MATCHING WITH MULTIPLE APPLICATIONS: A NOTE

JOHAN PHILIP

Dept. Mathematics, Royal Institute of Technology, 10044 Stockholm, Sweden.
email: johanph@math.kth.se

1. Introduction

The following matching problem has been considered in papers by Albrecht et al. (2003), [1], [2], and Tan (2003), [4]. There are u unemployed people who each send a applications to v vacancies. The problem is to estimate the number of jobs that become assigned. We propose an algorithm that handles this problem.

2. Notations and formulation.

The u senders each pick a receivers at random and sends one letter to each. There are v receivers. Every receiver that gets one or more letters, picks one at random and answers it. The problem is:

Find the probability that a sender gets at least one answer $= 1 - pr0$, where

$$pr0 = \text{Prob}\{\text{a sender gets no answer}\}.$$

Without loss of generality, we shall calculate the probability for sender #1. The receivers of letters from sender #1 will be numbered #1 through #a.

3. Preliminary calculation.

We have

$$p = \text{Prob}\{\text{an arbitrary sender hits an arbitrary receiver}\} = a/v.$$

We get

$$p_k = \text{Prob}\{\text{receiver \#1 gets } k \text{ letters from the competing } u-1 \text{ senders}\} = \binom{u-1}{k} p^k (1-p)^{u-1-k}. \quad (1)$$

By the random responding policy, we get

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$q_0 = \text{Prob}\{\text{receiver \#1 does not respond to sender \#1}\} =$

$$\sum_{k=1}^{u-1}(1-\frac{1}{k+1})p_k = 1 - \frac{1-(1-p)^u}{up}. \quad (2)$$

In paper [1], it was tacitly assumed that the number of letters received by #1, ., ., #a are independent random variables leading to the (incorrect) result

$$pr0 = {q_0}^a. \quad (3)$$

The fact that the number of letters received by two receivers are dependent was detected by Tan, [4]. She also gave the expression for $pr0$ for the case $a = 2$, and indicated the expression for bigger a.

4. The finite parameter case.

We shall describe a way of calculating $pr0$. The description will be carried out for $a = 3$, but the generalization to bigger a is immediate. We shall calculate the joint probability $p(k_1, k_2, k_3)$ that k_1, k_2, and k_3 competing letters arrive at receivers #1, #2, and #3, respectively. In analogy with (2), we then have

$$pr0 = \sum_{k_1=1}^{u-1}\sum_{k_2=1}^{u-1}\sum_{k_3=1}^{u-1}\frac{k_1}{k_1+1}\frac{k_2}{k_2+1}\frac{k_3}{k_3+1}p(k_1,k_2,k_3). \quad (4)$$

An outcome of the random sending process is defined by the binary indicator functions m_i of $a = 3$ binary variables

$$m_i(j_1,j_2,j_3) = \begin{cases} 1, & \text{if the competing sender \#i sends } j_r \text{ letters to receiver \#r} \\ 0, & \text{otherwise} \end{cases}$$

Here, $2 \le i \le u$, $1 \le r \le a$, and $j_r = 0, 1$, since a sender sends 0 or 1 letters to a receiver. This means that the domain of m_i has $2^a = 8$ elements and $m_i = 1$ on one of them

$$\sum_{j_1}\sum_{j_2}\sum_{j_3} m_i(j_1,j_2,j_3) = 1.$$

Define

$$m(j_1,j_2,j_3) = \sum_{i=2}^{u} m_i(j_1,j_2,j_3)$$

so that

$$\sum_{j_1}\sum_{j_2}\sum_{j_3} m(j_1, j_2, j_3) = u - 1. \tag{5}$$

The $2^a = 8 \quad m(j_1, j_2, j_3)$ take integer values in the range $(0, u-1)$. The number of competing letters arriving at receivers #1 - #3 are

$$\begin{aligned} k_1 &= \sum_{j_2}\sum_{j3} m(1, j_2, j3) \\ k_2 &= \sum_{j_1}\sum_{j3} m(j_1, 1, j3) \\ k_3 &= \sum_{j_1}\sum_{j2} m(j_1, j_2, 1). \end{aligned} \tag{6}$$

We need the probability that the outcome described by the numbers $m(j_1, j_2, j_3)$ occurs.
Define the probabilities $s(i), 0 \leq i \leq a$, pertaining to one sender and the group of receivers #1 - #a.

