: Thermostat

Consider the control problem of heating a room. Assume that a thermostat is used as a controller, but that we do not have an exact model of how the thermostat functions. It is only known that the thermostat turns on the radiator when the temperature is between 68 and 70 and it turns the radiator off when the temperature is between 80 and 82. This heating system can be modeled as the hybrid automaton in Figure 2, where x denotes the temperature. This hybrid

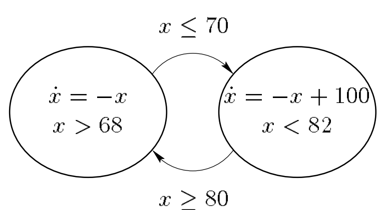

Fig. 2. Hybrid automaton modeling a thermostat and the heating of a room.

automaton is non-deterministic in the sense that for a given initial condition it accepts a whole family of different executions (solutions).

Example 3: Automatic Gear Box Control

Figure 3 shows a model of car with a gear box having four gears. The lateral

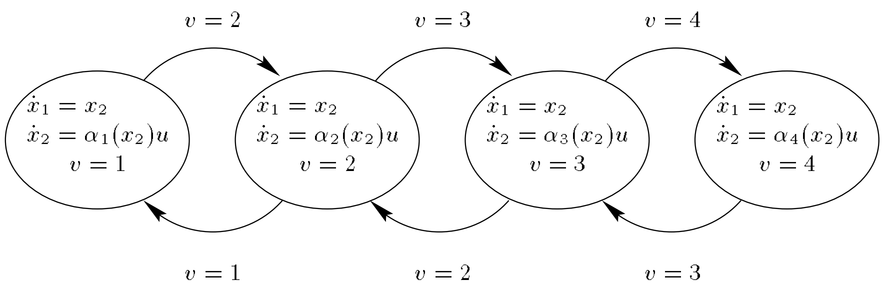

Fig. 3. A hybrid automaton modeling a car with four gears.

position is denoted x_1 and the velocity x_2. The model has two control signals: the gear $v \in \{1, \ldots, 4\}$ and the throttle $u \in [-1, 1]$. The functions α_v, shown in Figure 4, represent the efficiency of the corresponding gear. Several interesting

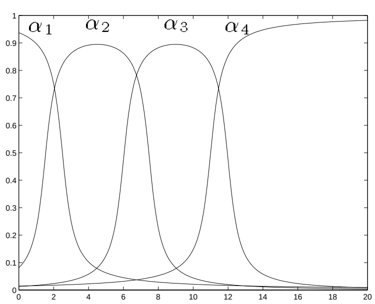

Fig. 4. The function $\alpha_v(x_2)$ represents the efficiency of gear $v \in \{1, \ldots, 4\}$.

control problems may be posed for this car model, including the following: What is the optimal control strategy to drive from $x_1 = a$ to $x_2 = b$ in minimum time? The problem is non-trivial if the reasonable assumption is included that each gear shift takes a certain amount of time. The solution, which will be discussed in another lecture, can also be modeled as a hybrid automaton.

Example 4: Swing-Up of an Inverted Pendulum

A pendulum on a cart is illustrated in Figure 5. It is possible to use a hybrid control strategy to swing-up the pendulum from a downward to an upright position and stabilize it. One strategy based on energy control is shown in Figure 6. Here $f(x) = (x_2, g \sin x_1 - u \cos x_1)^T$ and $z(x) = k[x_2^2/2 + g(\cos x_1 - 1)] \operatorname{sgn}(x_2 \sin x_1)$, where x_1 is the angle between the vertical and the pendulum, x_2 is the angular velocity, the control u is the acceleration of the pivot, g is the acceleration of

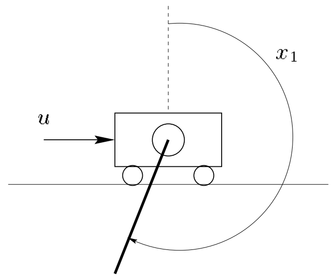

Fig. 5. Pendulum on a cart.

