

# THE PROBABILITY DISTRIBUTION OF THE DISTANCE BETWEEN TWO RANDOM POINTS IN A BOX.

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ABSTRACT. We determine the probability distribution for the distance between two random points in a box with sides  $a$ ,  $b$ , and  $c$ . The average of this distance is known before as Robbins's constant.

## 1. INTRODUCTION

We consider the random variable  $V =$  the distance between two random points in a box with sides  $a$ ,  $b$ , and  $c$ .  $V$  is one of the many random quantities studied in Geometric Probability, see eg. [6]. The classical Sylvester's problem considers the area of the convex hull of three random points in a convex set, in particular in a triangle, a square, or a circle. The three-dimensional variants of these problems have also been studied. These problems, which consider a random area in an area or a random volume in a volume are affinely invariant, meaning that the results are the same for a cube and for a box with different sides and even for a paralleliped. This is not the case in our present problem which considers a one-dimensional length in a three-dimensional set, explaining why our result depends on the box sides  $a$ ,  $b$ , and  $c$ .

The similar problem of the distance between two random points on the surface of a cube has been considered by Borwain et.al. [1] and Philip, [4]. See also Bailey et.al. [2].

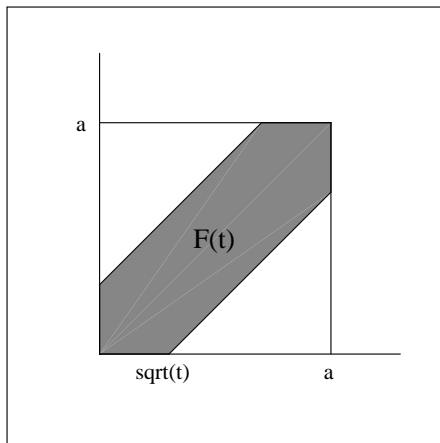
The expected value of the distance between two random points in a box was stated as a problem by D.P. Robbins, [5], and calculated by T.S. Bolin, [3]. The distribution function  $K(v) = \text{Prob}(V \leq v)$  for this random variable seems not to be known before.

A survey of the results in this area are collected by E. Weisstein on the web site [7].

Our method is to determine the distance distributions for each of the three directions and convolve them to get the distribution of  $V$ . The result is a long expression and we rely heavily on the use of a formula manipulating program, in our case *Maple 10*.

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FIGURE 1. The area  $F_a(t)$ .

## 2. NOTATION AND FORMULATION.

Let the two random points be  $(X_1, Y_1, Z_1)$  and  $(X_2, Y_2, Z_2)$ . We assume that  $X_1$  and  $X_2$  are independent and evenly distributed in the interval  $(0, a)$ . The same is assumed for  $Y_1$  and  $Y_2$  in  $(0, b)$  and for  $Z_1$  and  $Z_2$  in  $(0, c)$ . We assume also, without loss of generality, that the box sides satisfy  $0 < a \leq b \leq c$ .

We start by calculating the distribution function  $F_a(t) = \text{Prob}((X_1 - X_2)^2 \leq t)$  and from it the corresponding density function  $f_a(t) = \frac{dF_a(t)}{dt}$ . Then, the density  $g(s)$  corresponding to  $G(s) = \text{Prob}((X_1 - X_2)^2 + (Y_1 - Y_2)^2 \leq s)$  is obtained by convolving  $f_a$  and  $f_b$ . At last, the density  $h(u)$  corresponding to  $H(u) = \text{Prob}((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \leq u)$  is the convolution of  $f_c$  and  $g$ . The wanted distribution function for the distance is  $K(v) = H(v^2)$  with the density  $k(v) = 2v h(v^2)$ .

## 3. THE ONE-DIMENSIONAL DISTRIBUTION.

The probability that  $(X_1 - X_2)^2 \leq t$  is proportional to the area of the diagonal strip in Fig.1.

$$(1) \quad F_a(t) = \begin{cases} 1 - (1 - \sqrt{t}/a)^2, & 0 < t \leq a^2 ; \\ 1 & a^2 < t. \end{cases}$$

The density is

$$(2) \quad f_a(t) = \frac{1}{a\sqrt{t}} - \frac{1}{a^2}, \quad 0 < t \leq a^2 .$$

When we give the value of a function like  $f_a(t)$  in an interval, we tacitly assume that it is zero where it is not defined.