$s(0) = \text{Prob}\{\text{no letter hits the group}\}$
$s(i) = \text{Prob}\{\text{only letters \#1 through \#i hit the group}\}$ for $1 \leq i \leq a$.

Taking into account the possible permutations of the letters, we have

$$\sum_{i=0}^{a} \binom{a}{i} s(i) = 1.$$

The $s(i)$ are probabilities for sampling without replacement. We give the expressions for $a = 3$:

$$\begin{aligned} s(0) &= \frac{v-3}{v} \quad \frac{v-4}{v-1} \quad \frac{v-5}{v-2}, \\ s(1) &= \frac{3}{v} \quad \frac{v-3}{v-1} \quad \frac{v-4}{v-2}, \\ s(2) &= \frac{3}{v} \quad \frac{2}{v-1} \quad \frac{v-3}{v-2}, \\ s(3) &= \frac{3}{v} \quad \frac{2}{v-1} \quad \frac{1}{v-2}. \end{aligned}$$

The event that receivers #1 - #3 gets k_1,k_2, and k_3 letters, respectively, can happen in a multinomial number of ways with probabilities derivable from the $s(i)$. We get

$$p(k_1, k_2, k_3) = \sum_{\substack{\text{all indicator functions } m \\ \text{satisfying (5) and (6)}}} (u-1)! \prod \frac{s(j_1+j_2+j_3)^{m(j_1,j_2,j_3)}}{m(j_1,j_2,j_3)!} \tag{7}$$

The computer algorithm for doing the calculations generates all the 2^a-tiples m satisfying (5). According to Feller, [3], page 52, the number of such 2^a-tiples is $\binom{2^a+u-2}{u-1}$. For each such m, the algorithm calculates the k_j from (6). The corresponding term in the sum of (7) is calculated, multiplied by the factors $k_j/(k_j+1)$ of(4), and added to $pr0$.

5. The limit case

Paper [2] derives the limit value

$$pr0(\lambda) = \lim_{\substack{u=\lambda v \\ v\to\infty}} pr0,$$

where λ is a constant.
We obtain the limit value by letting $v \to \infty$ in (7). In the limit, we have $v^i s(i) \to$ constant. We shall show that the only terms that contribute to $pr0$ are those with one of the indices $j_r = 1$ and the others equal to 0. The resulting k_j are

$$\begin{aligned} k_1 &= m(1,0,0) \\ k_2 &= m(0,1,0) \\ k_3 &= m(0,0,1). \end{aligned}$$

The corresponding term in (7) is

$$s(0)^{u-1-k_1-k_2-k_3} \cdot \frac{(u-1)!}{(u-1-k_1-k_2-k_3)!} \cdot \frac{s(1)^{k_1+k_2+k_3}}{k_1!k_2!k_3!}.$$

When $v \to \infty$, this becomes proportional to

$$s(0)^u \cdot \frac{(us(1))^{k_1+k_2+k_3}}{k_1!k_2!k_3!}, \tag{8}$$

where $s(0)^u$ tends to an exponential and $us(1) \to a\lambda$. Any term of (7) with more than one $j_r = 1$ has factors $us(i)$, $i \geq 2$, which tend to zero when $v \to \infty$.
The expression (8) can be split into three factors each depending on only one k_j. This implies that (7) in the limit is the product of probabilities of independent events. It follows that (3) is correct in the limit. The limit $v \to \infty$ can be taken in (2) and inserted in (3) giving

$$pr0(\lambda) = \left(1 - \frac{1 - \exp(-a\lambda)}{a\lambda}\right)^a.$$

Two computer files (matching.bas and matching.exe) for doing the described calculations can be found on the authors homepage: www.math.kth.se/~johanph/

References

[1] Albrecht, J., P. Gauthier, and S. Vroman, 2003, Matching with Multiple Applications, Economics Letters, 78, 67-70.

[2] Albrecht, J., P. Gauthier, and S. Vroman, 2003, Matching with Multiple Applications: The limiting case, Georgetown University and Erasmus University, Rotterdam mimeo.

[3] Feller, W. 1950, An Introduction to Probability Theory and Its Applications, Vol. 1, (John Wiley)

[4] Tan, S. 2003, Matching with Multiple Applications: A Correction, Graduate Group in Economics, University of Pennsylvania