gravity, and $k > 0$ is a design parameter. The maximum acceleration is given by $u_{\max} = \max |u| = rg$, where $r > 0$ is a constant depending on the driving motor. The hybrid automaton consists of three discrete states: maximum acceleration to the left ($u = -u_{\max}$), maximum acceleration to the right ($u = u_{\max}$), and feedback stabilization. The first two swing-up the pendulum and the latter stabilize it around the upright position. Due to that the acceleration of the pivot is

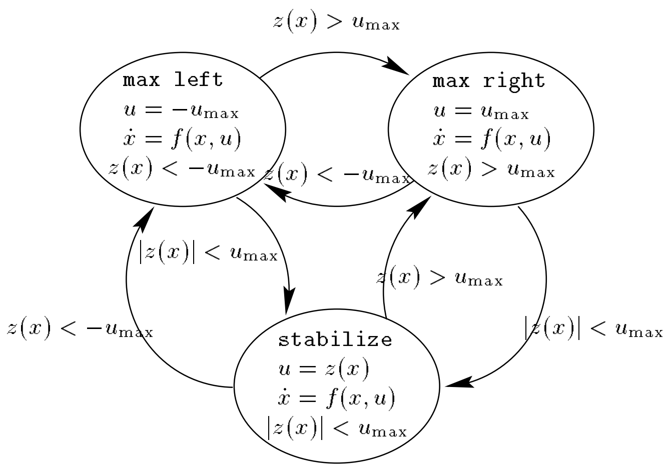

Fig. 6. Hybrid control strategy to swing up and stabilize an inverted pendulum on a cart.

limited to $u_{\max}$, more than one swing may be needed to swing-up the pendulum. Figure 7 shows an example where $u_{\max} = rg = 0.25g$, which leads to a swing-up with five swings.

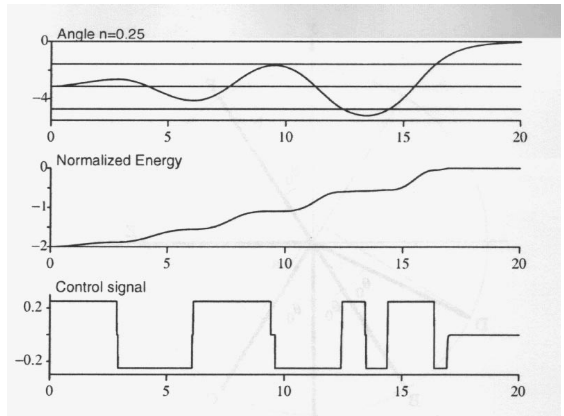

Fig. 7. Swing-up of an inverted pendulum on a cart. The angle x_1 is shown in the upper plot and the control signal u in the lower. The middle plot shows the energy of the pendulum; the hybrid control strategy is designed such that it drives the energy to zero.

Example 5: Computer-Controlled System

Most control systems today involve a computer. The block diagram in Figure 8 illustrates the standard set-up for a computer-controlled physical plant. If we assume the computer (together with the A/D and D/A converters) can be modeled as a finite state machine and the plant as a continuous dynamical system, then it is obvious that the closed-loop system is naturally represented by a hybrid automaton.

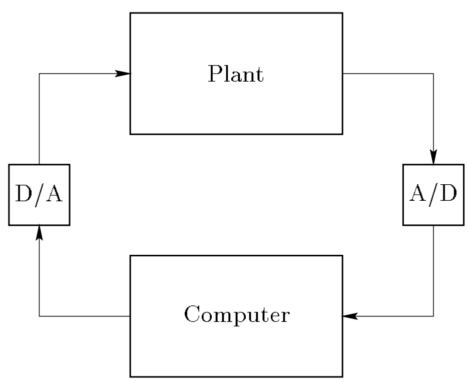

Fig. 8. Computer-controlled system.

