THE AREA OF A RANDOM TRIANGLE IN A REGULAR HEXAGON.

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Abstract. We determine the distribution function for the area of a random triangle in a regular hexagon.

1. Introduction

We shall denote the regular hexagon by K and the random triangle by T and shall consider the random variable $X = area(T)/area(K)$. It is well known that an affine transformation will preserve the ratio X. This follows from the fact that the area scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [13]. J. J. Sylvester considered the problem of a random triangle T in an arbitrary convex set K and posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown [2], that $\kappa = \frac{289}{3888}$ for K a regular hexagon.

A. Reńyi and R. Sulanke, [11] and [12], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. R. E. Miles [7] generalizes these asymptotic estimates for K a circle to higher dimensions. C. Buchta and M. Reitzner, [4], has given values of κ (generalized to three dimensions) for $n \geq 4$ points in a tetrahedron. H. A. Alikoski [2] has given expressions for κ when T is a triangle and K a regular r-polygon.

Here, we shall deduce the distribution function for X. We have done this before when K is a square in, [8] and [9] and when K is a triangle in [10]. The method used here is the same as in [9]. We hope to be able to present the result of applying the same method to a regular pentagon.

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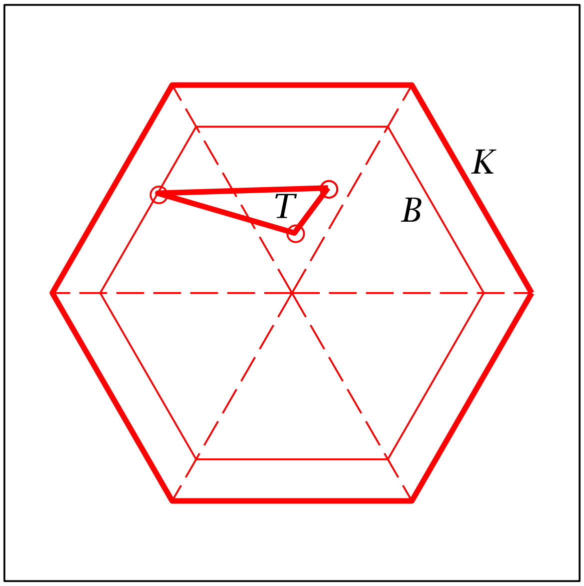

FIGURE 1. The regular hexagon K, the random triangle T and the shrunken hexagon B .

2. Notation and formulation.

We use a constant probability density for generating three random points in the regular hexagon K. Let T be the convex hull of the three points. Compare Figure 1. We shall determine the probability distribution of the random variable $X = area(T)/area(K)$.

Our method will be to shrink the hexagon around its midpoint until one of its sides hits a triangle point. The shrunken hexagon is denoted B. The random variable X that we study will be written as the product of two random variables

$$V = area(B)/area(K) \text{ and } W = area(T)/area(B).$$

One of the triangle points stops the shrinking and determines V. Since the density of the points is rotation invariant, we can use a local coordinate system at the stopping point with one axis along the side of the hexagon and one orthogonal to the side. It's the point's orthogonal coordinate that determines the shrink. The coordinate along this side is independent of the former and is consequently evenly distributed along the side. Since the coordinates of the other two triangle points are independent of the hitting point, it follows that V and W are independent.

We shall determine the distributions of V and W and combine them to get the distribution of $X = VW$.

3. The distribution of V.

V is the area of the shrunken hexagon B. The hexagon K is the sum of six equilateral triangles. Each of the three points of the random triangle sits in one of these equilateral triangles. Focussing on such an equilateral triangle, we measure the distance from the random point to the center of the hexagon othogonally to the side of the triangle that is part of the hexagon. Denote this distance by S. The distribution function for S is $L(s) = c \cdot s^2$, where c is a constant. Choosing a scale so that $s = 1$ on the boundary, we have $c = 1$. The largest of the three distances has the distribution function $L(s_{max})^3 = (s_{max})^6$. The area of B is $v = area(K) \cdot (s_{max})^2$. We get

$$G(v) = \text{Prob}(V < v) = (s_{max})^6 = v^3, \quad 0 \leq v \leq 1. \tag{1}$$

By the argument used here, this $G(v)$ holds for any regular r-polygon.

4. The distribution of W.

W is the area of a random triangle having one vertex on the boundary of a regular hexagon $= B$ and the other two vertices in the interior of the hexagon. Since the area ratio W is invariant under affine transformations of B, we will use the B shown in Figures 2 and 4, having $area(B) = 3$. Without loss of generality, we will number the three triangle vertices so that vertex one is the one sitting on the boundary and we let this boundary be the x-axis, so that vertex one is $(x, 0)$. The position of the second vertex is (x_2, y_2). Let l_0 be the line through vertices one and two. It contains one side of the triangle. Our calculations will be divided into five cases depending on where l_0 intersects B. More precisely, we shall number the sides clockwise around the hexagon beginning with the side along the y-axis as number one.

4.1. **Case 1.** Case 1, depicted in Figure 2, occurs when l_0 itersects side one of the hexagon i.e. the y-axis in the point $(0, y)$, $0 \leq y \leq 1$. This means that l_0 itersects two adjacent sides of B. The equation for l_0 is

$$l_0 : \eta = -\frac{y}{x}\,\xi + y.$$

Let $s = \sqrt{(x - x_2)^2 + {y_2}^2}$ be the distance between vertices one and two. For fixed x and y, the maximal value of s is $r_1 = \sqrt{x^2 + y^2}$.

The variable $W = area(T)/area(B) = area(T)/3$ will be less than w if the distance between l_0 and the third vertex is less than $6\,w/s$. To

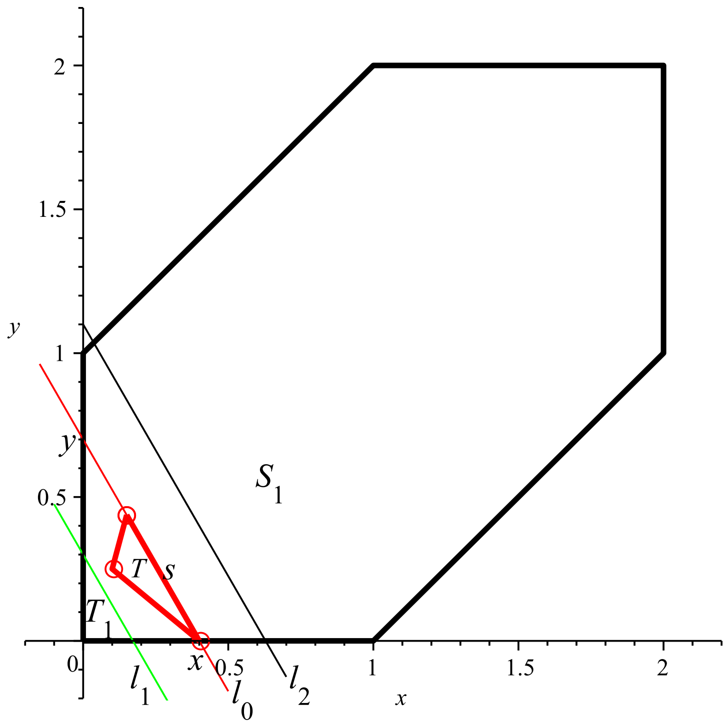

FIGURE 2. Case 1. The line l_0 trough vertices one and two intersects the left side of the hexagon.

avoid the factor 6 in numerous places below, we shall use the "normalized" double area $u = 6w$ in the calculations. The lines l_1 and l_2 have the distance u/s to l_0.

$$l_1 : \eta = -\frac{y}{x}\,\xi + y - \frac{u\,r_1}{s\,x}.$$

$$l_2 : \eta = -\frac{y}{x}\,\xi + y + \frac{u\,r_1}{s\,x}.$$

This means that the conditional probability $P(W \le u/6 \,|\, x,\, y,\, s)$ is proportional to the area between the lines l_1 and l_2 in the hexagon in Figure 2. The hexagon B has area 3 and we shall use the formula $3 - T_1 - S_1$ (see Figure 2) for this area and we shall average T_1 and S_1 over x, y, and s to get the contribution to $P(W \le u/6)$ from Case 1. In fact, when we consider all possible directions of l_0 in all our cases, it follows from a symmetry argument that the areas to the left of l_1 will be the same as those to the right of l_2. This implies that it suffices to calculate the areas to the left and then double the result. Thus, we will average $2\,T_1$ over x, y, and s and neglect S_1. Another way of putting this is to say that we treat T_1 here in Case 1 and S_1 in Case 5.

