THE AREA OF A RANDOM TRIANGLE IN A REGULAR PENTAGON AND THE GOLDEN RATIO.

JOHAN PHILIP

Abstract. We determine the distribution function for the area of a random triangle in a regular pentagon. It turns out that the golden ratio is intimately related to the pentagon calculations.

1. Introduction

We shall denote the regular pentagon by K and the random triangle by T and shall consider the random variable $X = area(T)/area(K)$. It is well known that an affine transformation will preserve the ratio X. This follows from the fact that the area scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [14]. J. J. Sylvester considered the problem of a random triangle T in an arbitrary convex set K and posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown [2], that $\kappa = \frac{1}{20} + \frac{\sqrt{5}}{90} \approx .074845.$ for K a regular pentagon.

A. Reńyi and R. Sulanke, [12] and [13], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. R. E. Miles [7] generalizes these asymptotic estimates for K a circle to higher dimensions. C. Buchta and M. Reitzner, [4], have given values of κ (generalized to three dimensions) for $n \geq 4$ points in a tetrahedron. H. A. Alikoski [2] has given expressions for κ when T is a triangle and K a regular r-polygon.

Here, we shall deduce the distribution function for X when K is a regular pentagon. We have done this before when K is a square in, [8]

Date: June 26, 2012.
1991 *Mathematics Subject Classification.* Primary: 60D05: Secondary: 52A22.
Key words and phrases. Area, random triangle, pentagon, Sylvester problem.

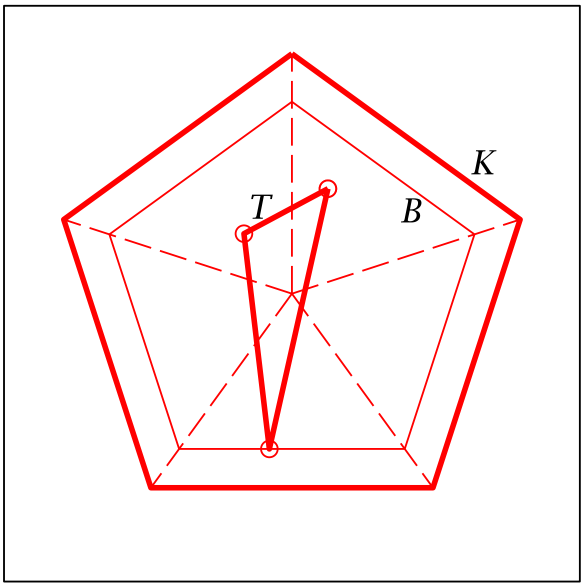

FIGURE 1. The regular pentagon K, the random triangle T and the shrunken pentagon B .

and [9], when K is a triangle in [10], and when K is regular hexagon in [11]. The method used here is the same as in [9] and [11].

2. Notation and formulation.

We use a constant probability density for generating three random points in the regular pentagon K. Let T be the convex hull of the three points. Compare Figure 1. We shall determine the probability distribution of the random variable $X = area(T)/area(K)$.

Our method will be to shrink the pentagon around its midpoint until one of its sides hits a triangle point. The shrunken pentagon is denoted B. The random variable X that we study will be written as the product of two random variables

$$V = area(B)/area(K) \text{ and } W = area(T)/area(B).$$

One of the triangle points stops the shrinking and determines V. Since the density of the points is rotation invariant, we can use a local coordinate system at the stopping point with one axis along the side of the pentagon and one orthogonal to the side. It's the point's orthogonal coordinate that determines the shrink. The coordinate along this side is independent of the former and is consequently evenly distributed along the side. Since the coordinates of the other two triangle points are independent of the hitting point, it follows that V and W are independent.

We shall determine the distributions of V and W and combine them to get the distribution of $X = VW$.

3. The distribution of V.

V is the area of the shrunken pentagon B. The pentagon K is the sum of five similar triangles. Each of the three points of the random triangle sits in one of these similar triangles. Focussing on such a similar triangle, we measure the distance from the random point to the center of the pentagon othogonally to the side of the triangle that is part of the pentagon. Denote this distance by S. The distribution function for S is $L(s) = c \cdot s^2$, where c is a constant. Choosing a scale so that $s = 1$ on the boundary, we have $c = 1$. The largest of the three distances has the distribution function $L(s_{max})^3 = (s_{max})^6$. The area of B is $v = area(K) \cdot (s_{max})^2$. We get

$$G(v) = \text{Prob}(V < v) = (s_{max})^6 = v^3, \quad 0 \le v \le 1. \tag{1}$$

By the argument used here, this $G(v)$ holds for any regular r-polygon.

4. The distribution of W.

W is the area of a random triangle having one vertex on the boundary of a regular pentagon $= B$ and the other two vertices in the interior of the pentagon. In the calculations, we will use the affine transforms of B shown in Figures 2 and 4. The affine transform in Figure 2 is chosen so that three adjacent vertices of the pentagon sit in the points (1,0), (0,0), and (0,1). Then, the remaining two vertices will sit in the points (1,a) and (a,1), where

$$a = \frac{1+\sqrt{5}}{2} \approx 1.618 \tag{2}$$

is the golden ratio.

The same number will occur naturally also in the affine transformation in Figure 4. The calculations below will lead to high powers of a in the expressions. A substantial part of the calculations in this paper consists of simplifying expressions with powers of a using the equation that it satisfies:

$$a^2 = a + 1\,. \tag{3}$$

When we did the calculations for a hexagon in [11], the constant imposed by the geometry was 2, so that its powers were handled by ordinary arithmetic.

Returning to the geometry, we will, without loss of generality, number the three triangle vertices so that vertex one is the one sitting on the boundary and we let this boundary be the x-axis, so that vertex

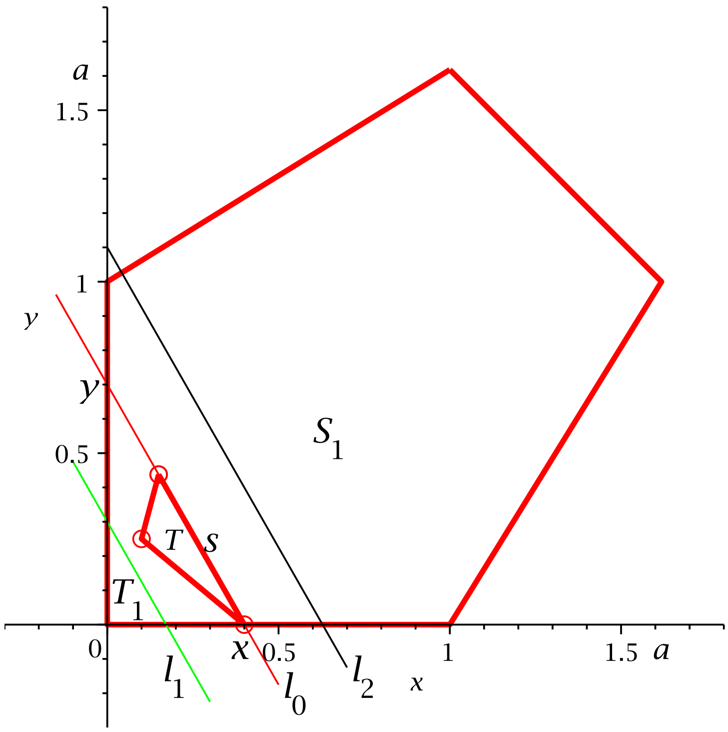

FIGURE 2. Case 1. The line l_0 through vertices one and two intersects the left side of the affine pentagon B_1.

one is $(x, 0)$, where $0 \leq x \leq 1$. In Figure 2, the position of the second vertex is (x_2, y_2). Let l_0 be the line through vertices one and two. It contains one side of the triangle having length $= s$, see the Figure. We get four geometrical cases depending on which side of B that l_0 intersects besides the x-axis. More precisely, we shall number the sides clockwise around the pentagon letting the side with x_1 have number 0. The case number will be the same as that of the intersected side.

4.1. **Case 1.** Case 1, depicted in Figure 2, occurs when l_0 itersects side one of the affine pentagon i.e. the y-axis in the point $(0, y)$, $0 \leq y \leq 1$. This means that l_0 itersects two adjacent sides of B. The equation for l_0 is

$$l_0 : \eta = -\frac{y}{x}\xi + y.$$

Let $s = \sqrt{(x - x_2)^2 + {y_2}^2}$ be the distance between vertices one and two. For fixed x and y, the maximal value of s is $r_1 = \sqrt{x^2 + y^2}$.