Example 6: Supervisory Controller

Figure 9 shows a block diagram for a supervisory controller. A decision maker chooses between m controllers, by for instance using some sort of adaptive updating mechanism. The controllers and the plant are continuous dynamical systems, so the switched control system can be represented by a hybrid automaton with m discrete states.

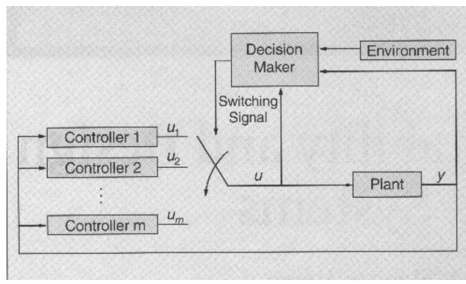

Fig. 9. In a switched controller the current control law is determined by a decision maker, which may take into account the current state of the process as well as other variables.

Example 7: Rocking Block

Figure 10 shows a block rocking due to seismic movements. The angular displace-

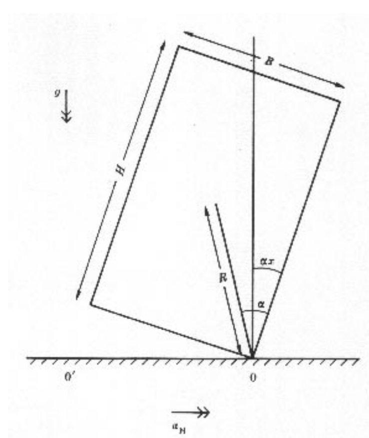

Fig. 10. Rocking block.

ment is denoted αx_1 ($= \alpha x$ in the figure), where $\tan\alpha = B/H$. Here $x_1 > 0$ corresponds to rotation about the point 0 and $x_1 < 0$ to rotation about $0'$. The angular velocity is denoted x_2. Assuming that the seismic movement is sinusoidal, the block is slender ($\alpha \ll 1$), and that at impact the energy loss is represented by a coefficient of restitution $c \in [0,1]$ ($x_2 := cx_2$), then the hybrid automaton in Figure 11 models the rocking block. Here $\beta > 0$ depends on the

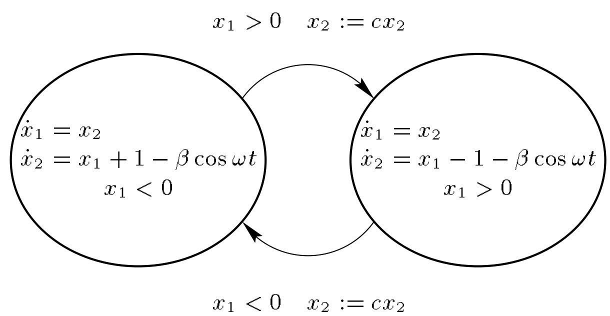

Fig. 11. Hybrid automaton modeling rocking block.

forcing amplitude and ω is its non-dimensional frequency.

The first return map of the attractor is shown in Figure 12. Here $T_k = (t_k - t_{k-1})2\pi/\omega$ is the (normalized) time between the $k-1$th and the kth impact. The coefficients are chosen as $\alpha = 0.001$, $c = 0.5$, $\beta = 4.6$, and $\omega = 2.7$.

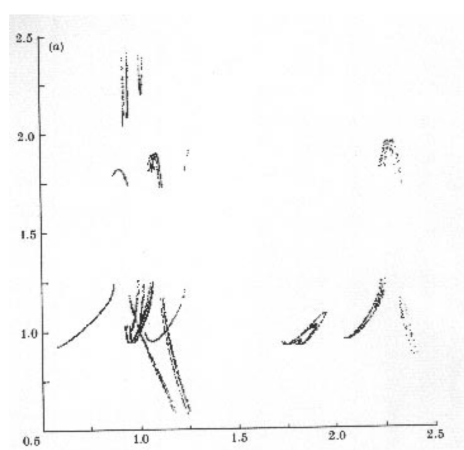

Fig. 12. First return map of the attractor, where $\Delta T_k = (t_k - t_{k-1})2\pi/\omega$ is the time between the $k-1$th and the kth impact divided by the period of the drive.