4. THE TWO-DIMENSIONAL DISTRIBUTION.

The probability density for the event  $(X_1 - X_2)^2 + (Y_1 - Y_2)^2 \leq s$  is the convolution  $g$  of  $f_a$  and  $f_b$

$$g(s) = \int f_a(s-t) f_b(t) dt.$$

Because of the different domains of definition of  $f_a$  and  $f_b$ , there are three cases

$$(3) \quad g_1(s) = \int_0^s f_a(s-t) f_b(t) dt, \quad 0 < s \leq a^2 .$$

$$(4) \quad g_2(s) = \int_{s-a^2}^s f_a(s-t) f_b(t) dt, \quad a^2 < s \leq b^2 .$$

$$(5) \quad g_3(s) = \int_{s-a^2}^{b^2} f_a(s-t) f_b(t) dt, \quad b^2 < s \leq a^2 + b^2 .$$

If  $a = b$ ,  $g_2$  is omitted. We get

$$(6) \quad g(s) = \begin{cases} -2 \frac{\sqrt{s}}{a^2 b} - 2 \frac{\sqrt{s}}{ab^2} + \frac{\pi}{ab} + \frac{s}{a^2 b^2}, & 0 < s \leq a^2 ; \\ -2 \frac{\sqrt{s}}{a^2 b} - \frac{1}{b^2} + \frac{2}{ab} \arcsin \left( \frac{a}{\sqrt{s}} \right) + \frac{2}{a^2 b} \sqrt{s - a^2}, & a^2 < s \leq b^2 ; \\ -\frac{1}{b^2} + \frac{2}{ab} \arcsin \left( \frac{a}{\sqrt{s}} \right) + \frac{2}{a^2 b} \sqrt{s - a^2} \\ -\frac{1}{a^2} + \frac{2}{ab} \arcsin \left( \frac{b}{\sqrt{s}} \right) + \frac{2}{ab^2} \sqrt{s - b^2} \\ -\frac{\pi}{ab} - \frac{s}{a^2 b^2}, & b^2 < s \leq a^2 + b^2 . \end{cases}$$

Since  $s$  is the the square of the distance, we get the density for the distance  $v = \sqrt{s}$  between two random points in a rectangle with sides  $a$  and  $b$  as  $g_v(v) = g(v^2) \frac{ds}{dv} = 2v g(v^2)$ . This density is shown in Figure 2.

The expectation of the distance between two random points in a rectangle is

$$(7) \quad E_{\text{rectangle}} = \int_0^{a^2+b^2} \sqrt{s} g(s) ds = \int_0^{\sqrt{a^2+b^2}} v g_v(v) dv.$$

The result is

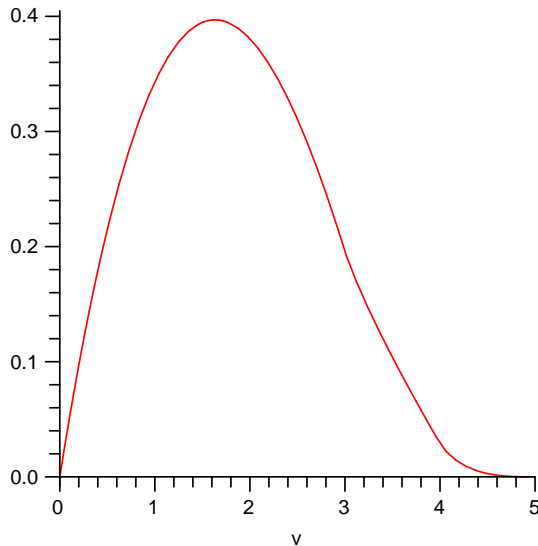


FIGURE 2. The density function  $g_v(v)$  for the distance between two random points in a rectangle with sides 3 and 4.

$$(8) \quad E_{\text{rectangle}} = \frac{a^2}{6b} \ln \left( \frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}} \right) + \frac{b^2}{6a} \ln \left( \frac{a}{b} + \sqrt{1 + \frac{a^2}{b^2}} \right) \\ + \frac{1}{15} \left( \frac{b^3}{a^2} + \frac{a^3}{b^2} + \left( 3 - \frac{a^2}{b^2} - \frac{b^2}{a^2} \right) \sqrt{a^2 + b^2} \right)$$