Putting $2\,T_1 = \alpha$ and and using the equation of l_1, we get

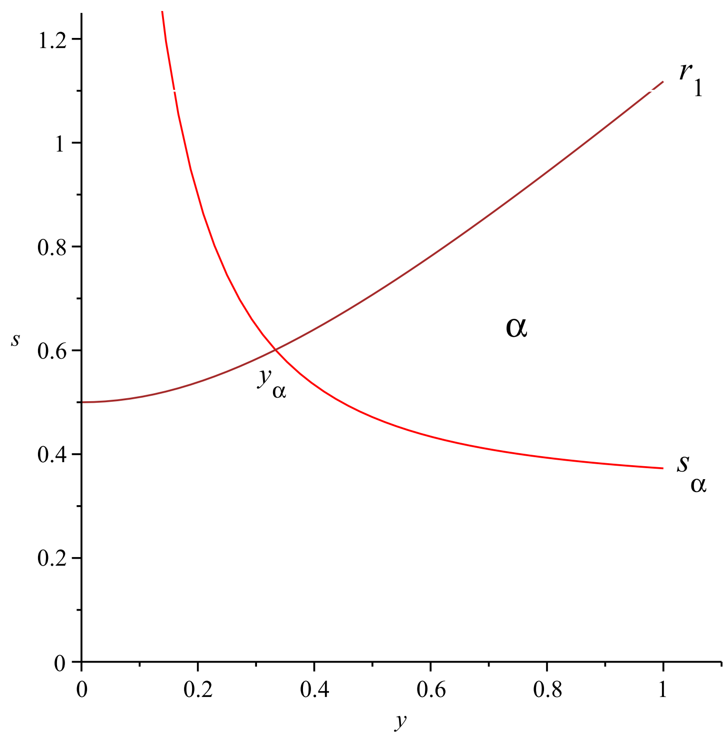

FIGURE 3. Area to integrate s and y over in Case 1 when $u = 1/6$ and $x = 1/2$.

$$\alpha = \frac{x}{y}\left(y - \frac{u\,r_1}{s\,x}\right)^2 \text{ if } s > \frac{u\,r_1}{x\,y}\ , \text{ otherwise } 0\ . \tag{2}$$

We shall determine the densities of x, y, and s. As we noted, x is evenly distributed over $(0, 1)$. The area to the left of l_0 is $xy/2$, so for fixed x, the density is the differential $\frac{x}{2}\,dy$. For fixed x and y consider the small triangle with vertices in $(x, 0)$, $(0, y)$, and $(0, y+dy)$. The fraction of the small triangle below s is $(\frac{s}{r_1})^2$ and the density is the differential $\frac{2s}{{r_1}^2}\,ds$. A calculation gives that the integral of the combined density $\rho_1 = x\,s/{r_1}^2$ over the whole range of (x, y, s) equals $\frac{1}{4}$. Divided by the area of the hexagon, it gives the probability $\frac{1}{12}$ for the occurence of Case 1.

Figure 3 shows the range in (s, y)-space to integrate over for fixed u and x. The increasing curve is the upper bound r_1 for s and the decreasing curve is its lower bound $s_\alpha = \frac{u\,r_1}{x\,y}$. The intersection of the lower and upper s-bounds is the lower bound $y_\alpha = u/x$ for y. We have $y_\alpha < 1$ when $x > u$.

The contribution from Case 1 is the weighted average of α:

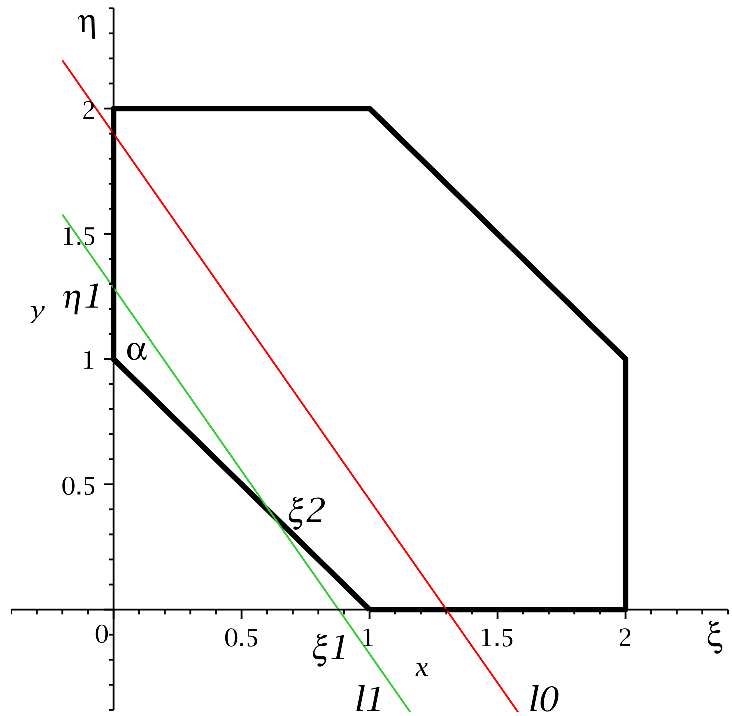

FIGURE 4. Case 2 when $u/s = .35$, $x = 1.3$, and $y = 1.9$.

$$h_1(u) = \int_u^1 x dx \int_{u/x}^1 {r_1}^{-2} dy \int_{u\, r_1/x\, y}^{r_1} \frac{x}{y} \left(y - \frac{u\, r_1}{s\, x}\right)^2 s ds. \tag{3}$$

Maple is helpful in solving integrals of this kind and delivers the following result valid for $0 \le u \le 1$

$$h_{11}(u) = -\frac{1}{3} u^3 + \frac{5}{4} u^2 - u + \frac{1}{12} - \frac{1}{2} u^2 \log(u) \left(1 - \log(u)\right). \tag{4}$$

4.2. **Case 2.** This case occurs when l_0 itersects sides zero and two of the hexagon, meaning that the intersected sides of B are separated by one side. We shall use the affine transformation of B shown in Figure 4 in the calculations. The expression for l_0 is the same as in Case 1, but here, $1 \le x \le 2$ and $1 \le y \le 2$. The maximal value of s and the density ρ_2 have the same expressions as in Case 1. They are $r_2 = \sqrt{x^2 + y^2}$ and $\rho_2 = xs/{r_2}^2$.

Let the intersections between l_1 and the coordinate axes be ξ_1 and η_1 respectively and let the intersection between l_1 and the line $\xi + \eta = 1$ have ξ-coordinate ξ_2. Compare Figure 4. The figure is drawn with $x < y$ and $0 < \xi_2 < 1$. The contribution to $P(W \le u/6)$ from Case 2 is twice the area in the hexagon to the left of l_1. We call this quantity

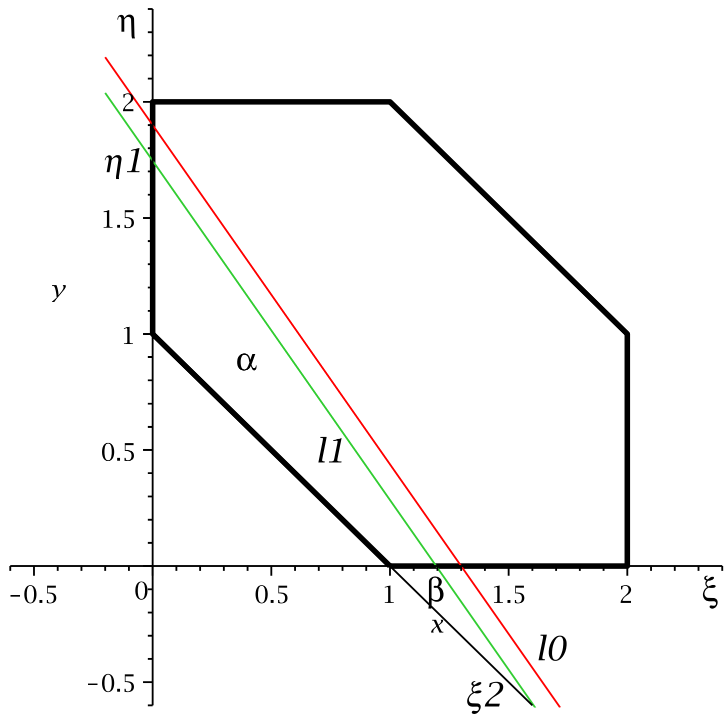

FIGURE 5. Case 2 when $u/s = .1$, $x = 1.3$, and $y = 1.9$.

