Introduction to higher mathematics

Jonas Sjöstrand

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Higher mathematics is more fun!

Technology studies involve lots of mathematics and many students experience a jump in difficulty from upper secondary school to the university. This text aims to relieve students from that feeling and to inspire to mathematical curiosity.

It is not merely a repetition of earlier math courses; rather it mainly contains material that is usually not covered properly either at upper secondary school or at the university. Thus, in order to get the most out of this text, you ought to remember basic mathematics from earlier courses quite well. If you need a repetition, the web page

`http://wiki.math.se/wikis/2009/bridgecourse1-ImperialCollege`

is a good place to start.

Innehåll

1 The language of mathematics

1.1 Symbols

We will practise writing, reading and speaking mathematically. In a math book, it may look like this:

$$\frac{4}{3}\pi R^3 \leq 8 \Rightarrow R < \sqrt[3]{2}$$

And this is how it is pronounced: $\Rightarrow$

Four thirds pi R cubed less than or equal to eight implies that R is less than the cube root of two.

This is a mathematical statement expressed first with symbols and then with English words. That it is a statement is evident from the symbol $\Rightarrow$. The arrow states that if the left side is true then so is the right side. It is pronounced *implies that*.

The left side starts with an expression with digits, letters, multiplication and division. The letter π always refers to the number $3.1415926535\ldots$ and π the letter R also refers to a number. The complete expression was written down 2200 years ago by Archimedes, perhaps the greatest mathematician through all times, and it is a formula for the volume of a sphere of radius R. By R^3 we mean $R \cdot R \cdot R$. The left side asserts that the volume of the sphere is at most 8 units of volume.

The right side states that the radius R is less than $1.2599\ldots$. That number is called the cube root of two since $1.2599\ldots \cdot 1.2599\ldots \cdot 1.2599\ldots = 2$.

If the volume of the sphere is at most 8 then its radius is less than $1.2599\ldots$ – this is the complete statement and it is actually true!

Here is Archimedes (to the left) and his tutor Euclid (to the right).

It is important to be able to translate between symbols and English, so here are some examples to practise on.

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

a plus b, squared, equals a squared plus two a b plus b squared.

$$\alpha^2 < 17 \Leftrightarrow -\sqrt{17} < \alpha < \sqrt{17}$$

$\Leftrightarrow$ *Alpha squared is less than seventeen if and only if alpha is between minus the square root of seventeen and plus the square root of seventeen.*

The double arrow is pronounced *if and only if.* Double inequalities are expressed most easily with *between.* The Greek alphabet is used by all mathematicians on Earth, so you will have to learn it now. But first some more symbols.

$$1! = 1,\ 2! = 2,\ 3! = 6,\ n! = n \cdot (n-1) \cdot (n-2) \cdots 1$$

$n!$ *One factorial is one, two factorial is two, three factorial is six, n factorial is n times n minus one times n minus two and so on down to one.*

You should know that $n!$ (Swe. *n-fakultet*) means the number of permutations of n objects. For instance, $3! = 3 \cdot 2 \cdot 1 = 6$ and indeed there are 6 orderings of three objects: $abc, acb, bac, bca, cab, cba$. How many orderings are there of ten objects? Answer: $10! = 3\,628\,800$.

Evidently, the expression $n!$ grows very rapidly. Exactly how rapidly it grows is given by the famous *Stirling's formula*, which we will look at now.

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{2}$$

$\approx$ *n factorial is approximately equal to the square root of two pi n times n over e to the power of n*

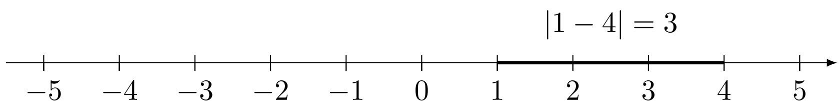
Precisely what *approximately equal to* means must be defined in the text. For 10! Stirling's formula yields the approximate value 3 598 696 with an error of less than one percent, and the larger n is, the smaller the relative error will be.

The letter e almost always refers to the number $2.718281828459045\ldots$ which, along with π, is the most important mathematical constant. It is surely remarkable that these decimal numbers are needed in a formula for $n!$, which is an integer! (You do not have to learn Stirling's formula yet.) e

$$|-\pi| = \pi,\ |2-3| = 1,\ |17-17| = 0$$

The absolute value of minus pi is pi, the distance between two and three is one, the distance between seventeen and seventeen is zero. $|x|$

The absolute value means that the minus sign is removed if there is any. It follows that the absolute value of a number minus another number is the distance between the numbers on the number line, no matter in which order the numbers are written.



In mathematics, as always, three dots $\cdots$ means that the reader must figure out what has been omitted.

$$1+3+5+\cdots+19 = 100, \quad 1+3+5+\cdots+n = \left(\frac{n+1}{2}\right)^2 \qquad (3)$$

One plus three plus five etc. up to nineteen equals 100. One plus three plus five etc. up to n is equal to n plus one half to the power of two. $\cdots$

When dot dot dot is not clear enough, you use the summation symbol, which is the capital sigma of the Greek alphabet. $\sum_{i=1}^{k}$

$$\sum_{i=1}^{k}(2i-1) = k^2 \qquad (4)$$

The sum of all numbers of the form two i minus 1 where i goes from one to k is equal to k squared.

Analogously, when many factors are multiplied you can use the product symbol, which is capital pi.

$$n! = n \cdot (n-1) \cdots 2 \cdot 1 = \prod_{i=1}^{n} i \tag{5}$$

$\prod_{i=1}^{n}$ *n factorial is the product of all numbers i from 1 to n*

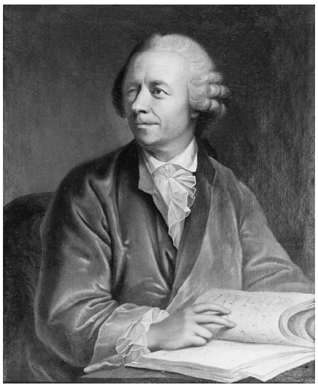
A famous formula where the product is not taken over all integers but only over all prime numbers is *Euler's product formula.* It was proved 1737 by Leonard Euler, the most productive mathematician of all times. He had thirteen children, by the way.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{\text{prime}\, p} \frac{1}{1 - 1/p^2} \tag{6}$$

∞ *The sum of one over n squared for all n from one to infinity is equal to the product of all one over one minus one over p squared for all primes p*

Here we come across a well-known symbol, ∞, showing that the sum has infinitely many terms. What about the product? Are there infinitely many primes? This question was answered already 2300 years ago by *Euclid* and later on you will repeat his achievement!

Here are Euler and the Greek alphabet:



α	alpha	ι	iota	ρ	rho
β	beta	κ	kappa	σ	sigma
γ	gamma	λ	lambda	τ	tau
δ	delta	μ	mu	υ	upsilon
ε	epsilon	ν	nu	φ	phi
ζ	zeta	ξ	xi	o	omikron
η	eta	o	omikron	ψ	psi
θ	theta	π	pi	ω	omega

Exercises

1.1: Tell something about $\alpha\rho\chi\iota\mu\eta\delta\eta\sigma$.

1.2: First read aloud and then try to understand what you just said.

$$\cos\varphi = 0 \Leftrightarrow \varphi = \pm\frac{\pi}{2} + 2n\pi \text{ for some integer } n$$

1.3: Read aloud, understand the statement, choose a pair of numbers ρ and σ and check whether the statement holds. Then, try to find a pair of numbers such that the statement is false.

$$\rho^2 + \sigma^2 + 2\rho\sigma \geq 2\rho + 2\sigma$$

1.4: Read aloud, understand, test with $x = -10$ and then justify the claim.

$$|x| = \sqrt{x^2}$$

1.5: Read aloud, understand, test with $n = 3$ and justify the claim. What does it say for $n = 0$?

$$(n+1)! = (n+1) \cdot n!$$

1.6: If $n = 0$ the product in (5) has no factors at all: $\prod_{i=1}^{0} i$. How should such a product be interpreted, with the previous exercise in mind?

1.7: In (3) and (4) an equation is written in three different ways. What should k and n be in order to make the three different equations mean the same thing? Does the equality hold for nineteen?

1.8: The sum and the product in Euler's formula have the same value, namely $\pi^2/6$. How much is this, roughly? If we keep only three terms in the sum, clearly we get a smaller value. Check that! Keeping only two factors in the product also results in a value that is too small. Check that too!

1.9: As we have written Euler's formula there is a two on both sides of the equality sign. Euler showed that the formula remains true if the twos are replaced by an arbitrary number greater than 1. Try to formulate this more general Euler's formula!

1.10: (Toughy) We saw that 10! is a seven-digit number. Use Stirling's formula to approximate the number of digits in 27!.

1.2 Inequalities

Equalities and inequalities are statements that can be true or false. Which one of the following statements is true?

$$2 \cdot 1 \cdot 0 < 3 \cdot 2 \cdot 1 \cdot 0$$

$$17x \leq 18x \text{ for any } x$$

$$|\lambda| \cdot |\mu| = |\lambda\mu| \text{ for any } \lambda \text{ and } \mu$$

$$\pi = 3.14$$

The first statement is false since both sides are zero. The second statement is true for positive x but false for $x = -1$. The last statement is false since π is only approximately equal to 3.14. But the third statement is true. The absolute values remove all minus signs and it does not matter whether this is done before or after the multiplication.

How can it be expressed that x lies at most one unit from the number 17 on the number line? We write either $16 \leq x \leq 18$ or $|x - 17| \leq 1$.

If one litre of mercury is heavier than one litre of water, then naturally two litres of mercury is heavier than two litres of water. In a formula this can be expressed like this:

$$x > y \Rightarrow 2x > 2y \quad \text{and more generally } x > y \Rightarrow ax > ay \text{ for any positive } a$$

But what if a is negative? We have that $13 > 1$ but $-26 < -2$. In general it holds that **multiplication by a negative number changes the direction of the inequality**. This is a trap when dealing with inqualities and there is a similar one for absolute values.