The area of the affine pentagon B_1 in Figure 2 is $area(B_1) = 1 + \frac{a}{2}$. The variable $W = area(T)/area(B_1) = area(T)/(1 + a/2)$ will be less than w if the distance between l_0 and the third vertex is less than $2(1 + a/2)\, w/s$. To avoid the factor $2(1 + a/2) = a + 2$ in numerous places below, we shall use the "normalized" double area $u = (a + 2)w$

in the calculations. The lines l_1 and l_2 have the distance u/s to l_0.

$$l_1 : \eta = -\frac{y}{x}\,\xi + y - \frac{u\,r_1}{s\,x}.$$

$$l_2 : \eta = -\frac{y}{x}\,\xi + y + \frac{u\,r_1}{s\,x}.$$

This means that the conditional probability $P(W \le u/(a+2) \,|\, x,\, y,\, s)$ is proportional to the area between the lines l_1 and l_2 in the pentagon in Figure 2. We shall use the formula $1 + a/2 - T_1 - S_1$ (see Figure 2) for this area and we shall average T_1 over x, y, and s to get the contribution to $P(W \le u/(a+2))$ from Case 1. The averaging of the area S_1 to the right of l_2 will be done in Case 4.

Putting $2\,T_1 = \alpha$ and and using the equation of l_1, we get

$$\alpha = \frac{x}{y}\left(y - \frac{u\,r_1}{s\,x}\right)^2 \text{ if } s > \frac{u\,r_1}{x\,y}\text{ , otherwise } 0\ . \tag{4}$$

We shall determine the densities of x, y, and s. As we noted, x is evenly distributed over $(0, 1)$. The area to the left of l_0 is $xy/2$, so for fixed x, the density is the differential $\frac{x}{2}\,dy$. For fixed x and y consider the small triangle with vertices in $(x, 0)$, $(0, y)$, and $(0, y + dy)$. The fraction of the small triangle below s is $(\frac{s}{r_1})^2$ and the density is the differential $\frac{2s}{{r_1}^2}\,ds$. We shall substitute s by $t = s/r_1$ so that $0 \le t \le 1$. We get $dt = ds/r_1$ and the combined (x, y, t)-density $\rho_1 = xt$. A calculation gives that the integral of ρ_1 over the whole range of (x, y, t) equals $\frac{1}{4}$. Divided by the area of the pentagon, it gives the probability $\frac{1}{2(a+2)} \approx .1382$ for the occurence of Case 1.

Figure 3 shows the domain in (y, t)-space to integrate over for fixed u and x. The decreasing curve is its lower bound $t_\alpha = \frac{u}{x\,y}$ below which $\alpha = 0$. The intersection of the lower and upper t-bounds is the lower bound $y_\alpha = u/x$ for y. We have $y_\alpha < 1$ when $x > u$.

The contribution from Case 1 is the weighted average of α:

$$h_{11}(u) = \int_u^1 x dx \int_{u/x}^1 dy \int_{u/x\,y}^1 \frac{x}{y}\left(y - \frac{u}{t\,x}\right)^2 t\,dt. \tag{5}$$

Maple is helpful in solving integrals of this kind and delivers the following result valid for $0 \le u \le 1$

$$h_{11}(u) = -\frac{1}{3}\,u^3 + \frac{5}{4}\,u^2 - u + \frac{1}{12} - \frac{1}{2}\,u^2\,\log(u)\,(1 - \log(u)). \tag{6}$$

4.2. **Case 2.** This case occurs when l_0 itersects sides zero and two of the pentagon, meaning that the intersected sides of B are separated by one side. We shall use the affine transformation B_2 shown in Figure 4 in the calculations. We have scaled B_2 so that $a \le x \le a + 1$ and $a \le y \le a + 1$. Alternatively, we could have chosen to have x and y in

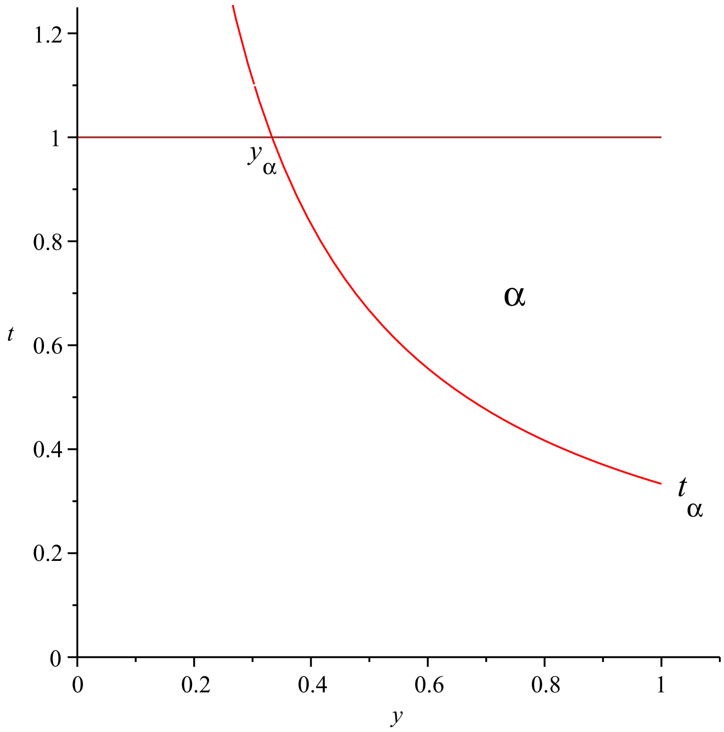

FIGURE 3. Area to integrate t and y over in Case 1 when $x/u = 3$.

the interval $(1, a)$. Neither of these natural choices gives B_2 the same area as B_1, so we have to do a change of scale later to merge the results from the different cases. The x- and y- axes are the same as in Case 1, so $\rho_2 = \rho_1$. Also the expression for l_0 is the same as in Case 1,

Let the intersections between l_1 and the coordinate axes be ξ_1 and η_1 respectively and let the intersection between l_1 and the line $\xi + \eta = a$ have ξ-coordinate ξ_2. Compare Figure 4. The figure is drawn with $x < y$ and $0 < \xi_2 < a$. The contribution to $P(W \leq u/(a+2))$ from Case 2 is twice the area in the pentagon to the left of l_1. We call this quantity α and in the figure it is

$$\alpha = (\eta_1 - a) \cdot \xi_2 = \frac{x}{y-x}\left(y - a - \frac{u}{xt}\right)^2.$$

Figure 5 shows the situation in Case 2 with a smaller u/t. Here, $\xi_2 > a$ and the area to the left of l_1 is instead $\alpha + \beta$, where

$$\beta = (\xi_1 - a) \cdot (a - \xi_2) = \frac{y}{x-y}\left(x - a - \frac{u}{yt}\right)^2.$$

Here, α extends outside the pentagon and β equals minus the part of α outside the pentagon. We have $\alpha \geq 0$ whenever $\eta_1 \geq a$, which is equivalent to $t \geq t_\alpha = \frac{u}{x \cdot (y-a)}$. Otherwise, $\alpha = 0$. We have $\beta \leq 0$

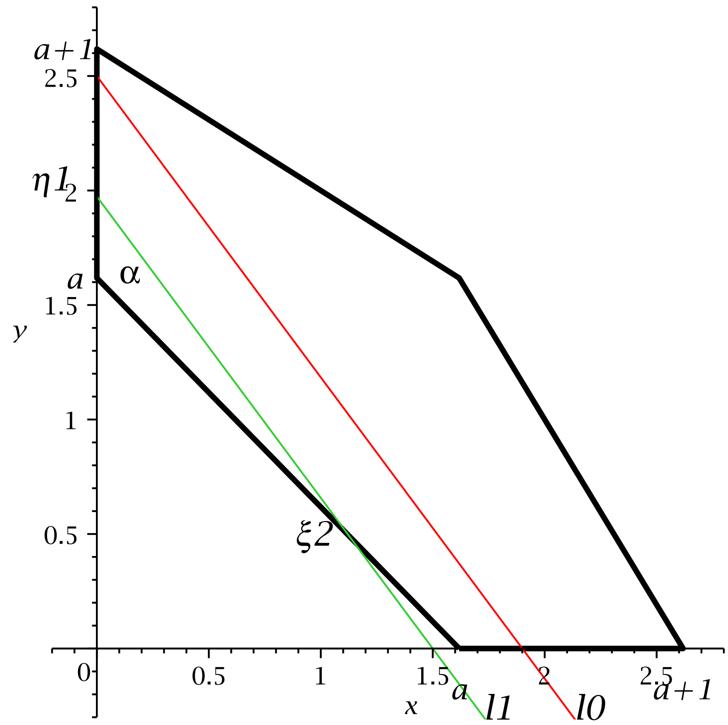

FIGURE 4. Case 2 when $u/t = 1$, $x = 1.9$, and $y = 2.5$.

whenever $\xi_1 \geq a$, which is equivalent to $t \geq t_\beta = \frac{u}{y\cdot(x-a)}$. Otherwise $\beta = 0$.