α and in the figure it is

$$\alpha = (\eta_1 - 1) \cdot \xi_2 = \frac{x}{y-x}\left(y - 1 - \frac{ur_2}{sx}\right)^2.$$

Figure 5 shows the situation in Case 2 with a smaller u/s. Here, $\xi_2 > 1$ and the area to the left of l_1 is instead $\alpha + \beta$, where

$$\beta = (\xi_1 - 1) \cdot (1 - \xi_2) = \frac{y}{x-y}\left(x - 1 - \frac{ur_2}{sy}\right)^2.$$

Here, α extends outside the hexagon and β equals minus the part of α outside the hexagon. We have $\alpha \geq 0$ whenever $\eta_1 \geq 1$, which is equivalent to $s \geq s_\alpha = \frac{u \cdot r_2}{x \cdot (y-1)}$. Otherwise, $\alpha = 0$. We have $\beta \leq 0$ whenever $\xi_1 \geq 1$, which is equivalent to $s \geq s_\beta = \frac{u \cdot r_2}{y \cdot (x-1)}$. Otherwise $\beta = 0$.

The areas to integrate s and y over are shown in Figure 6. Figures 4 and 5 are drawn for $y > x$, so we should consider only the part of Figure 6 where $y > x$. In fact, Figure 6 is valid also for $y < x$. The only difference is that $\beta \geq 0$ and $\alpha \leq 0$ for $y < x$. Incidentally, s_α and s_β intersect at $y = x$. Thus, we shall integrate α from s_α to r_2 and β from s_β to r_2. Noting that β equals α with x and y switched, one could hope that their integration would give the same result. However, ρ_2 is not symmetric in x and y. The reason is that the y-density is

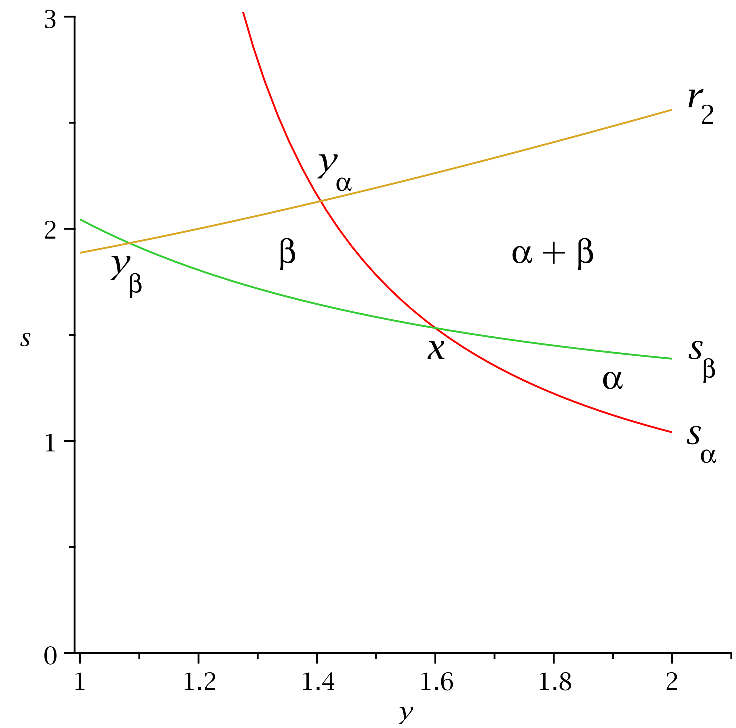

FIGURE 6. The areas to integrate s and y over in Case 2 when $u = .65$ and $x = 1.6$.

calculated for fixed x as is the s-density calculated for fixed x and y. The integrations shall be performed first in s, then in y and last in x. As long as $1 < y_\alpha < 2$ and $1 < y_\beta < 2$, we shall calculate

$$k_\alpha(u, x) = \int_{y_\alpha}^{2} dy \int_{s_\alpha}^{r_2} \rho_2\, \alpha\, ds \quad \text{and} \quad k_\beta(u, x) = \int_{y_\beta}^{2} dy \int_{s_\beta}^{r_2} \rho_2\, \beta\, ds\,.$$

Here, the intersection between s_α and r_2 is $y_\alpha = 1 + u/x$, and that between s_β and r_2 is $y_\beta = u/(x - 1)$. Whenever y_α and y_β are smaller than one, they shall be replaced by one and the integrals are zero when they are bigger than two. In Figure 7, we show the lines in (x,u)-space where y_α and y_β are one and two and indicate where k_α and k_β hold. In the area marked $k_{\beta,0}$ we have $y_\beta < 1$ so that the $k_\beta(u, x)$ given above is not valid and shall be replaced by

$$k_{\beta,0}(u, x) = \int_{1}^{2} dy \int_{s_\beta}^{r_2} \rho_2\, \beta\, ds\,,$$

where we have added the index 0 to indicate that the lower bound for y is at its bottom value.

We have

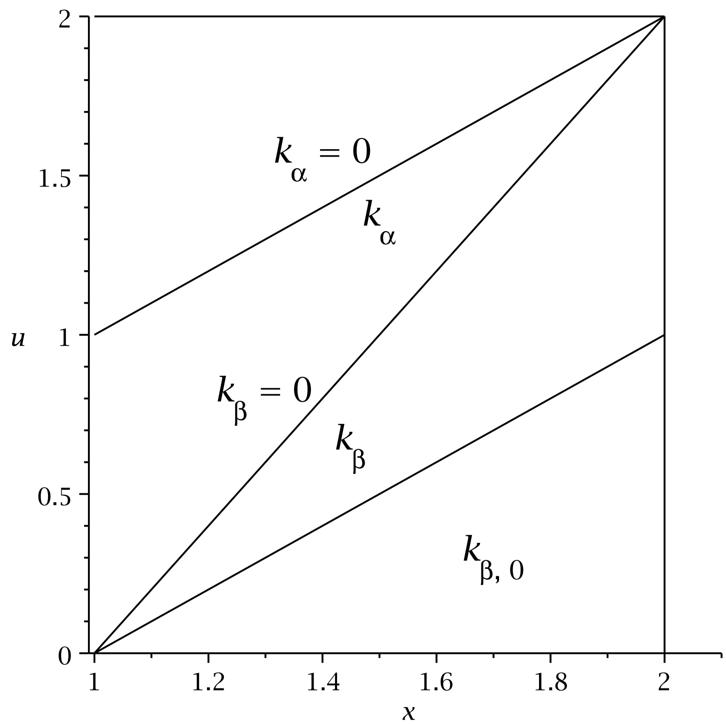

FIGURE 7. The x-intervals to integrate k_α, k_β, and $k_{\beta,0}$ over for different u in Case 2.

$$h_{21}(u) = \int_1^2 k_\alpha\,dx + \int_{1+u/2}^{1+u} k_\beta\,dx + \int_{1+u}^2 k_{\beta,0}\,dx, \qquad 0 \le u \le 1, \tag{5}$$

$$h_{22}(u) = \int_u^2 k_\alpha\,dx + \int_{1+u/2}^{2} k_\beta\,dx. \qquad 1 \le u \le 2.$$

To give an idea of what the evaluation of these integrals look like we give $h_{21}(u)$. We will supply any interested reader with *Maple*-files giving explicit expressions for other results.

(6)

$$
\begin{aligned}
h_{21}(u) = 2u^2 &\left[\operatorname{dilog}\left(\frac{u}{2}\right) - \operatorname{dilog}\left(\frac{3+\sqrt{1+4u}}{4}\right) - \operatorname{dilog}\left(\frac{3-\sqrt{1+4u}}{4}\right)\right. \\
&\left. + \ln(1+\sqrt{1+4u})^2 - \log(4u)\, \log(1+\sqrt{1+4u})\right] \\
&+ \frac{1}{60}(64u^2 - 18u - 1)\sqrt{1+4u}\left[2\log\left(\frac{3+\sqrt{1+4u}}{1+\sqrt{1+4u}}\right) + \log(\frac{u}{2-u})\right] \\
&+ \left[\left(\frac{3}{2} + 2\log\frac{u}{2}\right)u^2 + 3u - \frac{21}{20}\right]\log(2-u) \\
&+ \left[-\left(\frac{1}{2} + \log(2)\right)u^2 + \frac{1}{3}u + \frac{1}{60}\right]\log(u) \\
&+ \frac{1}{6}u^3 + \left(\frac{11}{2}\log(2)^2 - \frac{3}{2}\log(2) + \frac{37}{30}\right)u^2 - \left(\frac{8}{3}\log(2) + \frac{53}{15}\right)u \\
&+ \frac{16}{15}\log(2) + 1, \quad 0 \le u < 1.
\end{aligned}
$$

Here, dilog is *Maple's* dilog function. See Appendix A.