Example

Does $x < y$ imply that $|x| < |y|$ or is it the other way around?

If the numbers are positive the implication is obvious. If x is a large negative number the first inequality can be true while the second one is false. And if y is a large negative number the second inequality can be true while the first one is false, so no implication arrow holds.

Example

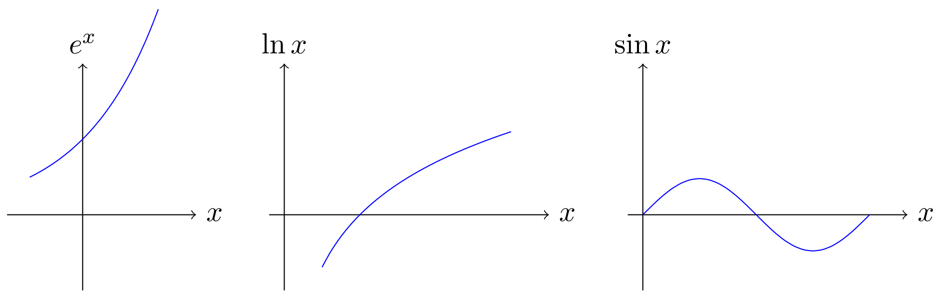
For which x does it hold that $5x + 2 \geq 2x + 5$?

We collect all x-terms on one side and constant terms on the other side. $5x - 2x \geq 5 - 2$, thus $3x \geq 3$. Here we can multiply by one third and obtain the answer $x \geq 1$.

That is fine, but if we had collected the x-terms on the right-hand side in stead we would have got $-3 \geq -3x$. Here we can multiply by $-1/3$ and obtain a sole x on the right-hand side. But since we multiply by a negative number the direction of the inequality changes and we get $1 \leq x$.

If $a = b$ of course $e^a = e^b$ and $\ln a = \ln b$ and $\sin a = \sin b$. But if $a < b$, is $e^a < e^b$ and $\ln a < \ln b$ and $\sin a < \sin b$? To answer this question we must look at the graphs of the functions.

The exponential and the logarithmic curves are growing, that is, uphill to the right. This makes it easy to see that $e^a < e^b$ och $\ln a < \ln b$. But the sine curve is alternately uphill and downhill, so $\sin a < \sin b$ does not always hold.



In a triangle, two sides together are longer than the third side. That inequality holds for triangles in the plane but we can imagine a triangle on the number line too. If the three corners are $0, x, y$ the side lengths are $|x|, |y|, |x - y|$ and the inequality tells us that triangle inequality

$$|x| + |y| \geq |x - y| \quad \text{for any } x \text{ and } y$$

The triangle inequality is equally true if the minus of the right-hand side is replaced by a plus, because changing the sign of y does not change the left-hand side.

positive square

An even more useful inequality is that $x^2 \geq 0$. "Squares are positive", you would like to say, but it is not true. "Squares are nonnegative" would be the correct thing to say.

Example

Prove that $a^2 + 2ab + 3b^2 \geq 0$ for any a and b.

One solution is based on the rule for squaring a binomial (1).

$$a^2 + 2ab + 3b^2 = a^2 + 2ab + b^2 + b^2 + b^2 = (a + b)^2 + b^2 + b^2$$

Now we have a sum of three squares and since all are nonnegative their sum is nonnegative.

Exercises

1.11: Write an inequality that holds for any x.

1.12: Write an inequality that holds for any x except $x = 0$.

1.13: Write an inequality that holds for any x except $x = 1$.

1.14: For which x does it hold that $5x - 7 > 2x + 5$?

1.15: For which x does it hold that $5x^2 - 7 \geq 2x^2 + 5$?

1.16: You know that $a < b$ and you are perfectly positive that $1 < 2$. Multiplication side by side yields $1 \cdot a < 2 \cdot b$. But this an illegal thing to do! Try to find two numbers such that $a < b$ is true but $a < 2b$ is false.

1.17: According to the text it is easy to see that $e^a < e^b$ if $a < b$. When you see that phrase in a math book you ought to be suspicious! Often it means that the mathematician has not found an easy argument for the assertion. You better do it now yourself. You may use that the exponential function is growing.

1.18: It works the other way around too. If $e^a < e^b$ then $a < b$. Justify that claim! Which implication arrow should there be between the inequalities? If you denote e^a by α and e^b by another convenient Greek letter, you will be able to express the result with logarithms!

1.19: (Toughy) For which x does it hold that $x^2 - 3x + 2 < 0$?

1.3 Set theory

Mathematical statements often look like this:

$$\sin n\pi = 0 \text{ for any integer } n$$

$$e^x > 0 \text{ for any real } x$$

With the notation $\mathbb{Z}$ for the set of integers and $\mathbb{R}$ for the set of real numbers, it can be written more compactly: $\mathbb{Z}, \mathbb{R}, \in$

$$\sin n\pi = 0 \text{ for any } n \in \mathbb{Z}$$

$$e^x > 0 \text{ for any } x \in \mathbb{R}$$

It can get even shorter with the symbol $\forall$ (for any): $\forall, \exists$

$$\sin n\pi = 0, \quad \forall n \in \mathbb{Z}$$

Sine n pi vanishes for any integer n.

$$\forall x \in \mathbb{R}, \quad e^x > 0$$

For any real x, e to the power of x is positive.

The $\forall$-part can be put at the beginning or at the end, whichever seems best in the given situation.

Together with $\forall x$ (for any x) you will often encounter $\exists y$ (exists a y) in statements of the following type.

$$\forall x > 0 \;\; \exists y \quad y^2 = x$$

For any positive number x there is a y whose square equals x.

$\mathbb{N}, \{\ldots | \ldots\}$ Other sets can be defined by set brackets, like this:[1]

$$\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}, \quad \text{the natural numbers} \tag{7}$$

$\varnothing$

$$\varnothing = \{\}, \quad \text{the empty set} \tag{8}$$

Intervals on the real axis can be either open or closed at the ends.

$$[0, 17] = \{x \in \mathbb{R} \mid 0 \leq x \leq 17\} \tag{9}$$

$[a, b]$ *The closed interval of real numbers between 0 and 17, boundary values included.*

$$(7, \infty) = \{x \in \mathbb{R} \mid 7 < x < \infty\} \tag{10}$$

(a, b) *The open interval of real numbers greater than seven.*

For finite sets the elements can be specified one by one.

$$A = \{1, \pi, 17\}, \qquad \varnothing = \{\}, \quad \text{the empty set} \tag{11}$$

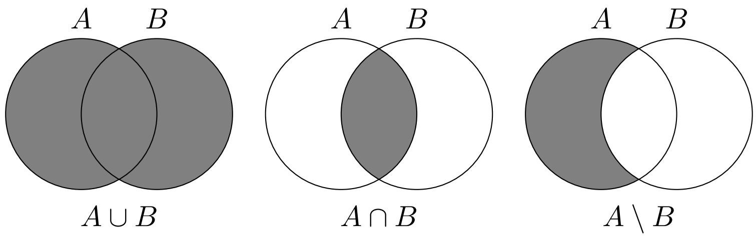
The most common set operations are *union*, *intersection* (Swe. snitt) and *difference*.

$A \cup B$ $$[0, 17] \cup (7, \infty) = [0, \infty)$$

$A \cap B$ $$[0, 17] \cap (7, \infty) = (7, 17]$$

$A \setminus B$ $$[0, 17] \setminus (7, \infty) = [0, 7]$$

Venn diagram With so called *Venn diagrams* the set operations can be visualized:



[1] A colon is sometimes used in stead of the vertical bar.

Natural numbers are a subset of the integers which in turn are a subset of the real numbers. $A \subset B$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \tag{12}$$

To more sets of numbers can be included in the chain. The *rational numbers* $\mathbb{Q}$ consists of all fractions, like $2/3$ and $-2014/17$. The *complex numbers* $\mathbb{C}$ have a real part and an imaginary part, as in $3-4i$ or $\pi+0.5i$. The imaginary unit i has the property that $i \cdot i = -1$ and that is all you need to know in this course. We will mostly stick to the real axis $\mathbb{R}$.

$\mathbb{Q}, \mathbb{C}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \tag{13}$$

But how is the *xy-plane* denoted, the plane we draw curves upon? Since each point (x, y) is specified by two real numbers, we can think of the plane as the set of pair of real numbers, and that set is denoted by $\mathbb{R} \times \mathbb{R}$. The first $\mathbb{R}$ is the x-axis and the second $\mathbb{R}$ is the y-axis. As a shorthand we may write $\mathbb{R}^2$. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

Exercises

1.20: What is claimed here? Is it true?

$$\forall x \in [1, 17], \ |x - 9| \leq 8$$

1.21: Use set brackets to define T, the set of natural numbers with two digits.

1.22: Describe in words the following set.

$$\mathbb{Z} \setminus \mathbb{N}$$

1.23: Write as an interval the set

$$\{x \mid |x - 4.2| < 0.8\}.$$

1.24: Find two sets A and B such that $A \cup B = \mathbb{R}$ and $A \cap B = \mathbb{Z}$.

1.25: Use set brackets to define J, the set of even natural numbers, and U, the set of odd natural numbers.

1.26: Write with symbols the following statement. *The square of an odd number is odd.* Hint: Use the previous exercise.

1.27: Write with symbols the following statement. *If the square of a natural number n is even, then n is even.*

1.28: (Toughy) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and is thus said to be *countable* or *enumerable*. $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ is also countable. Show that $\mathbb{Q}$ is countable too!