The areas to integrate t and y over are shown in Figure 6. Figures 4 and 5 are drawn for $y > x$, so we should consider only the part of Figure 6 where $y > x$. In fact, Figure 6 is valid also for $y < x$. The only difference is that $\beta \geq 0$ and $\alpha \leq 0$ for $y < x$. Incidentally, t_α and t_β intersect at $y = x$. Thus, we shall integrate α from t_α to 1 and β from t_β to 1. Noting that β equals α with x and y switched, one could hope that their integration would give the same result. However, ρ_2 is not symmetric in x and y. The reason is that the y-density is calculated for fixed x as is the t-density calculated for fixed x and y. The integrations shall be performed first in t, then in y and last in x. As long as $a < y_\alpha < a + 1$ and $a < y_\beta < a + 1$, we shall calculate

$$k_\alpha(u,x) = \int_{y_\alpha}^{a+1} dy \int_{t_\alpha}^{1} \rho_2\, \alpha\, dt \quad \text{and} \quad k_\beta(u,x) = \int_{y_\beta}^{a+1} dy \int_{t_\beta}^{1} \rho_2\, \beta\, dt\,.$$

Here, the intersection between t_α and 1 is $y_\alpha = a + u/x$, and that between t_β and 1 is $y_\beta = u/(x-a)$. Whenever y_α and y_β are smaller than a, they shall be replaced by a and the integrals are zero when they are bigger than $a + 1$. In Figure 7, we show the lines in (x, u)-space where y_α and y_β are between a and $a + 1$ and indicate where k_α and k_β hold. In the area marked $k_{\beta,0}$ we have $y_\beta < a$ so that the $k_\beta(u,x)$

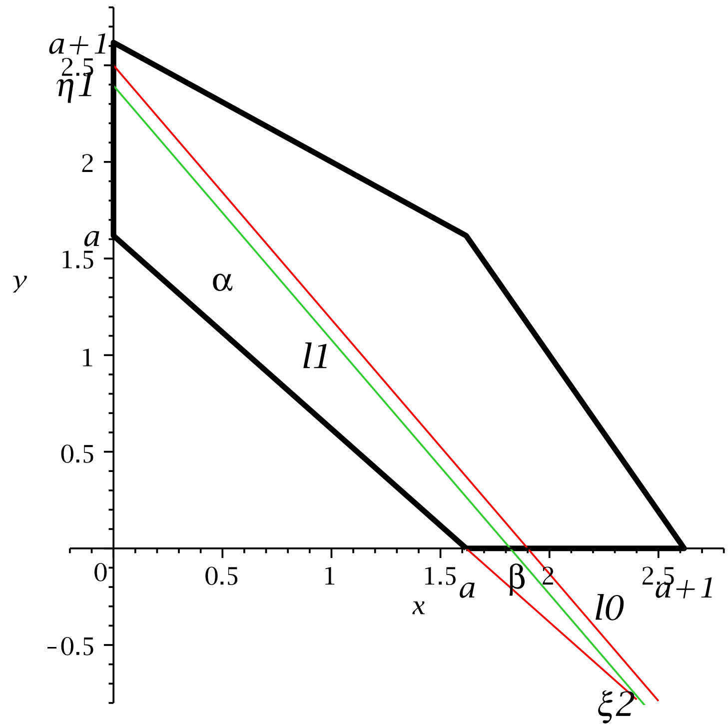

FIGURE 5. Case 2 when $u/t = .2$, $x = 1.9$, and $y = 2.5$.

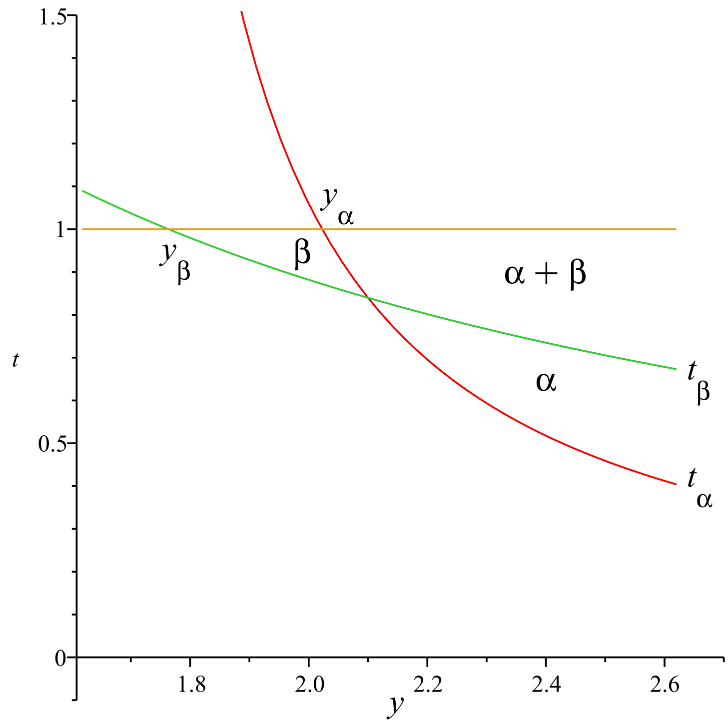

FIGURE 6. The areas to integrate t and y over in Case 2 when $u = .85$ and $x = 2.1$.

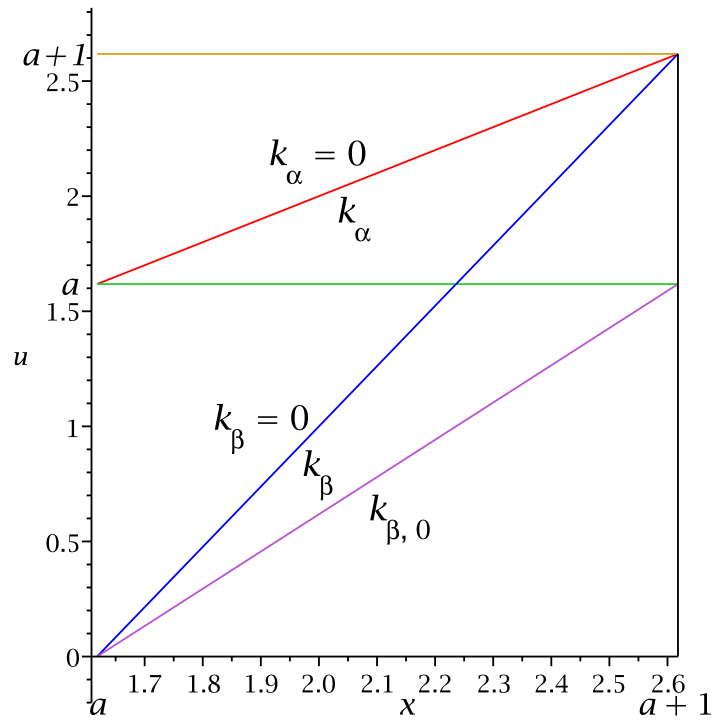

FIGURE 7. The x-intervals to integrate k_α, k_β, and $k_{\beta,0}$ over for different u in Case 2.

given above is not valid and shall be replaced by

$$k_{\beta,0}(u,x) = \int_a^{a+1} dy \int_{t_\beta}^1 \rho_2 \, \beta \, ds \,,$$

where we have added the index 0 to indicate that the lower bound for y is at its bottom value.

We have

(7)

$$h_{21}(u) = \int_a^{a+1} k_\alpha \, dx + \int_{a+u/a^2}^{a+u/a} k_\beta \, dx + \int_{a+u/a}^{a+1} k_{\beta,0} \, dx, \qquad 0 \le u \le a,$$

$$h_{22}(u) = \int_u^{a+1} k_\alpha \, dx + \int_{a+u/a^2}^{a+1} k_\beta \, dx. \qquad a \le u \le a+1.$$

To give an idea of what the evaluation of these integrals look like we give $h_{22}(u)$. This is the simplest of the h_{ij} and has got the form below after a considerable simplification of the result produced by *Maple*. We will supply any interested reader with *Maple*-files giving explicit expressions for other results. The function Li_2 is the dilogarithm, see the Appendix.