4.3. **Case 3.** This case occurs when l_0 itersects the bottom and top sides of B, see Figure 8. Denote the coordinate for the intersection with the top side by z. We let z increase from zero to one along the top side. The expression for l_1 is

$$\eta = \frac{2}{1+z-x}\,(\xi - x + \frac{ur_3}{2s}),$$

where $r_3 = \sqrt{2^2 + (1+z-x)^2}$. The density is $\rho_3 = 2s/{r_3}^2$. Like in Case 2, we describe the area to the left of l_1 as the difference between triangles. In Figure 8, α is twice the area of the big triangle with vertices in the points marked ξ_2, η_1, and the point $(0,1)$. The quantities called β, and γ are minus twice the areas marked β and γ in the figure, so that the quantity to be integrated is $\alpha+\beta+\gamma$. We have the expressions

$$
\begin{aligned}
\alpha &= \frac{1}{1-(z-x)^2}\,\left(1+z+x-\frac{ur_3}{s}\right)^2, \\
\beta &= -\frac{1}{2(1+z-x)}\,\left(2x-\frac{ur_3}{s}\right)^2, \\
\gamma &= -\frac{1}{2(1-z+x)}\,\left(2z-\frac{ur_3}{s}\right)^2.
\end{aligned}
$$

We have $\alpha \ge 0$ whenever $s > s_\alpha = \frac{ur_3}{1+z+x}$, otherwise 0, $\beta \le 0$ whenever $s > s_\beta = \frac{ur_3}{2x}$, otherwise 0, and $\gamma \le 0$ whenever $s > s_\gamma = \frac{ur_3}{2z}$, otherwise 0.

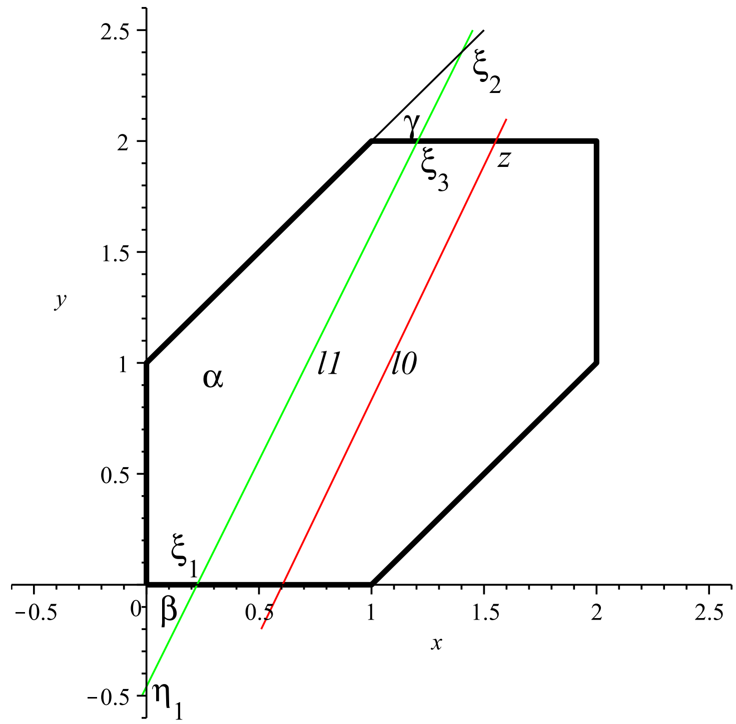

FIGURE 8. Case 3 when $u/s = 1/3$, $x = z = .6$.

The intersection of s_α and r_3 is $z_\alpha = u - 1 - x$. s_β and r_3 don't intersect but $s_\beta < r_3$ when $u < 2x$. s_γ and r_3 intersect at $z_\gamma = u/2$. We have $0 < z_\alpha < 1$ when $u - 2 < x < u - 1$. We shall calculate

$$k_\alpha(u,x) = \int_{z_\alpha}^1 dz \int_{s_\alpha}^{r_3} \rho_3\, \alpha\, ds \quad \text{and} \quad k_{\alpha,0}(u,x) = \int_0^1 dz \int_{s_\alpha}^{r_3} \rho_3\, \alpha\, ds\,,$$

$$k_\beta(u,x) = \int_0^1 dz \int_{s_\beta}^{r_3} \rho_3\, \beta\, ds \quad \text{and} \quad k_\gamma(u,x) = \int_{z_\gamma}^1 dz \int_{s_\gamma}^{r_3} \rho_3\, \gamma\, ds\,.$$

These functions shall be integrated over the x-intervals shown in Figure 9. We get

(7)

$$h_{31}(u) = \int_0^1 k_{\alpha,0}\, dx + \int_{u/2}^1 k_\beta\, dx + \int_0^1 k_\gamma\, dx, \qquad 0 \le u \le 1,$$

$$h_{32}(u) = \int_0^{u-1} k_\alpha\, dx + \int_{u-1}^1 k_{\alpha,0}\, dx + \int_{u/2}^1 k_\beta\, dx + \int_0^1 k_\gamma\, dx, \qquad 1 \le u \le 2.$$

$$h_{33}(u) = \int_{u-2}^1 k_\alpha\, dx, \qquad 2 \le u \le 3.$$

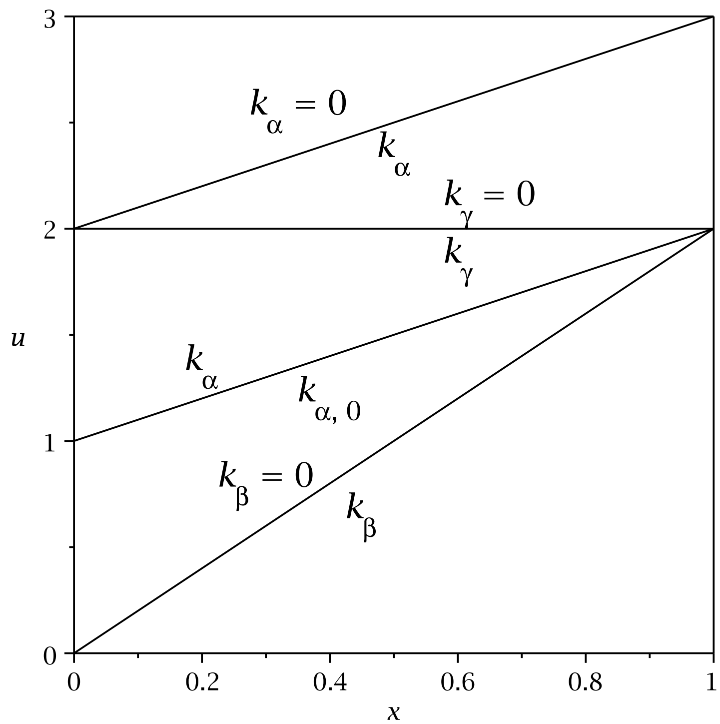

FIGURE 9. The x-intervals to integrate over for fixed u in Case 3.

We shall not burden this account with the explicit expressions.

4.4. **Case 4.** This case occurs when l_0 intersects sides zero and four. Like in Case 2, this implies that there is one side between the intersected sides and we shall use the same transformation of the hexagon as in Case 2. See Figure 4. The two cases are complementary. In Case 2, we studied the area to the left of l_1. Here we shall study the area to the right of l_1 in Figure 10.

The expression for l_1 is

$$\eta = -\frac{y}{x}\xi + y + \frac{ur_4}{sx},$$

where $r_4 = r_2 = \sqrt{x^2 + y^2}$. The density is $\rho_4 = \rho_2 = sx/{r_4}^2$. The signed double triangle areas in Figure 10 are

$$\alpha = (1 - \eta_2)(2 - \xi_2) = \frac{(y(x-2) - x + ur_4/s)^2}{x(y-x)},$$

$$\beta = (1 - \xi_2)(\xi_3 - 1) = \frac{(x(y-2) - y + ur_4/s)^2}{y(x-y)},$$

$$\gamma = \eta_2(2 - \xi_1) = -\frac{(y(x-2) + ur_4/s)^2}{xy},$$

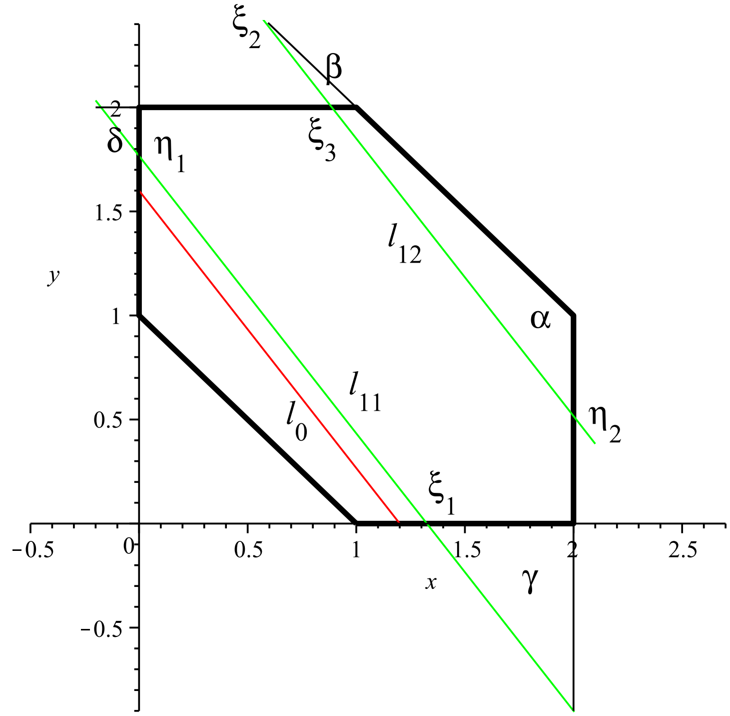

FIGURE 10. Case 4 when $x = 1.2$ and $y = 1.6$. To demonstrate the various triangles, two versions of l_1 are drawn, namely l_{11} for $u/s = .1$ and l_{12} for $u/s = 1$.