1.4 Modelling

Mathematics is a theoretical science, but strangely enough it is very useful for describing reality. Such a description is called a *mathematical model*. The physicist's goal is to construct a model for everything, the mathematician is then given the task to study the properties of the model, discover relations and solve equations.

Example

What is the weight of the air inside a football?

1. First choose denotations, one letter for each quantity.
   d (m), the radius of the football is 0.11
   V (m^3), the volume of the football
   ρ ($\mathrm{kg/m}^3$), the density of air is 1.2
   m (kg), the mass of the air

2. Write down relevant equations, often general laws.

   $V = \frac{4}{3}\pi r^3$ (the volume of a sphere according to Archimedes)

   $\rho = m/V$ (the definition of density)

3. Solve for the wanted quantity.

$$m = \rho \cdot \frac{4}{3}\pi r^3 \tag{14}$$

4. Plug in given values and do the calculation.

   $$m = 1.2 \cdot \frac{4}{3}\pi \cdot 0.11^3 = 0.0067$$

   Thus, the air inside the football weighs 6.7 grams.

The obtained result gives the truth in the model, not necessarily in reality. In reality, the air inside a football weighs approximately twice the value we obtained. The reason is that the air inside a football differs in one respect from the air the players breathe. Can you think of how?

It is vital that numerical values are not plugged in until the last step. By then we have a formula that is valid for any input, at least if we stick to SI units.

Example

The water boiler is broken! How long will it take to heat one litre of tea water with a ten-watt lamp?

1. First choose denotations, one letter for each quantity.
   V (m^3), the volume of the water is 0.001
   ρ (kg/m^3), the density of water is 1000
   c_p (J/kg·K), the specific heat capacity for water is 4200
   T (K), the increase in temperature is 80
   W (J), amount of energy
   P (J/s), the power is 10
   t (s), the heating time

2. Write down relevant equations.

   $$W = T \cdot c_p \cdot \rho \cdot V \quad \text{(needed energy)}$$
   $$W = P \cdot t \quad \text{(added energy)}$$

3. Solve for the wanted quantity.

   $$t = T \cdot c_p \cdot \rho \cdot V/P \tag{15}$$

4. Plug in given values and do the calculation.

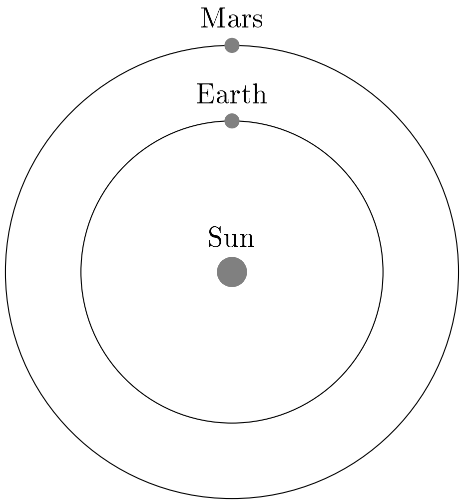
   $$t = 80 \cdot 4200 \cdot 1000 \cdot 0.001/10 = 33600$$

   It will take 33600 seconds, that is, nine hours and twenty minutes.

Relations in physics is most often like (15), a variable is *proportional* to some variables and *inversely proportional* to some others, or, as in (14), to some power of a variable. Often this information is enough to make it possible to draw interesting conclusions.

Example

The Earth or Mars, which one has the largest speed around the Sun?



1. First choose denotations, one letter for each quantity.
   r (m), the planet's distance to the Sun
   v (m/s), the planet's speed
   F (N), the Sun's gravitational force on the planet
   m (kg), the planet's mass
   a (m/s^2), the planet's acceleration

2. Write down relevant equations, often general laws.

$$F = m \cdot a \quad \text{(Newton's second law of motion)}$$
$$F = \text{constant} \cdot m/r^2 \quad \text{(gravity according to Newton)}$$
$$a = v^2/r \quad \text{(circular motion according to Newton)}$$

3. Solve for the wanted variable.

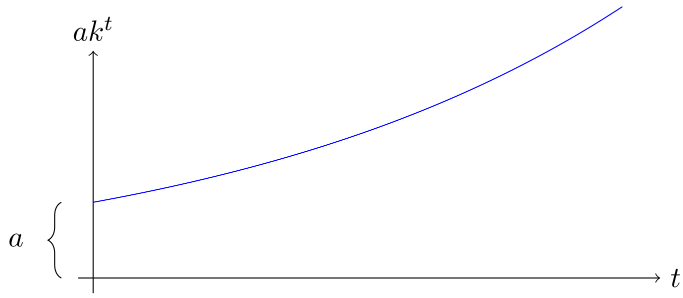
$$v^2 = r \cdot a = r \cdot F/m = r \cdot \text{constant}/r^2 = \text{constant}/r \tag{16}$$
$$v = \text{constant}/\sqrt{r} \tag{17}$$

4. Plug in given values and do the calculation.

   $r_M > r_J \;\Rightarrow\; v_M < v_J$

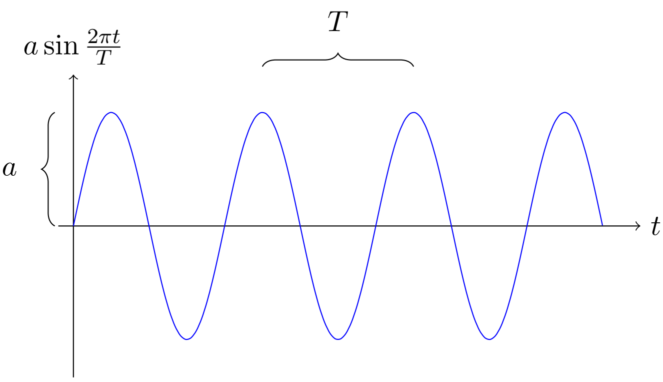
For phenomena depending on time t there are two especially common models. One is **exponential growth**, which means that something grows by a factor k each unit of time. If the initial value is a the formula looks like this:

$$a \cdot k^t \quad \textit{exponential growth} \tag{18}$$



The other is **periodic variation**, which means that something oscillates around a central value. If the value varies from a to $-a$ and back to a in the time period T, the formula is as follows.

$$a \sin \frac{2\pi t}{T} \quad \textit{periodic variation} \tag{19}$$



Here, $2\pi t/T$ should be interpreted in radians, and this is always the case unless the text explicitly says otherwise.

Example

Lily pads cover 1 m^2 of a lake of size 1 km^2. The covered area is doubled every day (24 hours). When is the lake completely covered?

1. First choose denotations, one letter for each quantity.
   a (m^2), covered area, initial value $= 1$
   A (m^2), covered area, final value $= 1\,000\,000$
   k, growth factor $= 2$
   t, number of days

2. Write down relevant equations, often general laws.

   $$A = a \cdot k^t$$

3. Solve for the wanted quantity.
   It is hard, but it is possible if we take the logarithm of both sides.

   $$\ln A = \ln a + t \ln k \tag{20}$$

   $$t = \frac{\ln A - \ln a}{\ln k} \tag{21}$$

4. Plug in given values and do the calculation.

   $$t = \frac{\ln 1\,000\,000}{\ln 2} \approx 20$$

If you know the powers of two, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, you see that ten doublings make something about one thousand times greater and hence ten more doublings about a million times greater. Then no logarithms are necessary.

Example

At high tide in the middle of the day, the water in the Port of Liverpool reaches the edge of the quay. The low-water mark is as much as eight metres lower. When in the afternoon has the water level dropped six metres?

1. First choose denotations, one letter for each quantity.
   Tide is a periodic phenomenon with the period $T = 12$ hours. Six hours before high water it was low water, and three hours before high water the water level was at its mean. Therefore, let us count the time t in hours after 9:00. In Liverpool the amplitude is $a = 4$ metres, so let us count the height h in metres above the mean water mark four metres below the edge of the quay.

2. Write down relevant equations, often general laws.

   $$h = a \sin \frac{2\pi t}{T}$$

3. Solve for the wanted quantity.
   This is too hard, we will have to do trial and error in stead.

4. Plug in given values and do the calculation.

   $$-2 = 4 \sin \frac{2\pi t}{12}$$
   $$\sin \frac{\pi}{6} t = -\frac{1}{2}$$

t	$\sin \frac{\pi}{6} t$
0	0
1	1/2
2	$\sqrt{3}/2$
3	1
4	$\sqrt{3}/2$
5	1/2
6	0
7	-1/2

Answer: Seven hours after 9:00, that is, 16:00.

Exercises

1.29: Why is the air inside the football heavier than the air the players breathe? Change the formula so that it takes this phenomenon into account.

1.30: You are to compute the height of a flagpole in a clever way by measuring its shadow (four metres) and a friend's shadow (one metre). If your friend's height is 1.75 metres, how long is the flagpole? Model the situation!

1.31: Saturn is just over nine times as far from the Sun as the Earth is. How long is a year on Saturn? Model and use (17)!

1.32: Use (15) to compute how long it will take to heat two litres of tea water in a water boiler with the power 2240 W.

1.33: After how many days is half the lake covered by lily pads?

1.34: A secret recipe for a lotion specifies nine different components and their volume amounts. With knowledge of the price per litre of these components, find a formula for the price per litre of the lotion if the profit should be 1000%.

1.35: Periodic processes can equally well be written as

$$a \cos \frac{2\pi t}{T}$$

but then t must be counted from another initial time. What would the initial time be in the tide example?

1.36: A cylindrical bottle with diameter 8 cm and height 20 cm weighs 1 kg. Will it do for sending messages in it?
Hint: An object floats if it is lighter than the same volume of water.

1.37: (Toughy) Write down a formula for ordinary fifty-period 230 V alternating voltage. Hint: The top value is $\sqrt{2}$ times the effective value.