(8)

$$
\begin{aligned}
&h_{22}(u) = \\
&= \frac{a\,u^2}{2}\left[\mathrm{Li}_2\left(\frac{a\sqrt{5}\,-\sqrt{a^2+4u}}{2a^2}\right) - \mathrm{Li}_2\left(\frac{a-\sqrt{a^2+4u}}{2a^2}\right)\right. \\
&\left.-\mathrm{Li}_2\left(\frac{\sqrt{a^2+4u}-a}{2}\right) - \mathrm{Li}_2\left(-\frac{\sqrt{a^2+4u}+a}{2}\right) + \mathrm{Li}_2\left(\frac{u}{a^2}\right)\right] \\
&- \frac{a\,u^2}{2}(\log(2a^2) - \log(a+\sqrt{a^2+4u}))\,\log(a^2-u) + \frac{u^2}{2}\log(u)^2 \\
&+ \left(2au^2\log(\frac{a+\sqrt{a^2+4u}}{2}) - 2(3a-1)au^2\log(a) + \frac{u^2}{2a} - \frac{a^3u}{3} - \frac{a^5}{60}\right)\log(u) \\
&- 2au^2\log(a+\sqrt{a^2+4u})^2 + \frac{au^2}{2}\log(2a^2)\log(a\sqrt{5}\,+\sqrt{a^2+4u}) \\
&- \frac{au^2}{2}\left(\log(a\sqrt{5}\,+\sqrt{a^2+4u}) + \log(\frac{a^2}{512})\right)\log(a+\sqrt{a^2+4u}) \\
&+ \frac{1}{60}(\log(4u) - 2\log(a+\sqrt{a^2+4u}))(a^4+18a^2u-64u^2)\sqrt{a^2+4u} \\
&- \frac{1}{15}a^2(2a^5-7a^4+20u+15au^2)\log(a) - \frac{5}{2}au^2\log(2)^2 - \frac{1}{12}au^2\pi^2 \\
&- \frac{1}{60}\left(20\frac{u^3}{a^2} + a^2(38a-113)u^2 - 2a^4(11a-41)u + a^6(4a-9)\right) \\
&+ a^2u^2(11-a)\log(a)^2, \quad a \le u \le a+1.
\end{aligned}
$$

4.3. **Case 3.** This case occurs when l_0 intersects sides zero and three. Like in Case 2, this implies that there is one side between the intersected sides and we shall use the same transformation of the pentagon as in Case 2. See Figures 4 and 5. Cases 2 and 3 are complementary. In Case 2, we studied the area to the left of l_1. Here we shall study the area S_1, see Figure 2, to the right of l_2. In Figure 8, we have drawn l_2 for $u/t = 1$. The relevant intersection points are $P_1 = (\xi_1, 0)$ and $P_2 = (0, \eta_1)$, where l_2 intersects the x, and y-axes, respectively. Futher, $P_3 = (\xi_3, \eta_3)$, and $P_4 = (\xi_4, \eta_4)$ where l_2 intersects the sides of the pentagon through (a, a).

The expression for l_2 is

$$\eta = -\frac{y}{x}\xi + y + \frac{u}{xt},$$

and the density $\rho_3 = \rho_2$. The double area to the right of l_2 in the pentagon will be written as the difference between three triangles as $\alpha + \beta + \gamma$. The triangle with vertices at (a, a), P_3, and P_4 has the double area α. The triangle with vertices at P_1, $(a+1, 0)$, and P_3 shall

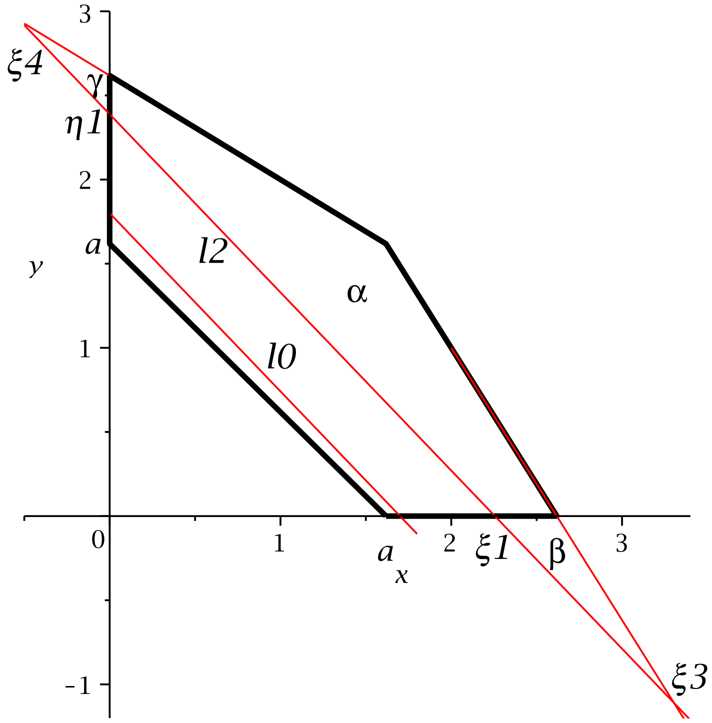

FIGURE 8. Case 3 when $x = 1.7$, $y = 1.8$, and $u/t = 1$.

be subtracted from α and has the signed double area $-\beta$. The triangle with vertices in P_2, $(0, a+1)$, and P_4 has the signed double area $-\gamma$. The expressions

$$\alpha = \frac{a}{(ax-y)(ay-x)}\,(a(x+y) - xy - u/t)^2,$$

$$\beta = -\frac{a}{y(ax-y)}\,(a^2y - xy - u/t)^2, \text{ and}$$

$$\gamma = -\frac{a}{x(ay-x)}\,(a^2x - xy - u/t)^2,$$

are valid where $\alpha > 0$ and $\beta < 0$, and $\gamma < 0$.

We have $\alpha > 0$ when $\xi_3 > a$ (and $\xi_4 < a$) which occurs when

$$t > t_\alpha = \frac{u}{ax + ay - xy}.$$

We have $\beta < 0$ when $\eta_3 < 0$ which occurs when

$$t > t_\beta = \frac{u}{y(a^2 - x)}.$$

We have $\gamma < 0$ when $\xi_4 < 0$ which occurs when

$$t > t_\gamma = \frac{u}{x(a^2 - y)}.$$

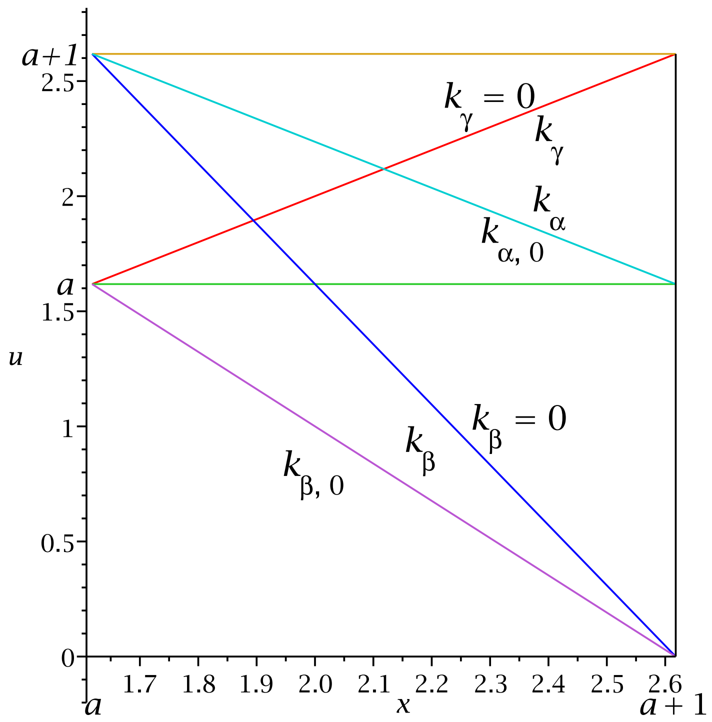

FIGURE 9. The intervals to integrate x over for fixed u in Case 3.