The rocking block system illustrates that low order hybrid systems may show complex behaviors.

Example 8: DC/DC Converter

Another hybrid system that show interesting dynamical behavior is the DC/DC buck converter, whose circuit diagram is shown in Figure 13. The constant input voltage is E and the output voltage v. The ramp generator gives a saw tooth signal v_r. It is easy to specify a hybrid automaton that models the DC/DC buck

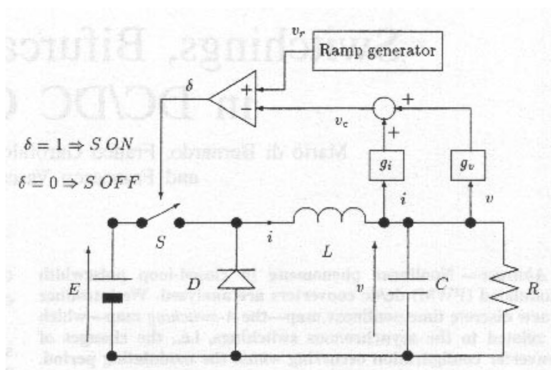

Fig. 13. Circuit diagram for a DC/DC buck converter and the corresponding hybrid automaton.

converter. It has two discrete states with linear continuous dynamics in each state. The left plot in Figure 14 shows a bifurcation diagram for when the input voltage E is varied and the output voltage v is measured at the beginning of each ramp cycle. The right plot shows the attractor for $E = 35$ in the v–i plane, where again v and i are measured at the beginning of each ramp cycle. Although the continuous dynamics is linear and only of second order, the hybrid system seems to show chaotic behavior.

Example 9: Hybrid Control of Helicopter

Many complex control tasks can be divided into discrete and continuous actions. One such problem is a helicopter search and rescue mission, which is illustrated in Figure 15. The overall solution is described by the discrete states of the hybrid automaton. In each discrete state, however, a quite involved continuous control problem is solved, for example, stabilizing the helicopter in the hovering mode during search.

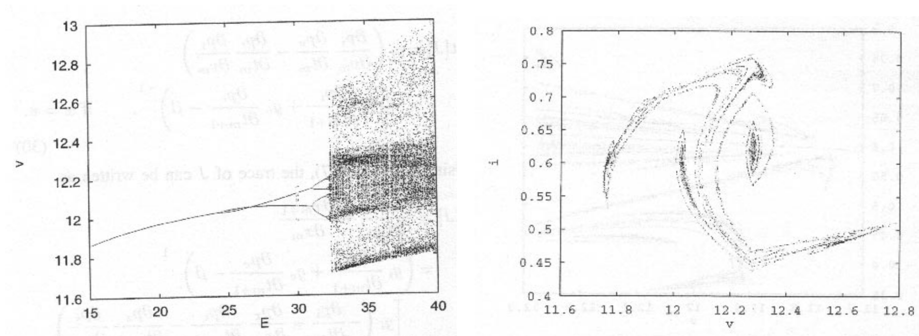

Fig. 14. Example of a bifurcation diagram of the output voltage by considering the stroboscopic map is shown to the left. To the right, a seemingly chaotic attractor is shown.

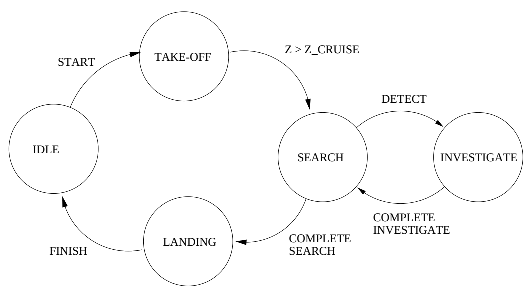

Fig. 15. Search and rescue mission for a helicopter modeled as a hybrid automaton.