For  $a = b = 1$ , this reduces to

$$(9) \quad E_{\text{unit square}} = \frac{1}{3} \ln (1 + \sqrt{2}) + \frac{1}{15} (2 + \sqrt{2}) \approx .52140543.$$

### 5. THE THREE-DIMENSIONAL DISTRIBUTION.

The density of the probability that  $(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \leq u$  is the convolution of  $f_c$  and  $g$

$$(10) \quad h(u) = \int f_c(u - s) g(s) ds.$$

We shall convolve  $f_c$  with each of the three components of  $g$  given in (6). Since the components have different domains of definition, this will also be the case for the convolutions. A closer study of (6) motivates the defining of the following  $g_{ij}$ .

$$\begin{aligned}
 (11) \quad g_{11} &= -2 \frac{\sqrt{s}}{a^2 b}, & 0 < s \leq b^2; \\
 g_{12} &= -2 \frac{\sqrt{s}}{a b^2} + \frac{\pi}{a b} + \frac{s}{a^2 b^2}, & 0 < s \leq a^2; \\
 g_{22} &= -\frac{1}{b^2} + \frac{2}{a b} \arcsin\left(\frac{a}{\sqrt{s}}\right) + \frac{2}{a^2 b} \sqrt{s - a^2}, & a^2 < s \leq a^2 + b^2; \\
 g_{32} &= -\frac{1}{a^2} + \frac{2}{a b} \arcsin\left(\frac{b}{\sqrt{s}}\right) + \frac{2}{a b^2} \sqrt{s - b^2}, & b^2 < s \leq a^2 + b^2; \\
 g_{33} &= -\frac{\pi}{a b} - \frac{s}{a^2 b^2}, & b^2 < s \leq a^2 + b^2.
 \end{aligned}$$

Then,

$$\begin{aligned}
 (12) \quad g_1 &= g_{11} + g_{12}, & 0 < s \leq a^2; \\
 g_2 &= g_{11} + g_{22}, & a^2 < s \leq b^2; \\
 g_3 &= g_{22} + g_{32} + g_{33}, & b^2 < s \leq a^2 + b^2.
 \end{aligned}$$

With this splitting, we can write  $g$  as the sum

$$(13) \quad g = g_{11} + g_{12} + g_{22} + g_{32} + g_{33},$$

Convolving each of the  $g_{ij}$  with  $f_c$ ,

$$(14) \quad h_{ij}(u) = \int f_c(u - s) g_{ij}(s) ds.$$

we can write

$$(15) \quad h = h_{11} + h_{12} + h_{22} + h_{32} + h_{33},$$

Notice that  $g_{32}$  is  $g_{22}$  with  $a$  and  $b$  switched. This applies also to their boundaries of definition implying that we can get  $h_{32}$  by switching  $a$  and  $b$  in  $h_{22}$ .

A detailed description of the resulting  $h(u)$  is given in the next section.

Like  $g$ , each  $h_{ij}$  has different analytical expressions  $h_{ijk}$  in three adjacent intervals. Remembering that  $a \leq b \leq c$ , and looking at e.g.  $h_{11}$ , we have

$$(16) \quad h_{11} = \begin{cases} h_{111}, & 0 < u \leq b^2; \\ h_{112}, & b^2 < u \leq c^2; \\ h_{113}, & c^2 < u \leq b^2 + c^2. \end{cases}$$