The lower t-bounds intersect the upper bound $t = 1$ at $y_\alpha = \frac{ax-u}{x-a}$, $y_\beta = \frac{u}{a^2-x}$, and $y_\gamma = \frac{a^2x-u}{x}$,. These intersections are not always between a and $a+1$ so we shall calculate

$$k_\alpha(u,x) = \int_a^{y_\alpha} dy \int_{t_\alpha}^1 \rho_3\,\alpha\,dt \quad \text{and} \quad k_{\alpha,0}(u,x) = \int_a^{a+1} dy \int_{t_\alpha}^1 \rho_3\,\alpha\,dt\,,$$

$$k_\beta(u,x) = \int_{y_\beta}^{a+1} dy \int_{t_\beta}^1 \rho_3\,\beta\,dt \;\;, \text{and} \;\; k_{\beta,0}(u,x) = \int_a^{a+1} dy \int_{t_\beta}^1 \rho_3\,\beta\,dt\,,$$

$$k_\gamma(u,x) = \int_a^{y_\gamma} dy \int_{t_\gamma}^1 \rho_3\,\gamma\,dt \;\;.$$

Notice that the y-integration of β goes from y_β to $a+1$ and that $y_\gamma < a+1$ for all x so that there is no $k_{\gamma,0}$.

These functions shall be integrated over the x-intervals shown in Figure 9 leading to the expressions in equation (9). We have written a^2 instead of $a+1$ in these expressions.

(9)

$$\begin{aligned} h_{31}(u) = &\int_a^{a^2} k_{\alpha,0}\, dx + \int_a^{a^2-u/a} k_{\beta,0}\, dx + \int_{a^2-u/a}^{a^2-u/a^2} k_\beta\, dx \\ &+ \int_a^{a^2} k_\gamma\, dx \qquad\qquad 0 \le u \le a, \\ h_{32}(u) = &\int_a^{a^3-u} k_{\alpha,0}\, dx + \int_{a^3-u}^{a^2} k_\alpha\, dx \\ &+ \int_a^{a^2-u/a^2} k_\beta\, dx + \int_u^{a^2} k_\gamma\, dx, \qquad\qquad a \le u \le a+1. \end{aligned}$$

The explicit expressions are not given here.

4.4. **Case 4.** This case occurs when l_0 intersects sides zero and four, meaning that it intersects adjacent sides like in Case 1 implying that Case 4 is complementary to Case 1. We shall calculate the area S_1 to the right of l_2 in Figure 2. Figure 10 describes the involved variables.

The expression for l_2 is the same as in Case 3 and the density is $\rho_4 = \rho_1 = xt$. The signed double triangle areas in Figure 10 are

$$\begin{aligned} \alpha &= \frac{a^2}{(x+ay)(x-y)}\,(ax+y-xy-u/t)^2, \\ \beta &= \frac{a^2}{(ax+y)(y-x)}\,(ay+x-xy-u/t)^2, \\ \gamma &= -\frac{a}{(x+ay)x}\,(xy-x+u/t)^2, \\ \delta &= -\frac{a}{(ax+y)y}\,(xy-y+u/t)^2. \end{aligned}$$

The function to integrate over t, y, and x is $\alpha+\beta+\gamma+\delta$.

Figure 10 is drawn with $x > y$ which makes $\alpha > 0$ and $\beta < 0$. With $x < y$, we have instead $\alpha < 0$ and $\beta > 0$. γ and δ are always negative.

As before, the above expressions hold respectively when

$$t \ge t_\alpha = \frac{u}{ax+y-xy}, \quad t \ge t_\beta = \frac{u}{ay+x-xy}$$
$$t \ge t_\gamma = \frac{u}{x(1-y)}, \quad t \ge t_\delta = \frac{u}{y(1-x)},$$

and are zero otherwise. The intersections between the lower t-bounds and 1 are $y_\alpha = \frac{u-ax}{1-x}$, $y_\beta = \frac{u-x}{a-x}$, $y_\gamma = \frac{x-u}{x}$, $y_\delta = \frac{u}{1-x}$.

These intersections are not always between zero and one so we shall calculate

$$k_\alpha(u,x) = \int_{y_\alpha}^1 dy \int_{t_\alpha}^1 \rho_4\, \alpha\, dt \quad \text{and} \quad k_{\alpha,0}(u,x) = \int_0^1 dy \int_{t_\alpha}^1 \rho_4\, \alpha\, dt\,,$$

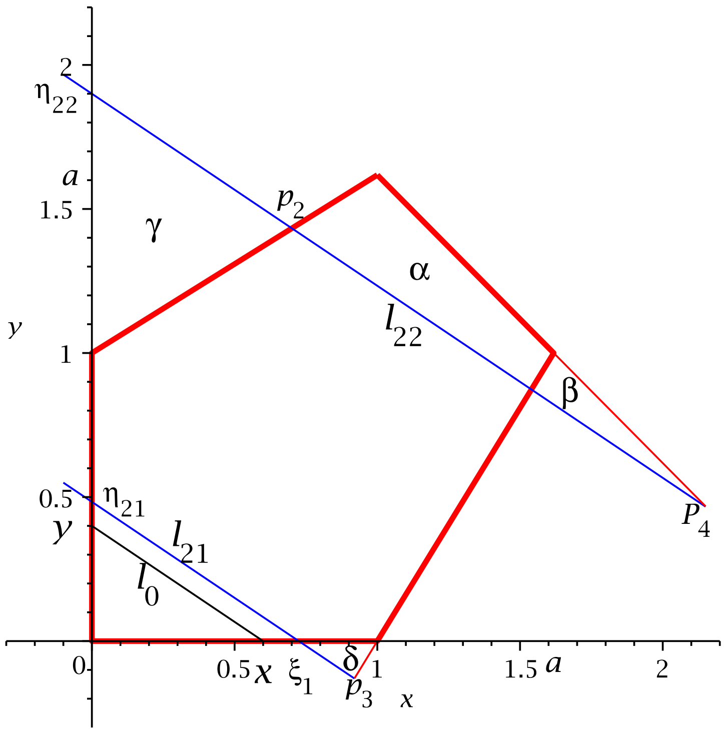

FIGURE 10. The pentagon in Case 4.

$$k_\beta(u,x) = \int_{y_\beta}^{1} dy \int_{t_\beta}^{1} \rho_4\, \beta\, dt \quad \text{and} \quad k_{\beta,0}(u,x) = \int_{0}^{1} dy \int_{t_\beta}^{1} \rho_4\, \beta\, dt\,,$$

$$k_\gamma(u,x) = \int_{0}^{y_\gamma} dy \int_{t_\gamma}^{1} \rho_4\, \gamma\, dt \quad \text{and} \quad k_\delta(u,x) = \int_{y_\delta}^{1} dy \int_{t_\delta}^{1} \rho_4\, \delta\, dt \ \ .$$

These functions shall be integrated over the x-intervals shown in Figure 11.

$$\begin{aligned}(10)\quad h_{41}(u) &= \int_0^{u/a} k_\alpha\, dx + \int_{u/a}^{1} k_{\alpha,0}\, dx + \int_0^u k_\beta\, dx \\ &\quad + \int_u^1 k_{\beta,0}\, dx + \int_u^1 k_\gamma\, dx + \int_0^{1-u} k_\delta\, dx, \qquad 0 \le u \le 1, \\ h_{42}(u) &= \int_{a(u-1)}^{u/a} k_\alpha\, dx + \int_{u/a}^{1} k_{\alpha,0}\ , dx + \int_0^1 k_\beta\, dx, \qquad 1 \le u \le a.\end{aligned}$$

The explicit expressions are not given here.

4.5. **Combination of cases.** Integrating the ρ_i over the whole t-, y- and x-range in each case gives $\frac{1}{4}$, $\frac{a^3}{4}$, $\frac{a^3}{4}$, $\frac{1}{4}$, respectively. Dividing these numbers by the area of the pentagon in each case gives the probability

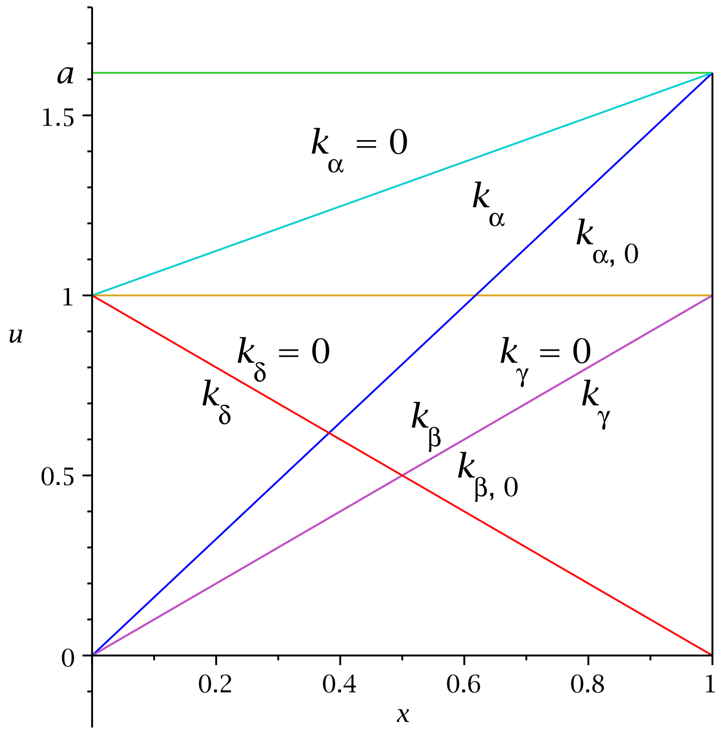

FIGURE 11. The intervals to integrate x over for fixed u in Case 4.

for the case. The area of the pentagon in Figure 2 is $T_{14} = \frac{a+2}{2}$ and in Figure 4 it is $T_{23} = \frac{3a+1}{2}$. The probabilities are $p_1 = p_4 = \frac{\sqrt{5}}{10\,a} \approx .1382$ and $p_2 = p_3 = \frac{a\sqrt{5}}{10} \approx .3618$.