2 Factorisation

When is it possible to write a polynomial as a product of polynomials of lower degree, for instance $x^4 - 1 = (x^2 + 1)(x^2 - 1)$, and why would you ever want to do it?

As a technology student you need to ask yourself these questions, and as we will see shortly they lead to unexpectedly beautiful mathematics! But first we will build an intuition for factorisation by looking at the corresponding question for ordinary integers: When is it possible to factor an integer, for example $153 = 9 \cdot 17$, and why would you ever want to do it? This, too, leads to beautiful mathematics.

2.1 Prime factorisation

If you have 90 caramels to distribute into a number of bags, how many bags could you have? A mathematician would formulate the question like this: Which positive integers divide 90?

To answer the question we first try to write 90 as a product of two smaller positive integers, $90 = 9 \cdot 10$ for instance. Both 9 and 10 can be further factorised so we obtain $90 = 3 \cdot 3 \cdot 2 \cdot 5$. The integers 2, 3 and 5 are only divisible by themselves and 1 and thus they cannot be written as a product of smaller integers. Such numbers are called *primes* and $90 = 3 \cdot 3 \cdot 2 \cdot 5$ is a *prime factorisation* of 90. The number 1 is per definition not a prime, so the smallest prime number is 2.

prime number

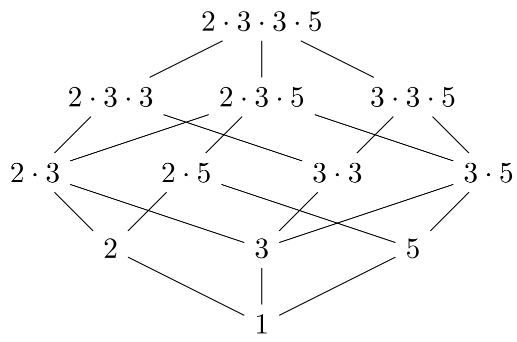
prime factorisation

What would have happened if we had chosen another factorisation of 90 to start with? Let us try $90 = 6 \cdot 15$ for instance. Both 6 and 15 can be further factorised so we obtain $90 = 2 \cdot 3 \cdot 3 \cdot 5$. Disregarding the order of the factors, this is exactly the same prime factorisation as before. That is not a coincidence – it will always turn out that way. Euclid showed the following famous theorem which we present without proof.

Theorem 2.1 (The fundamental theorem of arithmetic)**.** *Every integer greater than 1 can be written as a product of primes in a unique way (up to the order of the factors).*

the fundamental theorem of arithmetic

From the fundamental theorem of arithmetic it is easy to see that the divisors of 90 are the following numbers, with 90 itself at the top.



Prime numbers are also useful when you manipulate fractions. If the numerator and the denominator of a fraction have a common divisor, the fraction can be reduced. For instance,

reduction

$$\frac{180}{280} = \frac{18\cdot 10}{28\cdot 10} = \frac{18}{28}.$$

Since 2 divides both 18 and 28 we can reduce further and write $18/28 = 9/14$, but now no more reduction is possible. In stead of reducing first by 10 and then by 2 of course we could have reduced by 20 directly – the result would have been the same. This number 20 is the largest integer dividing both 180 and 280 and it is called the *greatest common divisor*. If both the numerator and the denominator have been prime-factorised, it is easy to find their greatest common divisor.

greatest common divisor

$$\frac{180}{280} = \frac{2\cdot2\cdot3\cdot3\cdot5}{2\cdot2\cdot2\cdot5\cdot7} = \frac{\cancel{2}\cdot\cancel{2}\cdot3\cdot3\cdot\cancel{5}}{\cancel{2}\cdot\cancel{2}\cdot2\cdot\cancel{5}\cdot7} = \frac{3\cdot3}{2\cdot7}.$$

Prime factors that are present in both the numerator and the denominator cancel out and the product of these is the greatest common divisor.

A third situation where prime factorisation is useful is when you want to simplify a root expression. Let us say that you have solved an exercise in a math book and got the answer $\sqrt{180}$. You check the answer in the book but it reads $6\sqrt{5}$. Not again! But after prime factorisation it is evident that

$$\sqrt{180} = \sqrt{2\cdot2\cdot3\cdot3\cdot5} = \sqrt{2\cdot2\cdot3\cdot3}\cdot\sqrt{5} = 2\cdot3\cdot\sqrt{5}$$

so you were right anyhow!

Exercises

2.1: Make a list over all prime numbers less than 50.

2.2: Find all positive divisors of 204.

2.3: Reduce 495/525 as much as possible by prime-factorising both the numerator and the denominator.

2.4: Simplify

$$\frac{\sqrt{450}-\sqrt{392}}{\sqrt{2}}$$

as much as possible.

2.5: (Toughy) The text says that it is "easy to see" from the fundamental theorem of arithmetic that the divisors of 90 are precisely the numbers in the figure. How can that be seen?

2.2 Polynomial division

A *polynomial* in x is an expression of the form polynomial

$$7x^3 - 2x^2 + 5x + 9.$$

The numbers 7, -2, 5 and 9 in front of the powers of x are called *coefficients* and they are not necessarily integers but might be any real numbers (or complex, if you want), as long as they are independent of x. The coefficient 9 in front of the invisible power x^0 is also called the *constant term* since that term is independent of x. The largest exponent 3 is the *degree* of the polynomial and the corresponding term $7x^3$ is called the *leading term*.

coefficients

constant term

degree

leading term

Adding two polyomials clearly yields another polynomial. For instance,

$$(3x^4 - 7x^2) + (x^4 + 1) = 4x^4 - 7x^2 + 1.$$

One polynomial minus another is also a polynomial, and even the product of two polynomials is a polynomial, as shown in the following example.

$$\begin{aligned}(3x^2 - x + 2)(5x + 1) &= 3x^2(5x + 1) - x(5x + 1) + 2(5x + 1)\\ &= (15x^3 + 3x^2) - (5x^2 + x) + (10x + 2)\\ &= 15x^3 - 2x^2 + 9x + 2.\end{aligned}$$

rational expression

But the quotient of two polynomials is called a *rational expression* and is typically not a polynomial. In this respect polynomial behave just like integers – those can be added, subtracted and multiplied, but if you divide two integers m and n a great amount of luck is required if their quotient m/n should be an integer too. If you happen to be so lucky we say that n *divides* m or that m *is divisible by* n, and the same terminology is used for polynomials. For example, $15x^3 - 2x^2 + 9x + 2$ is divisible by $5x + 1$ as we found above. Often we also say that $15x^3 - 2x^2 + 9x + 2$ *has the factor* $5x + 1$ since $15x^3 - 2x^2 + 9x + 2$ can be written as $5x + 1$ times a polynomial.

But how do you divide polynomials? Let us take the example

$$q(x) = \frac{x^3 - 6x^2 + 11x - 6}{x - 3}.$$

So our task is to find a polynomial $q(x)$ such that

$$(x - 3)q(x) = x^3 - 6x^2 + 11x - 6.$$

The method is to start by choosing $q(x)$ such that the left-hand side becomes *approximately* equal to the right-hand side and then adjust $q(x)$ step by step to make the approximation better and better.

- First we concentrate on the leading term x^3 on the right-hand side. To obtain the leading term x^3 on the left-hand side, $q(x)$ must have the leading term x^2. But $(x - 3)x^2 = x^3 - 3x^2$ so we get a quadratic term $-3x^2$ for free.

- The quadratic term on the right-hand side is $-6x^2$ so the left-hand side is lacking $-3x^2$. Then we are clever enough to let $q(x)$ include also the term $-3x$ and we try again: $(x - 3)(x^2 - 3x) = x^3 - 6x^2 + 9x$. Now both the cubic and the quadratic terms are correct, but we got a linear term $9x$ for free.

- We want the linear term $11x$ so we need another $2x$. Thus, we let $q(x)$ include also the term 2. Third time lucky: $(x - 3)(x^2 - 3x + 2) = x^3 - 6x^2 + 11x - 6$. Hurrah, it's correct!

That the constant term -6 happened to be just right in this example shows that the quotient is a polynomial. Usually when you try to divide polynomials the division does not come out even and you get a remainder (which is also a poynomial), but we will not concern ourselves with in this course.

It is possible to use long division for polynomials just as with numbers, but it is not mandatory. Ask your tutor to show an example if you are interested!

Exercises

2.6: Perform the polynomial division

$$\frac{x^3 - 7x^2 + 17x - 15}{x - 3}.$$

2.7: Perform the polynomial division

$$\frac{14x^4 - 23x^3 - 11x^2 + 23x - 3}{7x^2 + 6x - 1}.$$

2.3 Polynomial factorisation

Now we ask ourselves: What corresponds to prime numbers in the world of polynomials?

A non-constant[2] polynomial is *reducible* if it can be written as a product of two non-constant polynomials. If not, the polynomial is *irreducible* and those polynomials are the ones that correspond to prime numbers. There is also an equivalent to the fundamental theorem if arithmetic:

irreducible polynomial

Theorem 2.2. *Every polynomial can be written as a constant times a product of irreducible polynomials with leading coefficient 1, and that factorisation is unique up to the order of the factors.*

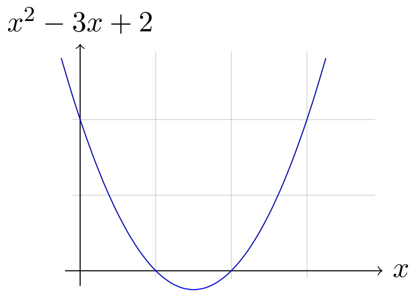
The proof belongs to a more advanced course.

[2]A polynomial is *constant* if it contains only a constant term.