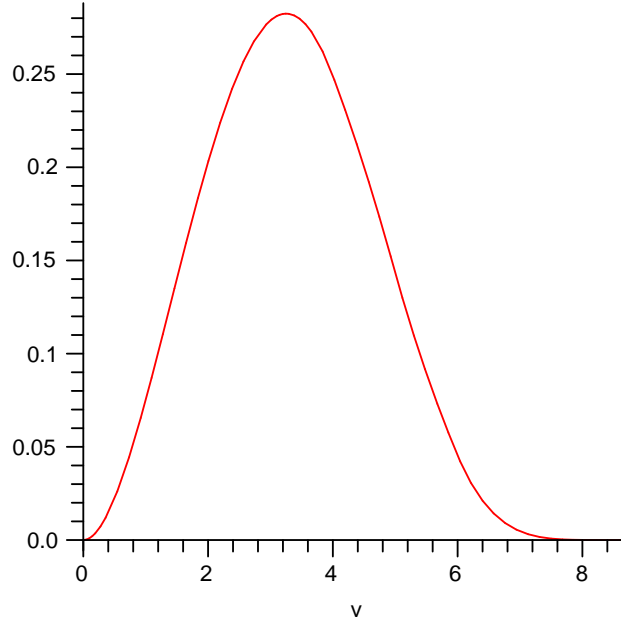


FIGURE 3. Density function  $k(v)$  for the distance between two random points in a box with sides 4, 5, and 6.

Altogether, there are eight breakpoints on the  $u$ -axis for the analytical description of  $h(u)$  and they are

$$(0, a^2, b^2, c^2, a^2 + b^2, a^2 + c^2, b^2 + c^2, a^2 + b^2 + c^2)$$

If  $c^2 < a^2 + b^2$  the points above are in increasing order. Otherwise, points 4 and 5 above should change place to show the intervals for the analytical expressions. This implies that we have two cases if we want to give  $h(u)$  for each interval. Writing  $h(u)$  as in (15) circumvents this difficulty. If  $a = b$  or  $b = c$  the number of intervals decreases to five and if they are all alike there are just three intervals left.

## 6. THE DENSITY OF THE SQUARE OF THE DISTANCE IN A BOX.

The practical integrations of type (14) are made with *Maple 10*. The *Maple* worksheet `habc.mw` is available at [www.math.kth.se/~johanph](http://www.math.kth.se/~johanph).

We give the result here as functions of  $u$ , which is the square of distance between two random points in a box with sides  $a$ ,  $b$ , and  $c$ . The density  $k(v)$  for the distance  $v = \sqrt{u}$  is  $k(v) = 2vh(v^2)$  and is shown in Figure 3.

(17)

$$h_{11} = \frac{1}{3a^2b^2c^2}.$$

$$\left\{ \begin{array}{l} -3\pi b c u + 4b u^{3/2}, \\ 4b^4 + 6b^2c\sqrt{u-b^2} - 6bcu \arcsin\left(\frac{b}{\sqrt{u}}\right), \\ 4b^4 + 6b^2c\sqrt{u-b^2} \\ + 6bcu(\arccos\left(\frac{c}{\sqrt{u}}\right) - \arcsin\left(\frac{b}{\sqrt{u}}\right)) \\ - 2b(2u+c^2)\sqrt{u-c^2}. \end{array} \right. \begin{array}{l} 0 < u \leq b^2, \\ b^2 < u \leq c^2, \\ c^2 < u \leq b^2 + c^2. \end{array}$$

(18)

$$h_{12} = \frac{1}{6a^2b^2c^2}.$$

$$\left\{ \begin{array}{l} 12\pi abc\sqrt{u} - 6\pi a(b+c)u + 8(a+c)u^{3/2} - 3u^2, \\ 5a^4 - 6\pi a^3b, \\ +12\pi abc\sqrt{u} + 8cu^{3/2} \\ -12\pi abc\sqrt{u-a^2} - 8c(u-a^2)^{3/2} \\ -12acu \arcsin\left(\frac{a}{\sqrt{u}}\right), \\ 5a^4 - 6\pi a^3b + 6\pi abc^2 - c^4 + 6(\pi ab + c^2)u + 3u^2 \\ -12\pi abc\sqrt{u-a^2} - 8c(u-a^2)^{3/2} \\ -4a(2u+c^2)\sqrt{u-c^2} \\ +12acu(\arccos\left(\frac{c}{\sqrt{u}}\right) - \arcsin\left(\frac{a}{\sqrt{u}}\right)), \end{array} \right. \begin{array}{l} 0 < u \leq a^2, \\ a^2 < u \leq c^2, \\ c^2 < u \leq a^2 + c^2. \end{array}$$