$$\delta = \xi_3(2 - \eta_1) = -\frac{(x(y-2) + ur_4/s)^2}{xy}.$$

The area to integrate over s, y, and x is $\alpha + \beta + \gamma + \delta$. With $y > x$ as in Figure 10, we have $\alpha \geq 0$, $\beta \leq 0$. With $y < x$ it is the other way around. γ and δ are always negative.

As before, the above expressions hold respectively when

$$s \geq s_\alpha = \frac{ur_4}{y(2-x) + x}, \quad s \geq s_\beta = \frac{ur_4}{x(2-y) + y}$$
$$s \geq s_\gamma = \frac{ur_4}{y(2-x)}, \quad s \geq s_\delta = \frac{ur_4}{x(2-y)},$$

and are zero otherwise. The intersections between r_4 and the lower s-bounds are $y_\alpha = \frac{u-x}{2-x}$, $y_\beta = \frac{2x-u}{x-1}$, $y_\gamma = \frac{u}{2-x}$, and $y_\delta = 2 - \frac{u}{x}$. These intersections are not always between one and two so we shall calculate

$$k_\alpha(u, x) = \int_{y_\alpha}^{2} dy \int_{s_\alpha}^{r_4} \rho_4 \, \alpha \, ds \quad \text{and} \quad k_{\alpha,0}(u, x) = \int_{1}^{2} dy \int_{s_\alpha}^{r_4} \rho_4 \, \alpha \, ds \,,$$

$$k_\beta(u, x) = \int_{1}^{y_\beta} dy \int_{s_\beta}^{r_4} \rho_4 \, \beta \, ds \quad \text{and} \quad k_{\beta,0}(u, x) = \int_{1}^{2} dy \int_{s_\beta}^{r_4} \rho_4 \, \beta \, ds \,,$$

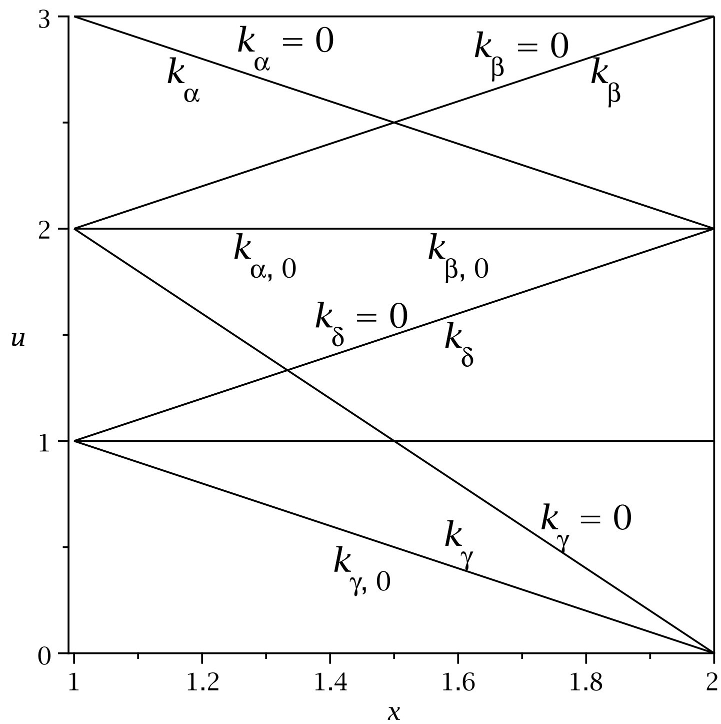

FIGURE 11. The intervals to integrate x over for fixed u in Case 4.

$$k_\gamma(u,x) = \int_{y_\gamma}^{2} dy \int_{s_\gamma}^{r_4} \rho_4\, \gamma\, ds \quad \text{and} \quad k_{\gamma,0}(u,x) = \int_{1}^{2} dy \int_{s_\gamma}^{r_4} \rho_4\, \gamma\, ds\,,$$

$$k_\delta(u,x) = \int_{1}^{y_\delta} dy \int_{s_\delta}^{r_4} \rho_4\, \delta\, ds \ \ .$$

Notice that the y-integrations of β and δ go from one to y_β and y_δ, respectively. We always have $y_\delta \le 2$.

These functions shall be integrated over the x-intervals shown in Figure 11.

$$
\begin{aligned}
(8)\qquad h_{41}(u) &= \int_1^2 k_{\alpha,0}\,dx + \int_1^2 k_{\beta,0}\,dx + \int_1^{2-u} k_{\gamma,0}\,dx \\
&\quad + \int_{2-u}^{2-u/2} k_\gamma\,dx + \int_1^2 k_\delta\,dx, && 0 \le u \le 1, \\
h_{42}(u) &= \int_1^2 k_{\alpha,0}\,dx + \int_1^2 k_{\beta,0}\,dx \\
&\quad + \int_1^{2-u/2} k_\gamma\,dx + \int_u^2 k_\delta\,dx, && 1 \le u \le 2. \\
h_{43}(u) &= \int_1^{4-u} k_\alpha\,dx + \int_{u-1}^2 k_\beta\,dx, && 2 \le u \le 3.
\end{aligned}
$$

The explicit expressions are not given here.

4.5. **Case 5.** This case occurs when l_0 intersects sides zero and five, meaning that it intersects adjacent sides like in Case 1. Case 5 is complementary to Case 1, and we shall use the hexagon in Figure 2. The line marked l_2 there will be our l_1 in Figure 12 and we shall consider the area to the right of l_1. In fact, Figure 12 has two versions of l_1 drawn for different values of u.

The expression for l_1 is

$$\eta = -\frac{y}{x}\xi + y + \frac{ur_5}{sx},$$

where $r_5 = r_1 = \sqrt{x^2+y^2}$. The density is $\rho_5 = \rho_1 = sx/{r_5}^2$. The signed double triangle areas in Figure 12 are

$$
\begin{aligned}
\alpha &= (2-\eta_1)(2-\xi_3) = \frac{(xy-2x-2y+ur_5/s)^2}{xy}, \\
\beta &= -(1-\eta_1)(2-\xi_4) = -\frac{(xy-x-2y+ur_5/s)^2}{x(x+y)}, \\
\gamma &= -(1-\xi_2(1-\xi_3) = -\frac{(xy-y-2x+ur_5/s)^2}{y(x+y)}, \\
\delta &= -(1-\xi_1)(1-\xi_4) = -\frac{(xy-y+ur_5/s)^2}{y(x+y)}, \\
\epsilon &= -\xi_2(1-\eta_2) = -\frac{(xy-x+ur_5/s)^2}{x(x+y)}.
\end{aligned}
$$

The function to integrate over s, y, and x is $\alpha+\beta+\gamma+\delta+\epsilon$. We always have α positive and the others negative.