Then the question remains how it can be decided whether a polynomial is irreducible. All first-degree polynomials are clearly irreducible, but not all quadratic ones. For instance, $x^2 - 3x + 2$ is reducible because it can be written as a product $x^2 - 3x + 2 = (x-1)(x-2)$ of two polynomials of lower degree. On the other hand, the quadratic polynomial $x^2 + 1$ is irreducible, since if we try to write it as $(x + a)(x + b)$ we must choose the numbers a and b such that $x^2 + 1 = x^2 + (a + b)x + ab$ and then $a + b$ must be zero and ab must be one. Try finding two numbers whose sum is zero and whose product is one!

Are there irreducible polynomials of higher degree than two?

To answer that question we must first examine another aspect of polynomials. Up to this point we have only thought of polynomials as expressions, as formal sums where each term is a coefficient times a power of x. But, obviously, a polynomial is also a *function* receiving a number as input and emitting a number as output. For instance, if we feed the function $p(x) = x^2 - 3x + 2$ with the number 4 it will output the number $p(4) = 4^2 - 3 \cdot 4 + 2 = 6$. Functions can in turn be thought of as graphs if you wish; here is the graph of $x^2 - 3x + 2$:



zero of a function

Those x that have $p(x) = 0$ are called *zeros* of the polynomial, and in the graph above we see that 1 and 2 seem to be zeros of $x^2 - 3x + 2$. To check this we may substitute 1 and 2 for x:

$$p(1) = 1^2 - 3 \cdot 1 + 2 = 0,$$
$$p(2) = 2^2 - 3 \cdot 2 + 2 = 0.$$

Correct! But if we remember the factorisation $x^2 - 3x + 2 = (x-1)(x-2)$ we will not need the graph to find the zeros. Indeed, they are standing there right under our very noses, and the check becomes almost trivial:

$$p(1) = (1-1)(1-2) = 0,$$
$$p(2) = (2-1)(2-2) = 0.$$

We have just oberved that if a polynomial has the factor $x - a$ then a is a zero of the polynomial. Actually the converse is also true, and we have the following theorem whose proof is omitted.

Theorem 2.3 (The factor theorem)**.** *A polynomial has a as a zero if and only if the polynomial has the factor $x - a$.*

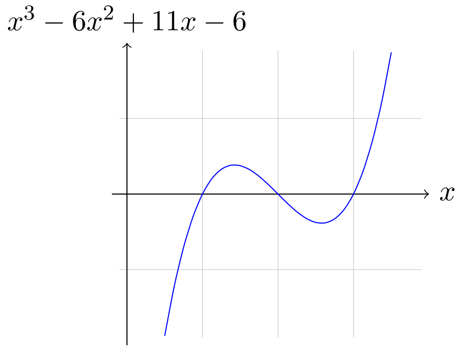
the factor theorem

The factor theorem is useful when you know one zero of a third-degree polynomial and want to find the remaining zeros if they exist.

Suppose, for instance, that we want to find all zeros of the the polynomial $x^3 - 6x^2 + 11x - 6$ and that we already know that 3 is a zero. Then the factor theorem tells us that $x^3 - 6x^2 + 11x - 6$ is divisible by $x - 3$ and we can perform the division

$$\frac{x^3 - 6x^2 + 11x - 6}{x - 3} = x^2 - 3x + 2.$$

Thus, $x^3 - 6x^2 + 11x - 6 = (x-3)(x^2 - 3x + 2)$ and the remaining zeros of $x^3 - 6x^2 + 11x - 6$ also have to be zeros of $x^2 - 3x + 2$. Those can be found by the pq-formula or by completing the square, or perhaps we remember the factorisation $x^2 - 3x + 2 = (x-1)(x-2)$. We conclude that the zeros of $x^3 - 6x^2 + 11x - 6$ are 1, 2 and 3. The function graph looks like this:



If we had access to an oracle that can find zeros of polynomials if they exist, then we would be able to factorise polynomials by applying the factor theorem repeatedly, like this:

- We start by asking the oracle for a zero of the polynomial $p(x)$ that we want to factorise. The oracle will answer with a number a_1.
- From the factor theorem we know that we can write $p(x) = (x - a_1)p_2(x)$ for some polynomial $p_2(x)$.
- Now we ask the oracle for a zero of the polynomial $p_2(x)$. The oracle answers that a_2 is a zero.
- From the factor theorem we know that we can write $p_2(x) = (x - a_2)p_3(x)$ for some polynomial $p_3(x)$.
- We may continue in this manner as long as the oracle is able to answer, and this will allow us to write $p(x) = (x - a_1)(x - a_2)\cdots(x - a_n)q(x)$ where $q(x)$ is a polynomial without zeros.

The following theorem, first proved by Carl Friedrich Gauss (1777–1855), is as close you can get to such an oracle.

the fundamental theorem of algebran

Theorem 2.4 (The fundamental theorem of algebra)**.** *Every non-constant polynomial has at least one complex zero.*

The theorem only holds if complex numbers are allowed as zeros; it might be that there is no real zero.

Here is a picture of Gauss:

If we allow complex numbers the oracle procedure above, together with the fundamental theorem of algebra, shows that polynomials of degree at least 2 can always be written as a product of first-degree polynomials (linear factors). Hence, only first-degree polynomials are irreducible over the complex numbers. The quadratic polynomial x^2+1, which normally is irreducible, becomes reducible when we allow complex numbers, because $x^2+1 = (x-i)(x+i)$.

Example

Let us find all complex zeros of the polynomial $x^2 - 2x + 5$ by using the pq-formula.

$$x = 1 \pm \sqrt{1-5} = 1 \pm \sqrt{-4} = 1 \pm 2i$$

Thus, the zeros are $1+2i$ and $1-2i$.

As you notice, the complex zeros of quadratic polynomials always occur in conjugate pairs, $a+bi$ and $a-bi$. It can be shown that this holds for any polynomial with real coefficients (but we omit the proof).

Theorem 2.5. *The non-real complex zeros of a polynomial with real coefficients occur in conjugate pairs.*

Let us return to the question we put in boldface above, before we got involved with complex numbers: Are there irreducible polynomials of higher degree than two?

Take the polynomial $p(x) = x^4 + 11x^2 + 10x + 50$ as an example. Is it irreducible?

By the fundamental theorem of algebra it has at least one complex zero, and one such zero happens to be $1+3i$. Since the zeros occur in conjugate pairs, $1-3i$ is a zero too. The factor theorem then yields that $\big(x-(1+3i)\big)$ and $\big(x-(1-3i)\big)$ are factors in $p(x)$ and since these factors are distinct and irreducible their product is also a factor in $p(x)$. (Indeed, both factors occur in the unique irreducible factorisation of $p(x)$.) Using the conjugate rule on the product we obtain

$$(x-1-3i)(x-1+3i) = (x-1)^2 - (3i)^2 = (x-1)^2 - 9i^2 = (x-1)^2 + 9$$

since $i^2 = -1$. This is a quadratic polynomial with real coefficients. Thus, $p(x)$ can be written as a product of two quadratic polynomials and hence $p(x)$ is reducible.

By the same kind of reasoning, it can be shown that every irreducible polynomial has degree 1 or 2.

Exercises

2.8: The polynomial $p(x) = 4x^3 - 20x^2 - x + 5$ has a zero for $x = 5$. Find all other zeros!

2.9: Is the polynomial $p(x) = x^3 + 3x^2 - x + 2$ divisible by $x + 2$?

2.10: (Toughy) Find all (real) zeros of the polynomial $p(x) = x^4 + x^3 + x^2 + x$.

2.4 Polynomial equations and their history

The fundamental theorem of algebra states that every nonconstant polynomial has at least one complex zero, but how do you go about finding one?

For first-degree polynomials it's a picnic: $ax + b = 0$ has the unique solution $x = -b/a$ (if $a \neq 0$).

Second degree equations are harder but were solved (by completing the square) by the Babylonians four thousand years ago. The pq-formula gives the solution directly:

$$x^2 + px + q = 0 \;\Leftrightarrow\; x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

In the early 16th century, competitions were held in solving third degree equations. When the Italian mathematician Scipione del Ferro (1465–1526) discovered a formula for the solution of the general third degree equation $x^3 + ax^2 + bx + c = 0$, the competitors turned to solving fourth degree equations, but soon thereafter Ludovico Ferrari (1522–1565) found a formula even for these.

Will somebody find a formula for fifth degree equations too? No, the Norwegian mathematician Niels Henrik Abel (1802–1829) proved that there is no such formula, in any case no formula using only algebraic operations.

But certainly some polynomial equations of degree five and higher are simple to solve algebraically. For example, the equation $x^5 = x$ has the solutions 0, 1, -1, i och $-i$. The French mathematician Évariste Galois

(1811–1832) analysed exactly which polynomial equations may be solved by algebraic operations.

Abel

Galois

Both Abel and Galois died very young. Abel contracted tuberculosis and died at 26, two days before information arrived that he had been appointed professor at the university of Berlin. Galois was an unappreciated genius who twice served time in prison. When released, he was challenged to duel with a rival. Luckily enough he wrote down some brilliant thoughts in a letter to a friend the night before being killed in the duel, just twenty years old.

Finally, it must be stressed that, in spite of the nonexistence of an algebraic formula solving equations of higher degree, solving such equations is a simple task for the working engineer. With *numerical methods* it is easy to find approximate solutions with any desired precision. That is also why the formulas for the third and fourth degree equations, discovered by medieval mathematicians, are now known to very few of us (by heart) – they are simply not useful any more. So if you want to be one of these very few – go ahead and learn them!

3 Proofs

Everybody can make assertions, but only mathematicians can prove them. The usual form of proof is when you move forward stepwise and via intermediate results finally reach your goal. This is what it looks like in principle:

Theorem: $A, B, C \Rightarrow Z$

Proof :
$$A, B, C \Rightarrow D$$
$$A, B, C, D \Rightarrow E$$
$$A, B, C, D, E \Rightarrow Z$$

A, B, C are called the *premises*, D, E are intermediate results and Z the conclusion. This is what a simple example looks like.