(19)

$$h_{22} = \frac{1}{3a^2b^2c^2}.$$

$$\left\{ \begin{array}{l} 0, \\ 3\pi a^2b(a+c) - 3a^4 - 6\pi abc\sqrt{u} + 3(a^2 + \pi bc)u \\ + (6\pi abc - 2(b+3c)a^2 - 4bu)\sqrt{u-a^2} \\ - 6abu \arcsin\left(\frac{a}{\sqrt{u}}\right), \\ 3a^2b(\pi a - b) - 4b^4 \\ - 12abc \arcsin\left(\frac{b\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-a^2}}\right) \sqrt{u} \\ - 6ac(a - \pi b) \sqrt{u-a^2} \\ - 6c\left(b^2 - a^2 + 2ab \arcsin\left(\frac{a}{\sqrt{a^2+b^2}}\right)\right) \sqrt{u-a^2-b^2} \\ - 6ab(a^2 + b^2) \arcsin\left(\frac{a}{\sqrt{a^2+b^2}}\right) \\ + 6bc(a^2 + u) \arcsin\left(\frac{b}{\sqrt{u-a^2}}\right), \\ 3a^2(a^2 - b^2 - c^2) - 4b^4 - 3a^2u \\ - 12abc\left(\arcsin\left(\frac{b\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-a^2}}\right) - \arccos\left(\frac{ac}{\sqrt{u-c^2}\sqrt{u-a^2}}\right)\right) \sqrt{u} \\ + 2b(a^2 + c^2 + 2u) \sqrt{u-a^2-c^2} \\ - 6c(b^2 - a^2 + 2ab \arcsin\left(\frac{a}{\sqrt{a^2+b^2}}\right)) \sqrt{u-a^2-b^2} \\ - 6ab(a^2 + b^2) \arcsin\left(\frac{a}{\sqrt{a^2+b^2}}\right) \\ + 6bc(a^2 + u) \left(\arcsin\left(\frac{b}{\sqrt{u-a^2}}\right) - \arccos\left(\frac{c}{\sqrt{u-a^2}}\right)\right) \\ + 6ab(c^2 + u) \arcsin\left(\frac{a}{\sqrt{u-c^2}}\right), \end{array} \right. \begin{array}{l} 0 < u \leq a^2, \\ a^2 < u \leq a^2 + b^2, \\ a^2 + b^2 < u \leq a^2 + c^2, \\ a^2 + c^2 < u \leq a^2 + b^2 + c^2 \end{array}$$



(20)

$$h_{32} = \frac{1}{3b^2 a^2 c^2}.$$

$$\left\{ \begin{array}{l} 0, \\ 3\pi a b^2 (b+c) - 3b^4 - 6\pi a b c \sqrt{u} + 3(b^2 + \pi a c)u \\ + (6\pi a b c - 2(a+3c)b^2 - 4a u) \sqrt{u-b^2} \\ - 6a b u \arcsin\left(\frac{b}{\sqrt{u}}\right), \\ 3a b^2 (\pi b - a) - 4a^4 \\ - 12a b c \arcsin\left(\frac{a\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-b^2}}\right) \sqrt{u} \\ - 6bc(b - \pi a) \sqrt{u-b^2} \\ - 6c \left( a^2 - b^2 + 2ab \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) \right) \sqrt{u-a^2-b^2} \\ - 6ab(a^2 + b^2) \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) \\ + 6ac(b^2 + u) \arcsin\left(\frac{a}{\sqrt{u-b^2}}\right), \\ 3b^2(b^2 - a^2 - c^2) - 4a^4 - 3b^2 u \\ - 12a b c \left( \arcsin\left(\frac{a\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-b^2}}\right) - \arccos\left(\frac{bc}{\sqrt{u-c^2}\sqrt{u-b^2}}\right) \right) \sqrt{u} \\ + 2a(b^2 + c^2 + 2u) \sqrt{u-b^2-c^2} \\ - 6c(a^2 - b^2 + 2ab \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right)) \sqrt{u-a^2-b^2} \\ - 6ab(a^2 + b^2) \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) \\ + 6ac(b^2 + u) \left( \arcsin\left(\frac{a}{\sqrt{u-b^2}}\right) - \arccos\left(\frac{c}{\sqrt{u-b^2}}\right) \right) \\ + 6ab(c^2 + u) \arcsin\left(\frac{b}{\sqrt{u-c^2}}\right), \end{array} \right. \begin{array}{l} 0 < u \leq b^2, \\ b^2 < u \leq a^2 + b^2, \\ a^2 + b^2 < u \leq b^2 + c^2, \\ b^2 + c^2 < u \leq a^2 + b^2 + c^2 \end{array}$$