As before, the above expressions hold respectively when

$$
s \ge s_\alpha = \frac{ur_5}{2x+2y-xy}, \quad s \ge s_\beta = \frac{ur_5}{x+y(2-x)}
$$
$$
s \ge s_\gamma = \frac{ur_5}{y+x(2-y)}, \quad s \ge s_\delta = \frac{ur_5}{y(1-x)},
$$

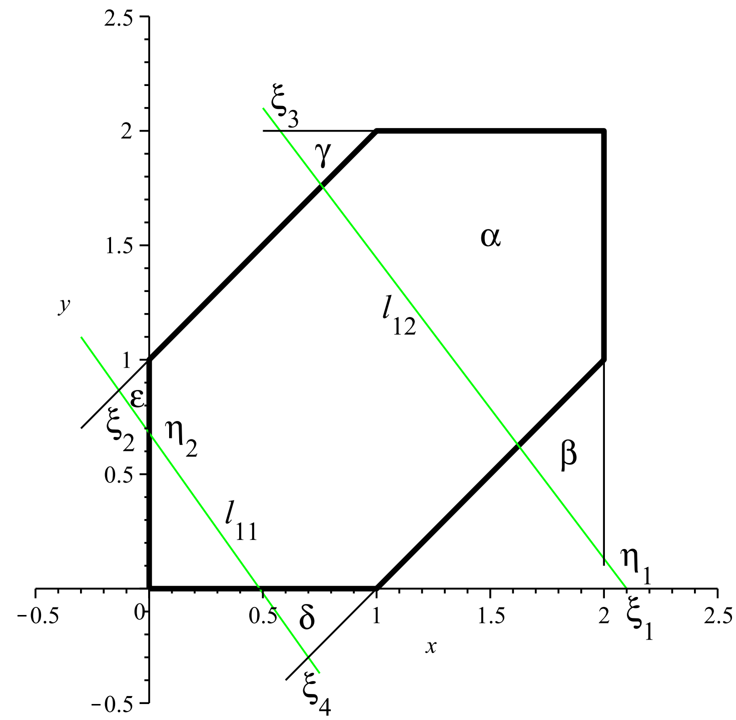

FIGURE 12. Case 5 when $x = .5$ and $y = .7$. To demonstrate the various triangles, two versions of l_1 are drawn, namely l_{11} for $u/s = .1$ and l_{12} for $u/s = 1.2$.

$$s \geq s_\epsilon = \frac{ur_5}{x(1-y)},$$

and are zero otherwise. The intersections between r_5 and the lower s-bounds are $y_\alpha = \frac{u-2x}{2-x}$, $y_\beta = \frac{u-x}{2-x}$, $y_\gamma = \frac{u-2x}{1-x}$, $y_\delta = \frac{u}{1-x}$, and $y_\epsilon = 1 - \frac{u}{x}$. These intersections are not always between zero and one so we shall calculate

$$k_\alpha(u,x) = \int_{y_\alpha}^{1} dy \int_{s_\alpha}^{r_5} \rho_5\, \alpha\, ds \quad \text{and} \quad k_{\alpha,0}(u,x) = \int_{0}^{1} dy \int_{s_\alpha}^{r_5} \rho_5\, \alpha\, ds\,,$$

$$k_\beta(u,x) = \int_{y_\beta}^{1} dy \int_{s_\beta}^{r_5} \rho_5\, \beta\, ds \quad \text{and} \quad k_{\beta,0}(u,x) = \int_{0}^{1} dy \int_{s_\beta}^{r_5} \rho_5\, \beta\, ds\,,$$

$$k_\gamma(u,x) = \int_{y_\gamma}^{1} dy \int_{s_\gamma}^{r_5} \rho_5\, \gamma\, ds \quad \text{and} \quad k_{\gamma,0}(u,x) = \int_{0}^{1} dy \int_{s_\gamma}^{r_5} \rho_5\, \gamma\, ds\,,$$

$$k_\delta(u,x) = \int_{y_\delta}^{1} dy \int_{s_\delta}^{r_5} \rho_5\, \delta\, ds \quad .$$

$$k_\epsilon(u,x) = \int_{0}^{y_\epsilon} dy \int_{s_\epsilon}^{r_5} \rho_5\, \epsilon\, ds \quad .$$

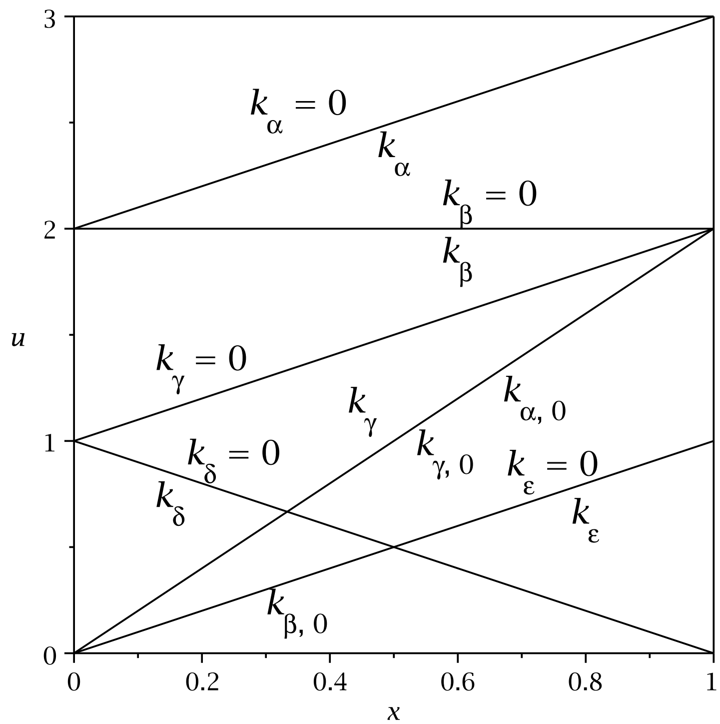

FIGURE 13. The intervals to integrate x over for fixed u in Case 5.

Notice that the y-integration of ϵ goes from zero to y_ϵ. We always have $y_\delta \geq 0$ and $y_\epsilon \leq 1$.

These functions shall be integrated over the x-intervals shown in Figure 13.

(9)

$$\begin{aligned}
h_{51}(u) &= \int_0^{u/2} (k_\alpha + k_\gamma)\,dx + \int_{u/2}^{1} (k_{\alpha,0} + k_{\gamma,0})\,dx \\
&\quad + \int_0^u k_\beta\,dx + \int_u^1 (k_{\beta,0} + k_\epsilon)\,dx + \int_0^{1-u} k_\delta\,dx, && 0 \leq u \leq 1, \\
h_{52}(u) &= \int_0^{u/2} (k_\alpha + k_\gamma)\,dx + \int_{u/2}^{1} (k_{\alpha,0} + k_{\gamma,0})\,dx \\
&\quad + \int_0^1 k_\beta\,dx, && 1 \leq u \leq 2. \\
h_{53}(u) &= \int_{u-2}^{1} k_\alpha\,dx, && 2 \leq u \leq 3.
\end{aligned}$$

The explicit expressions are not given here.

4.6. **Combination of cases.** The probabilities for the five cases are included in the ρ_i. Integrating the ρ_i over the whole s-, y- and x-range in each case gives $\frac{1}{4}$, $\frac{3}{4}$, 1, $\frac{3}{4}$, $\frac{1}{4}$, respectively. These numbers sum to 3, which is the area of the used hexagon. Dividing each integral by 3 produces the probability for the case.

Since the probabilities are included in the calculations, we just have to add the calculated $h_{nm}(u)$. We shall divide by 3 to have the probabilities add to one and divide by another 3 because our integrals are over the area 3 instead of 1. We get the probability distribution function $H(u)$ for twice the area of a random triangle in a hexagon, when one of the triangle vertices sits on the boundary of the hexagon:

$$
\begin{aligned}
&H_1(u) = 1 - \frac{1}{9}(h_{11} + h_{21} + h_{31} + h_{41} + h_{51}), && 0 \le u < 1,\\
&H_2(u) = 1 - \frac{1}{9}(h_{22} + h_{32} + h_{42} + h_{52}), && 1 \le u < 2,\\
&H_3(u) = 1 - \frac{1}{9}(h_{33} + h_{43} + h_{53}), && 2 \le u < 3.
\end{aligned}
\tag{10}
$$

5. Combination of the V- and W-distributions.

Let $F(x)$ be the distribution function for the triangle area X in a regular hexagon with unit area. We have $X \le x$ when $VW = UV/6 \le x$. Putting $x = y/6$, this happens when $UV \le y$ and we get

$$
\begin{aligned}
F(y/6) &= \int_0^3 G(y/u)\, dH(u) =\\
&= [G(y/u)H(u)]_0^3 - \int_y^3 H(u)\frac{d}{du}G(y/u)\, du =\\
&= G(y/3) - \int_y^3 H(u)\frac{d}{du}G(y/u)\, du, \quad 0 \le y \le 3.
\end{aligned}
\tag{11}
$$

The partial intergration in (11) is used to avoid integrating to the lower bound $u = 0$. Substituting $y = 6x$ in the result will give us the wanted $F(x)$. To write the result, we need the function

$$
\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt. \tag{12}
$$

This is the dilogarithm function discussed by Euler in 1768 and named by Hill, [5]. Some properties of $\mathrm{Li}_2(x)$ are given in Appendix A.