Example 10: Optimal Path Tracking

Consider the unicycle sketched in Figure 16 and described by the equations

$$\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\theta} &= \omega, \end{aligned}$$

where (x, y) defines the position of the vehicle and θ its heading. The velocity $v > 0$ is assumed to be constant. The angular velocity ω is the control signal. The problem to track a straight line from an arbitrary initial position in minimum time can be solved using Pontryagin's Maximum Principle. Figure 17 shows the solution presented as a hybrid automaton. The optimal solution consists of only three different actions `turn left`, `turn right`, and `go straight`. This means that the optimal path is surprisingly simple and consists of only circle segments and straight lines. The conditions for the discrete transitions in the hybrid automaton can be given explicitly, but are a bit too complicated to be presented here.

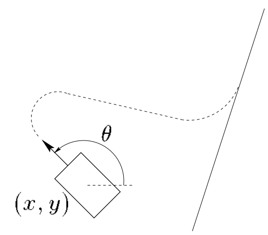

Fig. 16. Unicycle driving on minimum time to a wall.

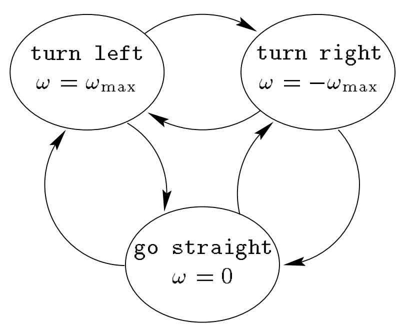

Fig. 17. Hybrid automaton modeling optimal path tracking for unicycle.

Background

Bouncing balls and similar mechanical systems are classical in the field of impact mechanics. In the context of hybrid automata, the bouncing ball example discussed here is for example studied in [5]. The automatic gear box example is adopted from [3], where the time optimal control problem is solved using linear programming. The swing-up strategy for the inverted pendulum is presented in [1]. It is based on controlling the energy of the pendulum. Supervisory control is for instance discussed in [7]. Rocking block dynamics is analyzed in [4]. It is mentioned that before the development of seismographs, peak accelerations in earthquakes were measured in Japan by observations of tombstone collapse. The unpredictable behavior illustrated in Figure 12 suggests that this method is not necessarily reliable. The DC/DC buck converter example is taken from [2]. Hybrid control of helicopters is discussed in [6]. The optimal route tracking problem is solved in [8], where it is also pointed out that the solution can be modeled as a hybrid automaton.

References

1. K. J. Åström and K. Furuta. Swinging up a pendulum by energy control. *Automatica*, 1999.
2. M. di Bernardo, F. Garofalo, L. Glielmo, and F. Vasca. Switchings, bifurcations, and chaos in DC/DC converters. *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, 45(2):133–141, 1998.
3. S. Hedlund and A. Rantzer. Optimal control of hybrid systems. In *Proceedings of IEEE Conference of Decision and Control*, December 1999.
4. S. J. Hogan. On the dynamics of rigid-block motion under harmonic forcing. *Proc. R. Soc. Lond. A*, 425:441–476, 1989.
5. K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry. On the regularization of Zeno hybrid automata. *System & Control Letters*, 38:141–150, 1999.
6. T. J. Koo, F. Hoffmann, B. Sinopoli, and S. Sastry. Hybrid control of an autonomous helicopter. In *Proceedings of IFAC Workshop on Motion Control*, Grenoble, France, 1998.
7. A. S. Morse. Supervisory control of families of linear set-point controllers, part i: Exact matching. *IEEE Transactions on Automatic Control*, 41:1413–1431, 1996.
8. P. Souères, A Balluchi, and A Bicchi. Optimal feedback conrol for route tracking with a bounded-curvature vehicle. Manuscript, 1999.