The calculated h_{nm} must be scaled before they are combined. The variable u ranges from 0 to a in cases 1 and 4 and from 0 to a^2 in cases 2 and 3. The first scaling is to replace u by $a\,u$ in the h_{nm} for cases 2 and 3. Then, the h_{nm} are divided twice by the area of the pentagon that they are integrated over, once to get the probabilities add to one and once because we are stydying the ratio between the area of the triangle and the pentagon. We get the probability distribution function $H(u)$ for twice the area of a random triangle in a pentagon, when one of the triangle vertices sits on the boundary of the pentagon:

(11)

$$H_1(u) = 1 - \frac{1}{{T_{14}}^2}(h_{11}(u) + h_{41}(u)) - \frac{1}{{T_{23}}^2}(h_{21}(au) + h_{31}(au)), \quad 0 \le u < 1,$$

$$H_2(u) = 1 - \frac{1}{{T_{14}}^2}h_{42}(u) - \frac{1}{{T_{23}}^2}(h_{22}(au) + h_{32}(au)), \qquad 1 \le u < a.$$

5. Combination of the V- and W-distributions.

Let $F(x)$ be the distribution function for the triangle area X in a regular pentagon with unit area. We have $X \leq x$ when $VW = UV/(a+2) \leq x$. Putting $x = y/(a+2) = y/a\sqrt{5}$, this happens when $UV \leq y$ and we get

$$\begin{aligned}
F(y/a\sqrt{5}\,) &= \int_0^a G(y/u)\,dH(u) = \\
&= [G(y/u)H(u)]_0^a - \int_y^a H(u)\frac{d}{du}G(y/u)\,du = \\
&= G(y/a) - \int_y^a H(u)\frac{d}{du}G(y/u)\,du = \\
&= \frac{y^3}{a^3} - 3\,y^3 \int_y^a u^{-4}\,H(u)\,du, \quad 0 \leq y \leq a.
\end{aligned} \tag{12}$$

The partial intergration in (12) is used to avoid integrating to the lower bound $u = 0$. Substituting $y = (a+2)x = a\sqrt{5}\,x$ in the result will give us the wanted $F(x)$ given in (14) and (15), with x ranging from 0 to $\frac{a}{a+2} = \frac{1}{\sqrt{5}}$.

To write the result, we need the function

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt. \tag{13}$$

This is the dilogarithm function discussed by Euler in 1768 and named by Hill, [5]. Some properties of $\mathrm{Li}_2(x)$ are given in Appendix A.

We will not carry out the integration (12) in detail. To avoid the factor $a+2$ in numerous places, we will write the result as a function of $y = (a+2)x = a\sqrt{5}\,x$ as it stands in (12). We have used both the constant a and the constant $\sqrt{5} = 2a-1$ to keep the expressions as short as possible. We display the result in Figure 12 in the form of the density $f(x) = \frac{dF}{dx}$ of the distribution. The range of x is $0 \leq x < \frac{1}{\sqrt{5}}$. We save the comments on our way to the result and on the result to the next sections.

(14)

$$F_1(\frac{y}{a\sqrt{5}}) =$$

$$\frac{24\sqrt{5}\,y^2}{5a^3}\left(a - \frac{2}{3}y\right)\left[\mathrm{Li}_2\left(\frac{1+\sqrt{1+4y/a}}{2a}\right) + \mathrm{Li}_2\left(\frac{1-\sqrt{1+4y/a}}{2a}\right)\right.$$

$$\left. + \log(\sqrt{1+4y/a}+1)\log(\sqrt{1+4y/a}-1) - \log(2)\log(2y/a)\right]$$

$$+ \frac{24\sqrt{5}\,y^2}{5a^3}\left(a^4 + \frac{2}{3}y\right)\left[\mathrm{Li}_2\left(\frac{a^2-\sqrt{a^4-4y}}{2a}\right) - \mathrm{Li}_2\left(\frac{1/a-\sqrt{a^4-4y}}{2a}\right)\right.$$

$$\left. - \log\left(\frac{\sqrt{a^4-4y}-1/a}{2a}\right)\log\left(\frac{a^2+\sqrt{a^4-4y}}{2a}\right)\right]$$

$$- \frac{24\sqrt{5}\,y^2}{5a^2}\mathrm{Li}_2\left(\frac{y}{a}\right) + \frac{4\sqrt{5}\,y^3}{5a^3}(\log(y))^2$$

$$+ \frac{\sqrt{5}}{75a^3}\left[(a^3 + 28a^2y - 324ay^2)\sqrt{(1+4y/a)} - a^3 - 30a^2y + 270ay^2\right]\log(y)$$

$$- \frac{\sqrt{5}}{75a^3}\left[(a^8 - 28a^4y - 324y^2)a^2\sqrt{a^4-4y}\right.$$

$$\left. +(a^2 + 28ay - 324y^2)a\sqrt{1+4y/a} - 2(41a + 26 - 30a^5y - 270a^2y^2)\right]\log(a-y)$$

$$+ \frac{2\sqrt{5}}{75a}(a^8 - 28a^4y - 324y^2)\sqrt{a^4-4y}\,\log\left(\frac{1+a\sqrt{a^4-4y}}{2a}\right)$$

$$+ \frac{2\sqrt{5}}{75a^2}(a^2 + 28ay - 324y^2)\sqrt{1+4y/a}\,\log\left(1 + \frac{\sqrt{1+4y/a}-1}{2y}\right)$$

$$- \frac{12\sqrt{5}\,y^2}{5a^3}(a\sqrt{5} + \frac{4}{3}y)(\log(a))^2 - \frac{4\sqrt{5}\,a}{75}(a^3 + \frac{11}{2}a^2 - 45ay - 135y^2)\log(a)$$

$$- \frac{4\sqrt{5}\,\pi^2y^2}{75a^3}(33a + 24 + 2y) + \frac{2\sqrt{5}}{25a}(2a+31)y - \frac{19}{5}\frac{y^2}{a^2},$$

$$0 \le y < 1.$$

(15)

$$
\begin{aligned}
&F_2(\frac{y}{a\sqrt{5}}) = \\
&1 - \frac{24\sqrt{5}\,y^2}{5a^3}\left(a^4 + \frac{2}{3}y\right)\left[\mathrm{Li}_2\left(\frac{a^2 - \sqrt{a^4-4y}}{2a}\right) - \mathrm{Li}_2\left(\frac{1/a - \sqrt{a^4-4y}}{2a}\right)\right. \\
&\left. + \log\left(\frac{a^2 + \sqrt{a^4-4y}}{2a}\right)\left(\log\left(a^2 - \sqrt{a^4-4y}\right) - \log\left(\sqrt{a^4-4y} - 1/a\right)\right)\right] \\
&+ \frac{8\sqrt{5}\,y^2}{5a^3}(3a^2 + y)\,\mathrm{Li}_2\left(\frac{y}{a}\right) \\
&+ \frac{24\sqrt{5}}{5a^3}\left(\frac{1}{4} + \frac{73}{180}a - \frac{1}{3}a^5 y - \frac{3}{2}a^3y^2 + a^4y^2\log(a)\right)\log(y) \\
&+ \frac{\sqrt{5}}{75a^2}\left(a(a^8 - 28a^4y - 324y^2)\sqrt{a^4-4y}\right. \\
&\qquad\left. -25 - 37a + (60 + 150a)y + 270y^2\right)\log(a-y) \\
&+ \frac{24\sqrt{5}\,y^2}{5a^3}\left(a - \frac{2}{3}y\right)\left(\log(2)\log(\frac{2y}{a})\right. \\
&\qquad\qquad\left. - \log(\sqrt{1+4y/a}+1)\log(\sqrt{1+4y/a}-1)\right) \\
&- \frac{\sqrt{5}}{75a^2}(a^2 + 28ay - 324y^2)\sqrt{1+4y/a}\left(\log(\frac{4y}{a}) - 2\log(1+\sqrt{1+4y/a})\right) \\
&- \frac{\sqrt{5}}{75a}(a^8 - 28a^4y - 324y^2)\sqrt{a^4-4y}\left(\log(y) - 2\log(\frac{a^2+\sqrt{a^4-4y}}{1/a+\sqrt{a^4-4y}})\right) \\
&+ \frac{\sqrt{5}\,a}{25}(270y^2 - 17a - 13)\log(a) + 2a^5\log(a)y - 24a(\log(a))^2y^2 \\
&+ \frac{4\sqrt{5}\,\pi^2y^2}{75a^3}(3a^2(3a-2) + y) - \frac{a\sqrt{5}}{25}(5 + a + 4(1-6a)y + 19y^2), \\
&\quad 1 \le y < a.
\end{aligned}
$$

It may look as if the terms of F_1 and F_2 haven't been collected in an optimal way. The reason is that we have written the expressions so that each term is finite in every point of its domain.