Theorem: If m and n are even numbers, then $m + n$ is even.
Proof:

$m = 2r$	(definition of even)
$n = 2s$	(definition of even)
$m + n = 2r + 2s$	(addition of equations)
$m + n = 2(r + s)$	(rule of algebra)
$m + n$ is even	(definition)

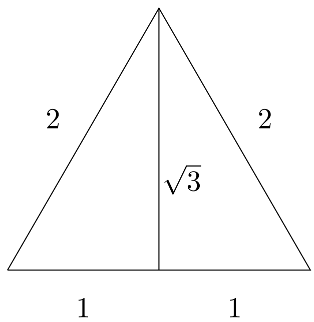
This is how most of the theorems in Euclid's geometry textbook Elementa are proved, for example Pythagoras's theorem that $a^2 + b^2 = c^2$ in a right triangle. But one of the most famous theorems, which is not about geometry, is proved in a backward manner that is very useful.

3.1 Proof by contradiction

Theorem: There are infinitely many primes.
Proof: Assume that the statement is false! If so, there are only finitely many primes $2, 3, 5, 7, 11, \ldots, p$ and we may call the largest one p. Every integer then has its prime factors among these $2, 3, 5 \ldots, p$. But consider the enormous number $q = 1 + 2 \cdot 3 \cdot 5 \cdots p$. It cannot be divisible by any of the numbers $2, 3, 5, 7, 11, \ldots, p$. We have arrived at a contradiction and that demonstrates that our assumption must have been wrong.

An intermediate result that is interesting in itself is called *helping theorem* or with a Greek word *lemma*. The proof of the lemma is often postponed until after the proof of the main theorem. Not completely logical but often easier to read.

Theorem: A triangular area with circumference six metres is at most $\sqrt{3}$ square metres.
Proof: An equilateral triangle with side length 2 splits into two right triangles with hypothenuse 2 and one leg 1. The other leg is $\sqrt{3}$ according to the Pythagorean theorem, for $1 + 3 = 4$. Thus the big triangle has area $\sqrt{3}$.



According to the next lemma, an equilateral triangle has the largest area of all triangles with a given circumference. That concludes the proof of the theorem.

Lemma: The largest triangle with a given circumference is equilateral.
Proof: Assume the opposite! Then, the largest triangle has some side a that is longer than another side b. Replace these two sides by a piece of string with length $a + b$ and let C be the common corner of the two sides. If C is moved while keeping the string taut, the circumference of the triangle doesn't change, and for symmetry reasons C will be furthest away from side c when $a = b$. The contradiction proves the lemma.

3.2 Induction

When you cut yourself a piece of cake, always leave half of it to latecomers! By this rule of behaviour, the first person will get $\frac{1}{2}$ cake and leave $1-\frac{1}{2} = \frac{1}{2}$.
Number two gets $\frac{1}{4}$ cake and leaves $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.
Number three gets $\frac{1}{8}$ cake and leaves $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$.

Evidently, the following equality holds.

$$1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^n} = \frac{1}{2^n}$$

Now, move all minus-terms over to the right and move the old term on the right side to the left side.

Theorem:

$$1 - \frac{1}{2^n} = \sum_{k=1}^{n} \frac{1}{2^k}, \quad \forall n \geq 1$$

This is not just an assertion, it is infinitely many assertions, one for each n. So, it seems to need infinitely many proofs, but in fact the whole theorem may be proved by one *induction proof* with only two steps. Let us write P_1 for the assertion when $n = 1$, P_2 for the asserton when $n = 2$ etc. Then, we only have to prove two things:

induction

P_1 is true	base case
$P_n \Rightarrow P_{n+1}, \quad \forall n$	induction step

Assertion P_1 means $1 - \frac{1}{2} = \frac{1}{2}$ and that is true.
Assertion $P_n \Rightarrow P_{n+1}$ means

$$1 - \frac{1}{2^n} = \frac{1}{2} + \cdots + \frac{1}{2^n} \;\Rightarrow\; 1 - \frac{1}{2^{n+1}} = \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

That is true too, for if you add $1/2^{n+1}$ to both sides in P_n, you get exactly P_{n+1}.

A proof by induction always consists of two parts – base case and induction step. If you forget about the base case, you may prove any nonsense.

Example

Prove that $n + n < 2n, \quad \forall n$.
P_n is $n + n < 2n$. Adding 2 to both sides gives $n + n + 2 < 2n + 2$ which we rewrite as $(n + 1) + (n + 1) < 2(n + 1)$, and that is P_{n+1}.

Whether to use proof by induction or some other method of proof is often a matter of taste.

Example

When n persons meet and everybody says *hej* to everybody else, that makes a total of $n(n - 1)$ *hej*.

Proof: Everybody says *hej* to $n-1$ others. For all n persons together that makes $n(n-1)$ *hej* .

Proof by induction: The base case $n=1$ is true. Assume that it is true for n persons and that person $n+1$ appears on the scene and salutes these n. The total increases to $n(n-1)+2n$, which may be written as $(n+1)n$, and that is exactly P_{n+1}.

Exercises

3.1: Prove that $n(n-1)$ is an even integer for all $n \geq 1$.

3.2: Prove the same assertion by induction.

3.3: Find the error in the following proof of $1=0$.

$$a=1 \;\Rightarrow\; a^2=a \;\Rightarrow\; a^2-1=a-1 \;\Rightarrow$$
$$\Rightarrow\; (a+1)(a-1)=1\cdot(a-1) \;\Rightarrow\; (a+1)=1 \;\Rightarrow\; a=0.$$

3.4: Prove that the ordinary arithmetic mean $(x+y)/2$ is greater than or equal to the geometric mean $\sqrt{xy}$.
Hint: $(\sqrt{x}-\sqrt{y})^2$.

3.5: If n persons meet and shake hands, how many handshakes will take place?

3.6: Prove that $n(n-1)(n-2)$ is divisible by 6 for all $n \geq 2$.

3.7: Prove by contradiction that all primes larger than 2 are odd.

3.8: You ride your bike to school and back home in 30 km/h but when it is windy, your speed increases in tailwind and decreases by the same amount in headwind. Prove that the total time is shortest in calm weather.

3.9: (Toughy) Prove the formula

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

4 Mixed exercises

4.1: Which of the following statements are true?

(a) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z},\ x \cdot y = 1$,

(b) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Q},\ x \cdot y = 1$,

(c) $\forall x \in \mathbb{Q}\ \exists y \in \mathbb{Q},\ x \cdot y = 1$.

4.2: Which of the following statements are true for all real numbers x and y?

(a) $|x - y| > 0 \Rightarrow x > y$,

(b) $x > y \Rightarrow |x - y| > 0$,

(c) $|x - y| > 0 \Leftrightarrow x > y$.

4.3: Which of the following statements are true?

(a) 17 is a prime,

(b) 15 and 21 have a common factor,

(c) $\sqrt{90}/15 = \sqrt{2/5}$.

4.4: Factorise the polynomial $p(x) = x^3 + x^2 - 17x + 15$ as much as possible. Hint: What is $p(1)$?

4.5: *A number is rational if and only if it is a quotient of two integers.* Write this in symbols!

4.6: Prove that all nonprime two-digit numbers are divisible by 2, 3, 5 or 7.

4.7: Write $\{x > 0 \mid |x^2 - 5| < 4\}$ as an interval.

4.8: Which of the following statements are true?

(a) $\forall n \in \mathbb{N}\ \exists y \in \mathbb{R},\ y < n$

(b) $\forall y \in \mathbb{R}\ \exists n \in \mathbb{N},\ y < n$

(c) $\forall n \in \mathbb{N}\ \exists y \in \mathbb{R},\ n < y$

(d) $\forall y \in \mathbb{R}\ \exists n \in \mathbb{N},\ n < y$

4.9: Find the error in the following proof by induction of the strange fact that everybody is called Lasse Svensson.

> For all positive integers n, we must show that in every group of n persons, each person is called Lasse Svensson. The base case when $n = 0$ is true, for each person in a zero-persons group is undeniably a Lasse Svensson. Now, assume that the statement is true for some n and consider a group of $n + 1$ persons. We will show that each person in this group is called Lasse Svensson. Remove an arbitrary person x from the group. The remaining persons form a group of n persons, so by the induction assumption, they are all called Lasse Svensson. Replace x and remove another person y from the group. The remaining n persons are of course all called Lasse Svensson, so in particular, x is a Lasse Svensson. We conclude that the statement is true for $n + 1$ and thus, by induction, for all n.

4.10: All integers with odd last digit are odd. Is the following proof of that statement correct?

> Let n be an arbitrary integer. We want to show that if the last digit in n is odd, then n itself must be odd. Assume the opposite: that n is odd but its last digit d is even. Assume that the other digits form the number k. Then we have $n = 10k + d$. As d is even, we have $d = 2m$ for some integer m. Thus, $n = 10k + 2m = 2(5k + m)$, contradicting our assumption that n is odd. The contradiction proves that the opposite statement is true.

4.11: Calculate $\sum_{n=0}^{100} (-1)^n$.

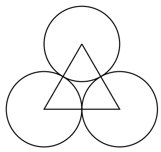
4.12: Calculate $\prod_{n=0}^{100}(-1)^n$.

4.13: Calculate $(1-k)(1+k+k^2+\cdots+k^{n-1})$. Use te result to derive a formula for $1+k+k^2+\cdots+k^{n-1}$ (a geometric series).

4.14: The capital in a scholarship fund gives an annual interest and pays out a fixed annual stipend. Model it!
Find the capital $x(t)$ after t years. Start with x_0 kronor.

4.15: Denote the average of a and b by m. Prove that $a^2+b^2 \geq 2m^2$. Hint: First prove the lemma $\exists s,\ a=m+s,\ b=m-s$.