$$(21) \quad h_{33} = \frac{1}{6 a^2 b^2 c^2}.$$

$$\left\{ \begin{array}{ll} 0, & 0 < u \leq b^2, \\ \begin{array}{l} 3(2\pi ab + b^2 + u)(u - b^2) \\ -4c(b^2 + 3\pi ab + 2u)\sqrt{u - b^2}, \end{array} & b^2 < u \leq a^2 + b^2, \\ \begin{array}{l} 3(a^2 + b^2)^2 - 3b^4 + 6\pi a^3 b \\ -4c(b^2 + 3\pi ab + 2u)\sqrt{u - b^2} \\ +4c(a^2 + b^2 + 3\pi ab + 2u)\sqrt{u - a^2 - b^2}, \end{array} & a^2 + b^2 < u \leq b^2 + c^2, \\ \begin{array}{l} 3(a^2 + b^2)^2 + c^4 + 6\pi ab(a^2 + b^2 - c^2) \\ -6(\pi ab + c^2)u - 3u^2 \\ +4c(a^2 + b^2 + 3\pi ab + 2u)\sqrt{u - a^2 - b^2}, \end{array} & b^2 + c^2 < u \leq a^2 + b^2 + c^2 \end{array} \right.$$

Putting  $a = b = c = 1$  in  $h$ , we get the density for the unit cube which is a reasonably long expression. We give the density  $k_{\text{cube}}(v) = 2v h(v^2)$  for the distance  $V$  between two random points in a unit cube.

(22)

$$k_{\text{cube}}(v) = \left\{ \begin{array}{ll} v^2(4\pi - 6\pi v + 8v^2 - v^3) & 0 < v \leq 1, \\ \begin{array}{l} (6\pi - 1)v - 8\pi v^2 + 6v^3 + 2v^5 + 24v^3 \arctan \sqrt{v^2 - 1} \\ -8v(1 + 2v^2)\sqrt{v^2 - 1} \end{array} & 1 < v \leq \sqrt{2}, \\ \begin{array}{l} (6\pi - 5)v - 8\pi v^2 + 6(\pi - 1)v^3 - v^5 \\ +8v(1 + v^2)\sqrt{v^2 - 2} \\ -24v(1 + v^2)\arctan \sqrt{v^2 - 2} + 24v^2 \arctan(v\sqrt{v^2 - 2}) \end{array} & \sqrt{2} < v \leq \sqrt{3}. \end{array} \right.$$

## 7. THE AVERAGE DISTANCE BETWEEN TWO RANDOM POINTS IN A BOX.

This average distance  $E(V)$  was given by T.S. Bolis [3] and we present his result in the next section. Here, we shall describe how we have calculated the same average. Of course, the results coincide. Since  $v = \sqrt{u}$ , we have

$$(23) \quad E(V) = \int \sqrt{u} h(u) du = \int \sqrt{u} \int f_c(u-s) g(s) ds du.$$

We start by calculating

$$(24) \quad m(s) = \int \sqrt{u} f_c(u-s) du,$$

and get

$$(25) \quad m(s) = \frac{s}{c} \log \left( \frac{c}{\sqrt{s}} + \sqrt{1 + \frac{c^2}{s}} \right) + \frac{2}{3} \frac{s^{3/2}}{c^2} + \frac{1}{3c^2} (c^2 - 2s) \sqrt{s + c^2}.$$

This function is well defined and increasing from  $\frac{c}{3}$  on the interval  $0 \leq s < \infty$ .

Then, we get  $E(V)$  as the sum of five integrals of the form

$$(26) \quad E_{ij} = \int m(s) g_{ij}(s) ds.$$

Moreover,  $E_{32}$  is  $E_{22}$  with  $a$  and  $b$  switched.