The density function $f(x) = \frac{dF}{dx}$ is plotted in Figure 12. Its domain is $0 \le x < \frac{1}{\sqrt{5}}$.

The first, second, and third moments of the distribution are obtained by integration

$$\alpha_1 = \int_0^{\frac{1}{\sqrt{5}}} x\,f(x)dx = \frac{4a+7}{180} = \frac{1}{20} + \frac{\sqrt{5}}{90} \approx .074845,$$

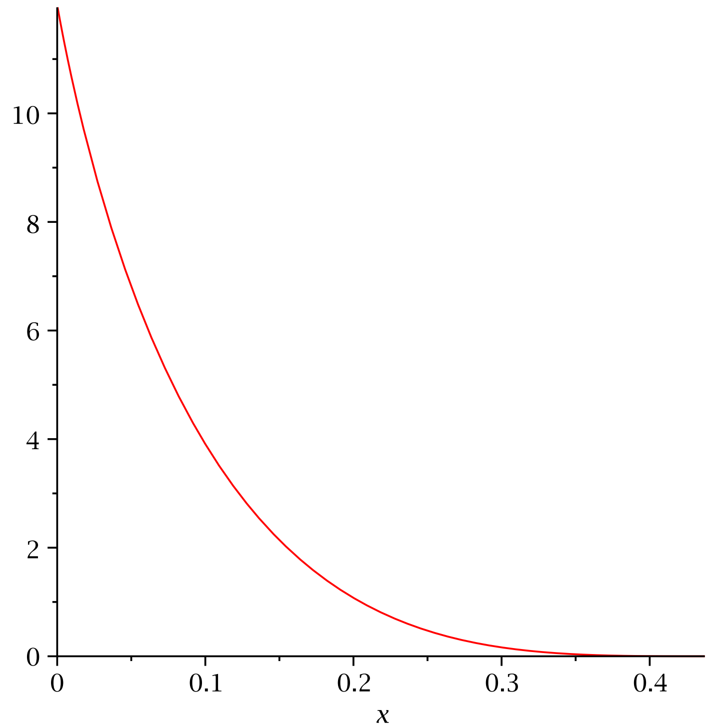

FIGURE 12. The density function $f(x) = \frac{dF(x)}{dx}$ for the area fraction of a random triangle in a regular pentagon. x ranges from 0 to $1/\sqrt{5}$.

$$\alpha_2 = \int_0^{\frac{1}{\sqrt{5}}} x^2 f(x)dx = \frac{4a+23}{3000} = \frac{1}{120} + \frac{\sqrt{5}}{1500} \approx .009824.$$

$$\alpha_3 = \int_0^{\frac{1}{\sqrt{5}}} x^3 f(x)dx = \frac{114a+199}{225000} = \frac{32}{28125} + \frac{19\sqrt{5}}{75000} \approx .001704.$$

6. THE CALCULATION OF THE h_{mn}.

The calculations leading to the h_{mn} leads to solving singular integrals. On several occasions we had to use limiting procedures when inserting the boundaries. We will not digress on this here but refer the reader to section 6 in [11], which discusses the used methods.

Usually, *Maple* answers with its dilog function dilog(x) $= \mathrm{Li}_2(1-x)$. During the simplifications of this paper, we had to deduce relations between the dilog functions. Here are some examples which hold when $a = \frac{\sqrt{5}+1}{2}$ and $0 < y < a$.

(16)

$$\operatorname{dilog}\left(\frac{a^2-\sqrt{a^4-4y}}{2a}\right)+\operatorname{dilog}\left(\frac{a^2+\sqrt{a^4-4y}}{2a}\right)-\operatorname{dilog}\left(\frac{y}{a}\right)$$
$$+\operatorname{dilog}\left(\frac{a^2-\sqrt{a^4-4y}}{2}\right)+\operatorname{dilog}\left(\frac{a^2+\sqrt{a^4-4y}}{2}\right)+\frac{1}{2}(\log a)^2=0$$

(17)

$$\operatorname{dilog}\left(\frac{2a-1-\sqrt{1+4y/a}}{2a}\right)+\operatorname{dilog}\left(\frac{2a-1+\sqrt{1+4y/a}}{2a}\right)$$
$$+\operatorname{dilog}\left(\frac{a}{2}(2a-1-\sqrt{1+4y/a})\right)+\operatorname{dilog}\left(\frac{a}{2}(2a-1+\sqrt{1+4y/a})\right)$$
$$+\operatorname{dilog}\left(\frac{y}{a}\right)+\log(y/a)\log(a-y)-\log(y)\log a+3(\log a)^2=\frac{1}{6}\pi^2$$

$$\log(a^2-\sqrt{3a-2})+2\log(a\sqrt{3a-2}+1)=\log(a)+3\log(2). \tag{18}$$

(19)

$$2\operatorname{dilog}\left(\frac{a^2-\sqrt{3a-2}}{2}\right)+2\operatorname{dilog}\left(\frac{a^2+\sqrt{3a-2}}{2}\right)$$
$$+4(\log(a\sqrt{3a-2}+1))^2-4\log(4a)\log(a\sqrt{3a-2}+1)=-(\log(4a))^2$$

Some dilogs involving a have explicit expressions like

$$\begin{aligned}
&\operatorname{dilog}(a)=\frac{(\log a)^2}{2}-\frac{\pi^2}{15}, \qquad \operatorname{dilog}(a^2)=(\log a)^2-\frac{\pi^2}{10},\\
&\operatorname{dilog}\left(\frac{1}{a}\right)=\frac{\pi^2}{15}-(\log a)^2,\\
&\operatorname{dilog}(-a)=\frac{4\pi^2}{15}-(\log a)^2-2i\pi\log a.
\end{aligned} \tag{20}$$

7. The golden ratio.

The appearance of the golden ratio $a=\frac{1+\sqrt{5}}{2}$ in the pentagon calculations stems from the fact that $\cos\frac{2\pi}{5}=\frac{1}{2a}$. We shall describe where it enters into the calculation. This occurs when we do the affine transformation of the regular pentagon to the pentagons in Figure 2 and Figure 4. The transformation to Figure 2 is chosen so that three adjacent vertices sit in $(1,0)$, $(0,0)$ and $(0,1)$. The "fourth" vertex, i.e. the one at the top in Figure 2, then sits in $(1,a)$. We do the same kind of transformation also for the square and the hexagon calculation. When done for a regular n-gon, the fourth vertex gets the position

$\left(1, \frac{\sin(3\pi/n)}{\sin(\pi/n)}\right)$. For $n = 4$, 5, 6, 8, and 10 this gives the y-coordinates of point four: 1, a, 2 , $1 + \sqrt{2}$, and $1 + a$, respectively. This means that a enters for $n = 5$ and 10.

Iterating the equation $a^2 = a + 1$, we get $a^3 = 2a + 1$, $a^4 = 3a + 2$, $a^5 = 5a + 3$, etc., which are linear expressions for the powers of a. The coefficients are the Fibonacci numbers. The iteration can also be run backwards giving $a^{-1} = a - 1$, $a^{-2} = 2 - a$, $a^{-3} = 2a - 3$, etc.. A great part of the calculations of this paper consists of simplifying expressions using these linear expressions. We have constructed two maple procedures, one for replacing powers of a with linear expressions and one for going the other way looking for linear expressions that can be replaced by powers of a. They have been indispensable in the calculations.