4.16: Coins lie tightly packed upon an enormous table top. Prove that the coins cover $\frac{\pi}{2\sqrt{3}}$ of the surface. Hint: It suffices to consider the triangle in the illustration.



4.17: What is stated here?

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists q \in \mathbb{Q},\ |x-q| < \varepsilon$$

Is it true? Check with $x=\pi$ and $\varepsilon = 0.01$.

4.18: $\frac{6!}{3!3!} = 20$ but what estimate does Stirling's formula give?

5 Solutions

—

1.1
Archimedes, perhaps the greatest mathematician of all times, found formulas for the area and volume of the sphere 2200 years ago.

—

1.2
Cosine phi is zero if and only if phi is plus minus pi over two plus two n pi, for some integer n. In other words, $\cos\varphi$ vanishes (is zero) for the following values of φ:

$$\ldots, +\frac{\pi}{2}-2\pi, -\frac{\pi}{2}-2\pi, +\frac{\pi}{2}, -\frac{\pi}{2}, +\frac{\pi}{2}+2\pi, -\frac{\pi}{2}+2\pi, +\frac{\pi}{2}+4\pi, -\frac{\pi}{2}+4\pi, +\frac{\pi}{2}+6\pi, \ldots$$

—

1.3
Rho squared plus sigma squared plus two rho sigma is greater than or equal to two rho plus two sigma. For $\rho = 1, \sigma = 1$ we obtain $4 \geq 4$, which is true. For $\rho = 1, \sigma = 0$ we obtain $1 \geq 2$, which is false.

—

1.4
The absolute value of x equals the square root of x squared. If x is positive this is clearly true. If x is negative, $|x|$ removes the minus sign and the minus sign also disappears in x^2 so the sign does not change anything.

—

1.5
With $n = 3$ we get $4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot (3 \cdot 2 \cdot 1)$. Correct! For $n = 0$ we get $1 = 1 \cdot 0!$. Apparently, zero factorial should be interpreted as one, so let us do that.

—

1.6
An empty product should be interpreted as 1.

—

1.7
$k = 10,\ n = 19$. A clever way of checking the sum with mental arithmetic is to add the first and last term $1 + 19 = 20$, the second and the second to last term $3 + 17 = 20$ etc. to the fifth pair $9 + 11 = 20$. Five twenties is a hundred, so it is correct.

—

1.8

$\pi^2/6 \approx 1.65$ but $1 + 1/4 + 1/9 \approx 1.36$ and

$$\frac{1}{1-1/4} \cdot \frac{1}{1-1/9} = 1.5. \tag{22}$$

—

1.9

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \quad = \prod_{\text{prime}\, p} \frac{1}{1-1/p^k} \quad \text{for all } k > 1$$

—

1.10

Since $e \approx 2.7$ we have $n/e \approx 10$. Mental arithmetic yields $\sqrt{2\pi n} \approx 13$. It follows that $27! \approx 13 \cdot 10^{27}$ which seems to have 29 digits.

—

1.11

$x < x + 1$ or why not $0 < 17$.

—

1.12

$0 < x^2$ or why not $x \neq 0$.

—

1.13

$0 < (x-1)^2$ (or, of course, $x \neq 1$).

—

1.14

Add 7 to both sides.	$5x > 2x + 12$
Add $-2x$ to both sides.	$3x > 12$
Divide both sides by 3.	$x > 4$

Answer: It holds for x larger than 4.

—

1.15

As before we obtain $x^2 > 4$. Now there are two cases to consider. If x is positive then $x > 2$. If x is negative then $x < -2$. Both can be expressed as $|x| > 2$.

Answer: It holds if the absolute value of x is greater than 2.

—

1.16

$a = -3,\ b = -2$

—

1.17

That the function $y = f(x)$ is growing means that it is uphill to the right, that is, going right on the x-axis from a till b makes the y-value increase from $f(a)$ to $f(b)$. Substituting e^x for $f(x)$ yields the inequality $e^a < e^b$.

—

1.18

$$a < b \;\Leftrightarrow\; e^a < e^b$$

We know that $a \geq b$ implies $e^a \geq e^b$, so the only possibility to obtain $e^a < e^b$ is that we have $a < b$. Put $e^a = \alpha$ and $e^b = \beta$. Then $a = \ln\alpha$ and $b = \ln\beta$ and the result can be written like this:

$$\alpha < \beta \;\Leftrightarrow\; \ln\alpha < \ln\beta$$

—

1.19

If we draw the curve $y = x^2 - 3x + 2$ we see that it goes below the x-axis between $x = 1$ and $x = 2$. This gives us the idea to write the inequality like this:

$$(x-1)(x-2) < 0$$

If the product of two numbers is negative, one of the numbers must be positive and the other one negative. This can only happen when $1 < x < 2$.

—

1.20

For any number between 1 and 17 the distance to 9 is at most 8. True!

—

1.21

$\{n \in \mathbb{N} \,|\, 10 \leq n \leq 99\}$

—

1.22

The set of all negative integers.

—

1.23

$(3.4, 5)$. And, by the way, it is probably better to write $\{x \;:\; |x - 4.2| < 0.8\}$ in stead of $\{x \mid |x - 4.2| < 0.8\}$.

—

1.24

The simplest choice is $A = \mathbb{R}$ and $B = \mathbb{Z}$. A more interesting solution is to let A be the union of all intervals of the type $[2r, 2r+1]$ and B the union of all intervals of the type $[2r-1, 2r]$, where r goes over all integers.

—

1.25
$J = \{2r \,|\, r \in \mathbb{N}\}$
$U = \{2r+1 \,|\, r \in \mathbb{N}\}$

—

1.26
$x \in U \;\Rightarrow\; x^2 \in U$

—

1.27
It is actually the same statement as the previous one, only expressed differently. $n \in \mathbb{N} \;\Rightarrow\; \big(n^2 \in J \;\Rightarrow\; n \in J\big)$

—

1.28
Start the enumeration with all fractions that can be written with the digits 0 and 1.

$$\mathbb{Q} = \{\frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \cdots\}.$$

Continue with those that can be written with 0, 1 and 2 (but do not repeat yourself).

$$\mathbb{Q} = \{\frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{-2}{1}, \frac{2}{1}, \frac{-1}{2}, \frac{1}{2}, \cdots\}.$$

All fractions in $\mathbb{Q}$ will eventually be enumerated.

—

1.29
Its pressure is approximately twice as high. If p (atm) is the air pressure inside the football, the formula is as follows.

$$m = p \cdot \rho \cdot \frac{4}{3}\pi r^3$$

—

1.30
L_f (m), the length of the flagpole
S_f (m), the length of the flagpole's shadow
L_k (m), the height of your friend
S_k (m), the length of your friend's shadow

$$\frac{L_f}{S_f} = \frac{L_k}{S_k}$$
$$L_f = L_k \cdot S_f / S_k$$
$$L_f = 1.75 \cdot 4/1 = 7$$

Hence, the flagpole is seven metres.

—

1.31

r_J (km), the distance between the Earth and the Sun
v_J (km/year), the speed of the Earth
s_J (km), the orbit length of the Earth
t_J (years), the orbit time of the Earth = 1
r_S (km), the distance between Saturn and the Sun
v_S (km/year), the speed of Saturn
s_S (km), the orbit length of Saturn
t_S (years), the orbit time of Saturn

$v_J = \text{konstant}/\sqrt{r_J}$ (by (17))
$s_J = 2\pi r_J$ (the circumference of a circle)
$v_J \cdot t_J = s_J$ (speed times time = length)
$v_S = \text{constant}/\sqrt{r_S}$ (by (17))
$s_S = 2\pi r_S$ (the circumference of a circle)
$v_S \cdot t_S = s_S$ (speed times time = length)
$r_S = 9 r_J$, (given in the text)

Solving for the wanted variable yields

$$t_S = 9\sqrt{9}\, t_J = 27.$$

Thus, the Saturn year is just more than 27 years on Earth.

—

1.32

$$t = T \cdot c_p \cdot \rho \cdot V/P = 80 \cdot 4200 \cdot 1000 \cdot 0.002/2240 = 300$$

It will take five minutes.

—

1.33

If it takes twenty days to cover the whole lake, clearly it takes nineteen days to cover half of it.

—

1.34

Let p_k (crowns/l) be the price per litre of the kth component and let v_k (l) be its volume according to the recipe. Then, the recipe will produce $v = \sum v_k$ litres of lotion at a cost of $c = \sum p_k v_k$, and so the cost per litre of the lotion is c/v. To make a thousand percent profit the lotion should be sold at the price

$$11 \frac{\sum p_k v_k}{\sum v_k}$$

per litre. How cheap!

—

1.35

A glimpse at the cosine curve reveals that it has a maximum at $t = 0$. Thus, t should be counted from 12:00.

—

1.36

The bottom area of the bottle is $\pi r^2 = 16\pi \approx 50.265$ so its volume is $20 \cdot 50.265 = 1005.3$ litres. The weight of the displaced water is 1005 g and the bottle itself weighs only 1000 g, so it will float. However, the message in the bottle cannot be too large, no more than five grams, and that is exactly the weight of an ordinary A4 sheet.