We will not carry out the integration (11) in detail, but will give as result the density function $f(x) = \frac{dF}{dx}$. We save the comments on our way to the result and on the result to the next section.

(13)

$$\begin{aligned}
f_1(x) = {} & 48x(1-6x)\left[\mathrm{Li}_2\left(\frac{1-\sqrt{1+24x}}{4}\right)+\mathrm{Li}_2\left(\frac{1+\sqrt{1+24x}}{4}\right)\right.\\
&\left.-\log\left(1+\sqrt{1+24x}\right)^2+\log(24x)\,\log\left(1+\sqrt{1+24x}\right)\right]\\
&+48x[\mathrm{Li}_2(1-3x)+(3x+4)\,\mathrm{Li}_2(3x/2)]\\
&+(52x-\frac{1}{3})\sqrt{1+24x}\,\left[2\log\left(\frac{1+\sqrt{1+24x}}{3+\sqrt{1+24x}}\right)+\log(1-3x)\right]\\
&-\left[360x^2+48x\,(1-\log(3x))-5\right]\,\log(1-3x)\\
&-8(15x+2)\,(2-3x)\,\log(2-3x)+72\,x^2\log(24x)^2\\
&+48(1-\log(2))\,x\log(24x)-\left[\frac{1}{3}+(52x-\frac{1}{3})\sqrt{1+24x}\right]\,\log(3x)\\
&+8\,\pi^2\,x^2+4\left[18\log(2)^2+36\,\log(2)-7\pi^2-8\right]x\\
&+\frac{94}{3}\,\log(2)+12,\\
&\quad 0\le x<\frac{1}{6}.
\end{aligned}$$

(14)

$$\begin{aligned}
f_2(x) = {} & 48x(1-6x)\,\log\left(\frac{1+\sqrt{1+24x}}{24x}\right)\,\log\left(1+\sqrt{1+24x}\right)\\
&+\left(52x-\frac{1}{3}\right)\,\sqrt{1+24x}\,\left(\log(24x)-2\,\log\left(1+\sqrt{1+24x}\right)\right)\\
&-8\,(15x+2)\,(2-3x)\,\log(2-3x)+48x(3x+4)\,\mathrm{Li}_2(3x/2)\\
&+\left[\frac{1}{3}-72(8\,\log(2)+5)\,x^2-48(1-\log(2))x\right]\,\log(24x)\\
&+8\,(135\log(2)+144\log(2)^2-2\pi^2)\,x^2\\
&+4\left[96\,\log(2)-12\log(2)^2-5\pi^2-13\right]\,x+26\log(2)+\frac{46}{3},\\
&\quad \frac{1}{6}\le x<\frac{1}{3}.
\end{aligned}$$

$$
\begin{aligned}
f_3(x) = {} & 144\,x(2x+3)\left[\frac{\pi^2}{9} - \mathrm{Li}_2\left(\frac{1+\sqrt{9-24x}}{4}\right) - \mathrm{Li}_2\left(\frac{1-\sqrt{9-24x}}{4}\right)\right. \\
& \left. + \log(2)\ \big(2\log(12x) - 3\log(3+\sqrt{9-24x})\big)\right] \\
& + 48\,x\,[\mathrm{Li}_2(1-3x) - (3x+4)\,\mathrm{Li}_2(3x/2)] \\
& + 6\,[48x\,(2x+3)\log(3+\sqrt{9-24x}) - (52x+3)\sqrt{9-24x} \\
& - (2x+3)\,24x\,\log(24x)]\log(1+\sqrt{9-24x}) \\
& + 3\,[-48\,x\,(2x+3)\log(3+\sqrt{9-24x}) + (52x+3)\sqrt{9-24x} \\
& + 16x\,\log(24x) + 48x(4x+5)\log(2) \\
& + 120x^2 - 80x - 7]\log(3x-1) \\
& + 8(15x+2)(2-3x)\log(2-3x) \\
& + 9\,(52x+3)\,\log(2)\,\sqrt{9-24x} \\
& + (24\,x\,(\log(2)-2)+11)\,\log(2) + 12\,x - 6, \\
& \frac{1}{3} \le x < \frac{3}{8}.
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
f_4(x) = {} & 144\,x(2x+3)\left[\frac{\pi^2}{9} - \mathrm{Li}_2\left(\frac{1+i\sqrt{24x-9}}{4}\right)\right. \\
& -\mathrm{Li}_2\left(\frac{1-i\sqrt{24x-9}}{4}\right) + \frac{1}{2}\log(2)\,\log(3x/2) \\
& \left. -2\left(\arctan(\sqrt{24x-9}) - \frac{\pi}{3}\right)\arctan\!\left(\frac{1}{3}\sqrt{24x-9}\right)\right] \\
& + 48\,x\,[\mathrm{Li}_2(1-3x) - (3x+4)\,\mathrm{Li}_2(3x/2)] \\
& + 6\,(52x+3)\sqrt{24x-9}\,\arctan(\sqrt{24x-9}) \\
& + 3\,[-8x(6x+7)\log(3x/2) + 16x\log(2) \\
& + 120x^2 - 80x - 7]\log(3x-1) \\
& + 8\,(15x+2)(2-3x)\log(2-3x) - 2\pi(52x+3)\sqrt{24x-9} \\
& + (24\,x\,(\log(2)-2)+11)\,\log(2) + 12x - 6, \\
& \frac{3}{8} \le x < \frac{1}{2}.
\end{aligned} \tag{16}
$$

The functions $f_3(x)$ and $f_4(x)$ are the same but written in different ways because $\sqrt{9-24x}$ is imaginary when $x > 3/8$. The only remaining imaginary arguments are in the terms on the first and second line of $f_4(x)$. Since Li_2 is analytic, the sum of these two terms with complex conjugate arguments is real. See the Appendix.

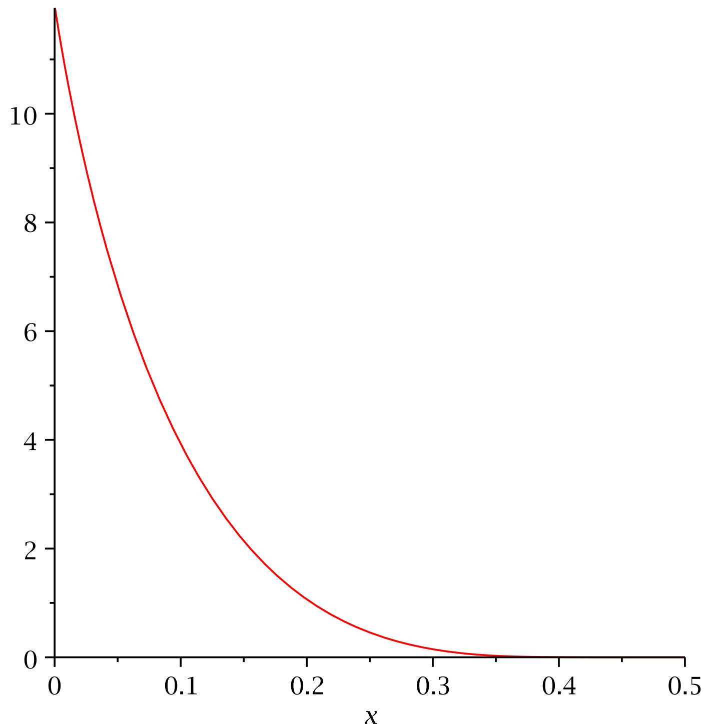

FIGURE 14. The density $f(x)$ for the area fraction of a random triangle in a regular hexagon.

In may look as if the terms of $f_3(x)$ haven't been collected in an optimal way. The reason is that we have written the expression so that each term is finite in every point of its domain.

The first, second, and third moments of the distribution are obtained by integration

$$\alpha_1 = \int_0^{\frac{1}{2}} x\, f(x) dx = \frac{289}{3888} = \frac{17^2}{2^4 \cdot 3^5} = .07433,$$

$$\alpha_2 = \int_0^{\frac{1}{2}} x^2\, f(x) dx = \frac{25}{2592} = \frac{5^2}{2^5 \cdot 3^4} = .009645.$$

$$\alpha_3 = \int_0^{\frac{1}{2}} x^3\, f(x) dx = \frac{57709}{34992000} = \frac{57709}{2^7 \cdot 3^7 \cdot 5^3} = .001649.$$

6. The calculation of H_{21}.

We want to describe in more detail how the calculations are carried out. The obstacles are similar in all cases and we have chosen to discuss them for H_{21}, compare (6).