8. Comparison with triangle, square, and hexagon results.

We have given the distribution functions for the area of a triangle in a triangle, in a square, and in a regular hexagon in references [8], [10], [11], respectively. Even though the expressions for the densities f_3, f_4, f_5, and f_6, in the four cases are widely different, their graphs are almost identical. All four take the value 12 at the origin and their derivatives at the origin are $-16\pi^2$. Their second derivatives are infinite at the origin. Their domains of definition are $(0, 1)$ for the triangle, $(0, \frac{1}{2})$ for the square and the hexagon, and $(0, \frac{1}{\sqrt{5}})$ for the pentagon. We have plotted all four in Figure 13. Only f_3 for the triangle can be distinguished from the other three. To give an idea of the difference between the curves, we give their values at $x = .1$: $f_3(.1) \approx 3.652$, $f_4(.1) \approx 3.848$, $f_5(.1) \approx 3.908$, and $f_6(.1) \approx 3.926$.

9. Concluding comment.

We have not shown any integral calculations in detail. In principle, they are elementary, which doesn't mean that they don't require a substantial effort. The calculations would not have been possible without some tool like *Maple* or Mathematica for handling the large number of terms that come out of the integrations. This doesn't mean that *Maple* performs the integrations automatically. Often, we had to split up the integrands into parts and treat each part in a special way. We had to do some partial integrations manually. A substantial part of the work has been spent on simplifying the integrals that *Maple* produces.

It should be pointed out that *Maple* isn't reproducible in the sense that it doesn't always give exactly the same answer. The terms often come in a different order when you rerun a calculation. This implies that partial results must be saved.

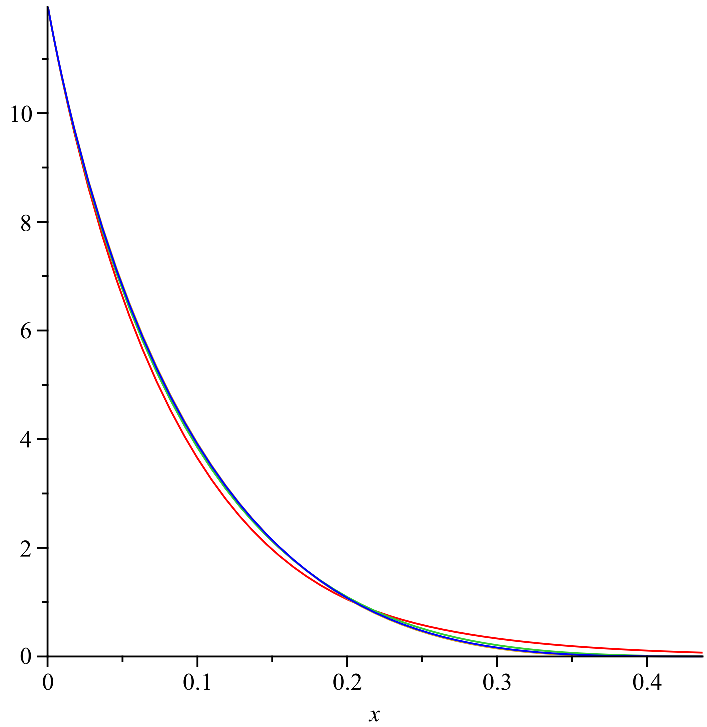

FIGURE 13. The density functions for the area of a triangle in a triangle, square, pentagon, and hexagon. It is the triangle density that is descernable from the other three, which are very close.

We will supply any interested reader with *Maple* files describing the calculations.

APPENDIX A

The dilogarithm function $\mathrm{Li}_2(x)$ is defined in [6] and [16] for complex x as

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt. \tag{21}$$

When x is real and greater than unity, the logarithm is complex. A branch cut from 1 to ∞ can give it a definite value.

We have the series expansion

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}\ , \quad |x| \leq 1. \tag{22}$$

This implies that $\mathrm{Li}_2(\mathrm{x})$ is analytic in the unit circle. Although the series (22) is only convergent for $|x| \leq 1$, the integral in (21) is not restricted to these limits and $\mathrm{Re}(\mathrm{Li}_2(\mathrm{x}))$ is defined and is real on the whole real axis. We use this function for $|x| < 1$.

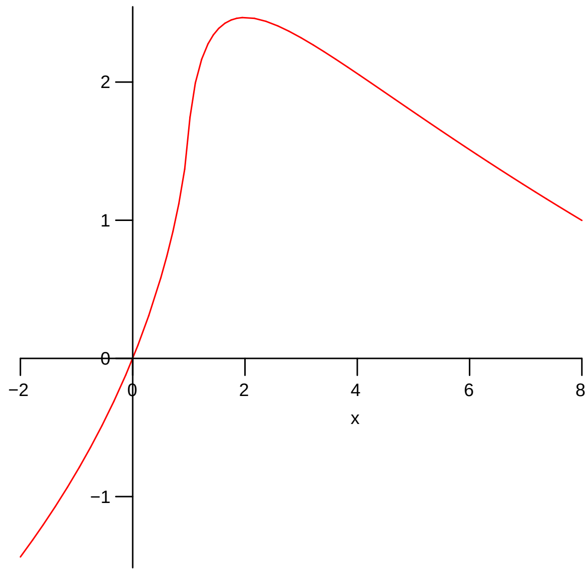

FIGURE 14. The function $\mathrm{Re}(\mathrm{Li}_2(\mathrm{x}))$.

The definition of the dilogarithm function has varied a little from author to author. Maple has the function $\mathrm{polylog}(2, x)$ which is defined by the series expansion (22) for $|x| \leq 1$ otherwise by analytic continuation. Maple also has a function $\mathrm{dilog}(x) = \mathrm{Li}_2(1 - x)$ defined on the whole real axis. Maple's dilog function is the same as the dilog function given in [1], page 1004.

$\mathrm{Re}(\mathrm{Li}_2(\mathrm{x}))$ is increasing from $\mathrm{Re}(\mathrm{Li}_2(0)) = 0$ via $\mathrm{Re}(\mathrm{Li}_2(1)) =\pi^2/6$ to its maximum $\mathrm{Re}(\mathrm{Li}_2(2)) = \pi^2/4$.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* Dover Publications, Inc., New York, 1965

[2] H. A. Alikoski, Über das Sylvestersche Vierpunktproblem *Ann. Acad. Sci. Fenn.* 51 (1938), no. 7, pp. 1-10.

[3] W. Blaschke, *Vorlesungen über Differentialgeometrie Vol 2.* Springer, Berlin 1923.

[4] C. Buchta and M. Reitzner, *The convex hull of random points in a tetrahedron: Solution of Blaschke's problem and more general results J. reine angew. Math.* 536 (2001), 1-29.

[5] C. J. Hill, Journal für Math. (Crelle) 3:107, 1828

[6] L. Lewin, *Polylogarithms and Associated functions* North Holland, New York, Oxford, 1981.

[7] R. E. Miles, Isotropic Random Simplices *Advances in Appl. Probability* 3 (1971), pp. 353-382.

[8] J. Philip, The Area of a Random Convex Polygon. TRITA MAT 04 MA 07 (2004). `www.math.kth.se/~johanph/area12.pdf`.

[9] J. Philip, The area of a random triangle in a square. TRITA MAT 10 MA 01 (2010). `www.math.kth.se/~johanph/squaref.pdf`.

[10] J. Philip, The Area of a Random Convex Polygon in a Triangle. TRITA MAT 05 MA 04 (2005). `www.math.kth.se/~johanph/area21.pdf`.

[11] J. Philip, The area of a random triangle in a regular hexagon. TRITA MAT 10 MA 12 (2010). `www.math.kth.se/~johanph/hexagon.pdf`.

[12] A. Rényi and R. Sulanke, Über die konvexe Hülle von n zufällig gewählte Punkten *Z. Wahrscheinlichkeitstheorie* 2(1963), pp. 75-84.

[13] A. Rényi and R. Sulanke, Über die konvexe Hülle von n zufällig gewählte Punkten, II *Z. Wahrscheinlichkeitstheorie* 3(1964) pp. 138-147.

[14] L. Santaló, *Integral Geometry and Geometric Probability* Encyclopedia of mathematics and its Applications, Addison-Wesley 1976

[15] M. Trott, The Area of a Random Triangle in a Square, 1.10.1 in *The Mathematica GuideBook for Symbolics*, Springer-Verlag, pp. 298-311, 2006

[16] Digital Library of Math. Fcns. `http://dlmf.nist.gov`

Department of Mathematics, Royal Institute of Technology, S-10044 Stockholm Sweden

E-mail address: `johanph@kth.se`