—

1.37

The period T is a fiftieth of a second, so the formula is as follows.

$$U = 230 \cdot \sqrt{2}\,\sin(100\pi t)$$

—

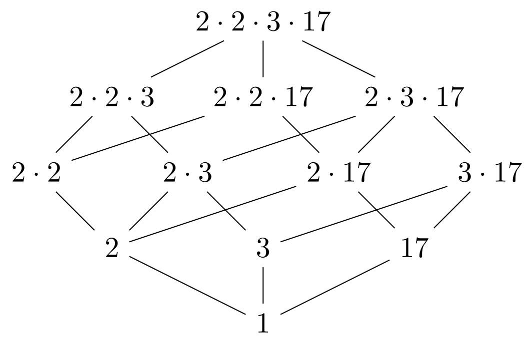
2.1

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

—

2.2

The prime factorisation of 204 is $204 = 2 \cdot 2 \cdot 3 \cdot 17$ and here are the divisors:



2.3

$$\frac{495}{525} = \frac{3 \cdot 3 \cdot 5 \cdot 11}{3 \cdot 5 \cdot 5 \cdot 7} = \frac{\cancel{3} \cdot 3 \cdot \cancel{5} \cdot 11}{\cancel{3} \cdot \cancel{5} \cdot 5 \cdot 7} = \frac{33}{35}.$$

2.4

$$\begin{aligned}\frac{\sqrt{450} - \sqrt{392}}{\sqrt{2}} &= \frac{\sqrt{2 \cdot 3 \cdot 3 \cdot 5 \cdot 5} - \sqrt{2 \cdot 2 \cdot 2 \cdot 7 \cdot 7}}{\sqrt{2}} \\ &= \frac{3 \cdot 5 \cdot \sqrt{2} - 2 \cdot 7 \cdot \sqrt{2}}{\sqrt{2}} = 3 \cdot 5 - 2 \cdot 7 = 1.\end{aligned}$$

2.5

If m divides 90 it is possible to write 90 as a product $90 = m \cdot n$ for some positive integer n. If m and n have the prime factorisations $m = p_1 p_2 \cdots p_k$ and $n = q_1 q_2 \cdots q_\ell$, then 90 has the prime factorisation $p_1 p_2 \cdots p_k q_1 q_2 \cdots q_\ell$. Since prime factorisation is unique, it must be possible to find the prime factors $p_1, p_2, \ldots p_k$ among the numbers 2, 3, 3 and 5.

2.6

The quotient is $x^2 - 4x + 5$.

2.7

The quotient is $2x^2 - 5x + 3$.

—

2.8
By the factor theorem, $p(x)$ has the factor $x-5$, so we perform the division

$$\frac{4x^3-20x^2-x+5}{x-5}=4x^2-1.$$

The other zeros of $p(x)$ are precisely the zeros of $4x^2-1$. According to the rule of conjugation, $4x^2-1=(2x-1)(2x+1)$, which may be written as $4(x-\frac{1}{2})(x+\frac{1}{2})$, so the zeros are $\frac{1}{2}$ and $-\frac{1}{2}$.

—

2.9
By the factor theorem, we only need to investigate whether -2 is a zero of $p(x)$. But $p(-2)=(-2)^3+3(-2)^2-(-2)+2=-8+3\cdot 4+2+2=8\neq 0$, so -2 isn't a zero and thus $p(x)$ isn't divisible by $x+2$.

—

2.10
Since x is a factor, the polynomial has a zero in $x=0$ and may be written as $p(x)=(x^3+x^2+x+1)x$. So the other zeros of $p(x)$ are precisely the zeros of x^3+x^2+x+1. Had this been a second-degree polynomial, the pq-formula would have solved the problem, but as we know no similar formula for third-degree equations, we must use trial and error. And we are lucky, for -1 turns out to be a zero, for $(-1)^3+(-1)^2+(-1)+1=(-1)+1+(-1)+1=0$. The factor theorem tells us that x^3+x^2+x+1 has a factor $x+1$, so we perform the division

$$\frac{x^3+x^2+x+1}{x+1}=x^2+1.$$

Obviously x^2+1 has no real zero (a square is positive) so $p(x)=x(x+1)(x^2+1)$ has only these two zeros: 0 and -1.

—

3.1
Either n or $n-1$ is an even number, thus divisible by 2.

—

3.2
Base case: $n=1 \;\Rightarrow\; n(n-1)=0$ and zero is even.
Induction step: $n(n-1)=2r \;\Rightarrow\; (n+1)n=2r+2n=2(r+n)$.

—

3.3
The error is the division by $(a-1)$ since that is zero.

—

3.4
$(\sqrt{x}-\sqrt{y})^2 \geq 0$ (anything squared is nonnegative).
Men $(\sqrt{x}-\sqrt{y})^2 = x+y-2\sqrt{xy}$ (squaring rule).
Thus $x+y-2\sqrt{xy} \geq 0$.
Rewrite this as $x+y \geq 2\sqrt{xy}$, divide by two and you are finished.

—

3.5
One handshake corresponds to two *hej* , so the example in the text provides the answer $n(n-1)/2$.

—

3.6
Out of three consecutive integers, one must be a multiple of 3 and at least one must be a multiple of 2. So the result follows, but of course a proof by induction is also possible.
Base case: $n=2 \;\Rightarrow\; n(n-1)(n-2)=0$ and zero is divisible by 6.
Induction step: $n(n-1)(n-2)=6r \;\Rightarrow\; (n+1)n(n-1)=6r+3n(n-1)$
And as $n(n-1)$ is even, it is divisible by 6.

—

3.7
Assume that for some prime $p>2$ we have $p=2r$. Since a prime is divisible only by itself and by 1, we must have $p=2$. The contradiction provides the proof.

—

3.8
If the wind speed is v km/h and the distance is s km, the time is

$$\frac{s}{30+v}+\frac{s}{30-v}=\frac{(30-v)s+(30+v)s}{(30+v)(30-v)}=\frac{60s}{900-v^2}$$

With $v=0$, the denominator is maximised and, consequently, the time minimised.

—

3.9
Base case: $n=1 \;\Rightarrow\; \frac{1}{1\cdot 2}=1-\frac{1}{2}$
Induction step:

$$\frac{1}{1\cdot 2}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$
$$\Rightarrow\; \frac{1}{1\cdot 2}+\cdots+\frac{1}{(n+1)(n+2)}=1-\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}=1-\frac{1}{n+2}$$

—

4.1
None of the statements is true, for if x is zero, there is no number y such that $x \cdot y = 1$.

—

4.2
Only statement (b) is true.

—

4.3
All three statements are true.

—

4.4
$p(x) = (x-1)(x-3)(x+5)$.

—

4.5

$$x \in \mathbb{Q} \Leftrightarrow \exists p, q \in \mathbb{Z},\ x = \frac{p}{q}$$

—

4.6
A nonprime number may be written as a product $p \cdot q$, where p and q are integers greater than 1. If $p \cdot q < 100$, one of p and q has to be less than 10. So, either p or q must belong to the set $\{2, 3, \ldots, 9\}$, and all numbers in that set are divisible by 2, 3, 5 or 7.

—

4.7
The inequality $|x^2 - 5| < 4$ may be written as $1 < x^2 < 9$ and for positive real numbers, this means the interval $(1, 3)$.

—

4.8
(a), (b), (c) are true, (d) is false.

—

4.9
The error is in the step where you remove "another person y from the group". For if $n+1 = 1$ there is no "other person", there is only x. So, the induction step from $n = 0$ to $n = 1$ fails and therefore the whole induction fails.

—

4.10
No, the opposite statement to "if the last digit is odd, then n itself is odd"

isn't "for some odd number n, its last digit is even" but "for some even number n, its last digit is even".

—

4.11
$1-1+1-1+\cdots+1=1$.

—

4.12
$1\cdot(-1)\cdot 1\cdot(-1)\cdot\cdots\cdot 1=1$ as there are fifty minuses.

—

4.13
We have that

$$(1-k)(1+k+k^2+\cdots+k^{n-1}) = 1+k+k^2\cdots+k^{n-1}-k-k^2-k^3-\cdots-k^n = 1-k^n.$$

Dividing both sides by $1-k$, we derive a formula for the geometric series:

$$1+k+k^2+\cdots+k^{n-1} = \frac{1-k^n}{1-k} \text{ om } k\neq 1.$$

—

4.14
We introduce the notation p for the interest rate and s for the annual stipend. The capital after one year is $x(1)=x_0(1+p)-s$, after two years it becomes $x(2)=x(1)(1+p)-s=x_0(1+p)^2-s(1+p)-s$, after three years it becomes $x(3)=x(2)(1+p)-s=x_0(1+p)^3-s(1+p)^2-s(1+p)-s$ and so on. After t years, the fund capital is

$$x(t)=x_0(1+p)^t-s\big(1+(1+p)+\cdots+(1+p)^{t-1}\big)=x_0(1+p)^t-s\cdot\frac{(1+p)^t-1}{p},$$

where we have used the formula for a geometric series.

—

4.15
Put $s=\frac{a-b}{2}$. Since $m=\frac{a+b}{2}$, we then have $a=m+s$ and $b=m-s$. Now $a^2+b^2=(m+s)^2+(m-s)^2=2m^2+2s^2\geq 2m^2$ as squares are nonnegative.

—

4.16
In the triangle, there are three sixths of a circle, which together have the area $\frac{3}{6}\pi r^2$, where r is the circle radius. The triangle is equilateral with area $\frac{2\sqrt{3}}{2}r^2$. The fraction of the surface that is covered by coins thus is $\frac{\pi}{2\sqrt{3}}$.

—

4.17

For every real number and every positive distance there is a rational number within that distance. It is true! For example, the distance of $\frac{314}{100} \in \mathbb{Q}$ from $\pi = 3.14159\ldots$ is less than 0.01.

—

4.18

Stirling's formula gives

$$(2n)! \approx \sqrt{2\pi \cdot 2n}\left(\frac{2n}{e}\right)^{2n}$$
$$n! \cdot n! \approx 2\pi n\left(\frac{n}{e}\right)^{2n}$$

so

$$\frac{(2n)!}{n! \cdot n!} \approx \frac{2^{2n}}{\sqrt{\pi n}}$$

and putting in $n = 3$ we get approximately $\frac{64}{3.1} \approx 20.5$.