First, when integrating with respect to s, one can substitute s by $t = s/r_2$. Doing so, r_2 disappears from the calculation. The resulting integral with respect to t is in principal the same for all the triangles α, β,.. etc. and has the form

$$\int_{t_\alpha}^{1} a\,(b-ct)^2\,\frac{dt}{t} = \ldots - ab^2\,\log\left(|t_\alpha|\right)\ldots \quad ,$$

where a, b, c, and t_α are functions of u, x, and y. Any singularities of a and b and any zeroes of t_α will be singular points in the result. Here, a has the factor $(x-y)^{-1}$.

Continuing to the integration with respect to y, we refer to Fig.5, which is drawn for $y > x$. If y approaches x, $\xi_2 \to \infty$ so that $\alpha \to \infty$ and $\beta \to -\infty$. However $\alpha + \beta$, which is the quantity to be integrated remains finite at $y = x$.

To keep down the number of cases to be considered, we integrate α and β separately and hope that the integrals shall be finite even if the integrands are not. Only in one place did we have to take the limit below for two triangles together, but else, we could do them separately. Since y is integrated across x , we encounter singular integrals of the form

$$\lim_{\epsilon\to 0+}\left(\int_{y_\alpha}^{x-\epsilon} + \int_{x+\epsilon}^{2}\right)\phi(u,x,y)\,\frac{1}{x-y}dy,$$

for some function ϕ.

Instead of doing these limts, we let *Maple* do the integration formally as if there were no singularities. *Maple* treats x and y as complex variables and produces

$$\int \phi(u,x,y)\,\frac{1}{x-y}dy = \ldots - \phi(u,x,y)\,\log\left(x-y\right) + \ldots \quad .$$

Taking the limit above on this expression gives

$$\ldots - \phi(u,x,x)\,\lim_{\epsilon\to 0+}\left(\log\left(\epsilon\right) - \log\left(-\epsilon\right)\right) = \ldots\, i\pi\phi(u,x,x) + \ldots \quad .$$

Maple answers with the correct real result plus an erronous imaginary part. Our tactic is to ignore the singularities and remove the imaginary part at the end of the calculation.

In the following integration with respect to x, it turns out that we have a term $\psi(u,x)\,\log\left(u+x-x^2\right)$ to integrate. The log here has a zero for $x = \frac{1}{2}(1+\sqrt{1+4u})$ with no simple geometrical interpretation. We don't bother about this and just remove any resulting imaginary term.

Much effort is spent on simplifying the integrals that *Maple* produces so we want to describe a simplification technique used over and over again. The H_{21} produced by *Maple* has the following dilog terms all multiplied by u^2.

$$\begin{aligned} tmp_1 = &- \operatorname{dilog}\left(\frac{-3+\sqrt{1+4u}}{-1+\sqrt{1+4u}}\right) - \operatorname{dilog}\left(\frac{3+\sqrt{1+4u}}{1+\sqrt{1+4u}}\right) \\ &+ \operatorname{dilog}\left(\frac{-1+\sqrt{1+4u}-2u}{-1+\sqrt{1+4u}}\right) - \operatorname{dilog}\left(\frac{-1+\sqrt{1+4u}-u}{-1+\sqrt{1+4u}}\right) \\ &- \operatorname{dilog}\left(\frac{1+\sqrt{1+4u}+u}{1+\sqrt{1+4u}}\right) + \operatorname{dilog}\left(\frac{1+\sqrt{1+4u}+2u}{1+\sqrt{1+4u}}\right) . \end{aligned}$$

Candidates for our simplifications are terms like this which have a power of u as a common factor and consists of functions that we would like to be spared from in the result. Our technique is to take the derivative $D_u(tmp_1)$, simplify the obtained expression and then integrate it to get tmp_2, satisfying

$$tmp_2 = tmp_1 + C ,$$

where C = constant. C is determined by insertion of a value of u, usually one of the boundary values of the domain of definition. In the present case we had to form $\lim_{u\to 0+}(tmp_2 - tmp_1)$ to get C. The result is

$$\begin{aligned} tmp_2 = &- \operatorname{dilog}\left(\frac{u}{2}\right) - 2\operatorname{dilog}\left(\frac{3+\sqrt{1+4u}}{4}\right) - 2\operatorname{dilog}\left(\frac{3-\sqrt{1+4u}}{4}\right) \\ &+ \text{logarithm terms} + \text{constant} . \end{aligned}$$

For the simplification of the derivative, we use ordinary logarithmic and trigonometric formulas and relations for dilog functions from [1] and [6].

7. Concluding comment.

We have not shown any integral calculations in detail. In principle, they are elementary, which doesn't mean that they don't require a substantial effort. The calculations would not have been possible without some tool like *Maple* or Mathematica for handling the large number of terms that come out of the integrations. This doesn't mean that *Maple* performs the integrations automatically. Often, we had to split up the integrands into parts and treat each part in a special way. We had to do some partial integrations manually. A substantial part of the work has been spent on simplifying the integrals that *Maple* produces.

It should be pointed out that *Maple* isn't reproducible in the sense that it doesn't always give exactly the same answer. The terms often come in a different order when you rerun a calculation. An integration leading to e.g. $\operatorname{dilog}(\lambda(u))$ for some $\lambda(u)$ as in tmp_1 above may come out as $-\operatorname{dilog}(1-\lambda(u))$ + logarithmic terms or as $-\operatorname{dilog}(1/\lambda(u))$ + logarithmic terms. All three answers are correct, but the simplification of them is not the same. This implies that partial results must be saved.

We will supply any interested reader with *Maple* files describing the calculations.

Appendix A

The dilogarithm function $\mathrm{Li}_2(x)$ is defined in [6] for complex x as

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt. \tag{17}$$

When x is real and greater than unity, the logarithm is complex. A branch cut from 1 to ∞ can give it a definite value.

We have the series expansion

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}\ , \quad |x| \leq 1. \tag{18}$$

This implies that $\mathrm{Li}_2(\mathrm{x})$ is analytic in the unit circle. Consequently, the first term of $f_4(x)$ is real since the two $\mathrm{Li}_2(\mathrm{x})$ there have complex conjugate arguments. Ref. [6], gives the expression

$$\mathrm{Li}_2(re^{i\theta}) + \mathrm{Li}_2(r^{-i\theta}) = -\int_0^r \frac{\log\left(1 - 2t\cos\theta + t^2\right)}{t}\,dt, \text{ when } 0 \leq r \leq 1.$$

Although the series (18) is only convergent for $|x| \leq 1$, the integral in (17) is not restricted to these limits and $\mathrm{Re}(\mathrm{Li}_2(\mathrm{x}))$ is defined and is real on the whole real axis. We use this function for $|x| < 1$.

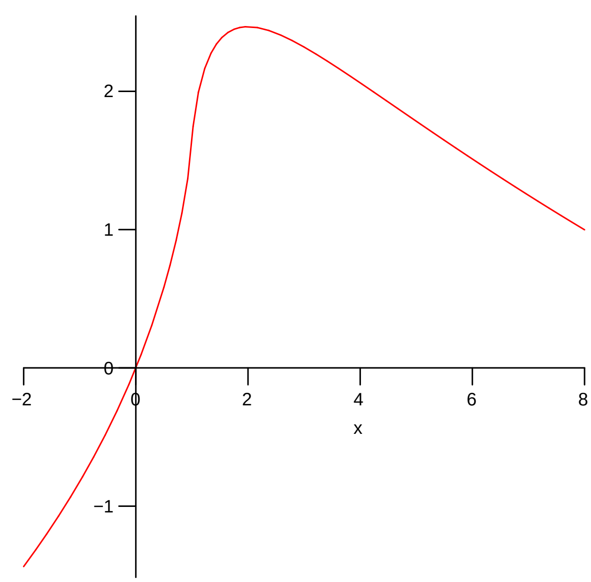

Figure 15. The function $\mathrm{Re}(\mathrm{Li}_2(\mathrm{x}))$.

The definition of the dilogarithm function has varied a little from author to author. Maple has the function polylog$(2,x)$ which is defined by the series expansion (18) for $|x| \leq 1$ otherwise by analytic continuation. Maple also has a function dilog$(x) = \mathrm{Li}_2(1-x)$ defined on the whole real axis. Maple's dilog function is the same as the dilog function given in [1], page 1004.

$\mathrm{Re}(\mathrm{Li}_2(\mathrm{x}))$ is increasing from $\mathrm{Re}(\mathrm{Li}_2(0)) = 0$ via $\mathrm{Re}(\mathrm{Li}_2(1)) = \pi^2/6$ to its maximum $\mathrm{Re}(\mathrm{Li}_2(2)) = \pi^2/4$.

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