The second moment of the distance is easily calculated as

$$(27) \quad \alpha_2 = \int_0^{a^2} t f_a(t) dt + \int_0^{b^2} t f_b(t) dt + \int_0^{c^2} t f_c(t) dt = \frac{1}{6} (a^2 + b^2 + c^2).$$

Our calculations are described in the *Maple* worksheet `Eabc.mw`, which is available at [www.math.kth.se/~johanph](http://www.math.kth.se/~johanph).

## 8. ROBBINS'S CONSTANT

Robbins stated this problem in [5]. Bolis, [3] presented a solution obtained by splitting the box into three similar cones and integrating over a cone in spherical coordinates.

Notice that the box in [3] has the sides  $2a$ ,  $2b$ , and  $2c$ . The result given below is that in [3] divided by two.

Define:

$$(28) \quad \begin{aligned} r &= \sqrt{a^2 + b^2 + c^2} \\ r_1 &= \sqrt{b^2 + c^2} \\ r_2 &= \sqrt{c^2 + a^2} \\ r_3 &= \sqrt{a^2 + b^2}. \end{aligned}$$

Then, the average can be written in the following symmetric and condensed form

$$\begin{aligned}
(29) \quad E(V) &= \frac{1}{15} r - \frac{7}{90 a^2} (r - r_1) r_1^2 - \frac{7}{90 b^2} (r - r_2) r_2^2 - \frac{7}{90 c^2} (r - r_3) r_3^2 \\
&+ \frac{4}{315 a^2 b^2 c^2} (a^7 + b^7 + c^7 - r_1^7 - r_2^7 - r_3^7 + r^7) \\
&+ \frac{1}{30 a b^2 c^2} \left( b^6 \operatorname{arsinh} \left( \frac{a}{b} \right) + c^6 \operatorname{arsinh} \left( \frac{a}{c} \right) - r_1^2 (r_1^4 - 8 b^2 c^2) \operatorname{arsinh} \left( \frac{a}{r_1} \right) \right) \\
&+ \frac{1}{30 a^2 b c^2} \left( c^6 \operatorname{arsinh} \left( \frac{b}{c} \right) + a^6 \operatorname{arsinh} \left( \frac{b}{a} \right) - r_2^2 (r_2^4 - 8 c^2 a^2) \operatorname{arsinh} \left( \frac{b}{r_2} \right) \right) \\
&+ \frac{1}{30 a^2 b^2 c} \left( a^6 \operatorname{arsinh} \left( \frac{c}{a} \right) + b^6 \operatorname{arsinh} \left( \frac{c}{b} \right) - r_3^2 (r_3^4 - 8 a^2 b^2) \operatorname{arsinh} \left( \frac{c}{r_3} \right) \right) \\
&- \frac{2}{15 a b c} \left( a^4 \arcsin \left( \frac{bc}{r_2 r_3} \right) + b^4 \arcsin \left( \frac{ca}{r_3 r_1} \right) + c^4 \arcsin \left( \frac{ab}{r_1 r_2} \right) \right).
\end{aligned}$$

The following relation holds for the arsinh function

$$(30) \quad \operatorname{arsinh}(y) = \frac{1}{2} \log \left( y + \sqrt{1 + y^2} \right).$$

Inserting  $a = b = c = 1$ , we get the value for the unit cube

$$\begin{aligned}
(31) \quad E_{\text{unit cube}} &= \frac{1}{105} \left( 4 + 17\sqrt{2} - 6\sqrt{3} + 21 \log(1 + \sqrt{2}) + 84 \log(1 + \sqrt{3}) \right. \\
&\quad \left. - 42 \log(2) - 7\pi \right) \approx .661707182.
\end{aligned}$$

## 9. COMMENT.

Even if *Maple* is very helpful in doing the calculations of this paper, there are several things it doesn't do. The success of the calculations relies on manual simplification of trigonometric expressions and on manual factorization of polynomials.

We tested our method on the corresponding four-dimensional problems. When trying to calculate the four-dimensional distribution or average distance, we encountered integrals that we cannot solve like

$$\int \frac{\arctan x}{\sqrt{1 - a x^2} \sqrt{1 - b x^2}} dx, \text{ where } a > 0 \text{ and } b > 0.$$

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