THE AVERAGE VOLUME OF A RANDOM TETRAHEDRON IN A TETRAHEDRON.

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ABSTRACT. We calculate the average size of the volume of a random tetrahedron inside a mother tetrahedron. The result is not new, but the method is different from that of previous papers.

1. Introduction

Four points are generated at random inside a tetrahedron A. Let T be the tetrahedron spanned by the random points. We shall consider the random variable $X = volume(T)/volume(A)$. It is well known that any affine transformation will preserve the ratio X. This follows from the fact that the volume scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation. This means that our results hold for any shape of the tetrahedron A.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [12]. J. J. Sylvester considered the plane problem of a random triangle T in an arbitrary bounded convex set K and posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown, [2] that $\kappa = \frac{11}{144}$ for K a square.

A. Reńyi and R. Sulanke, [10] and [11], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. H. A. Alikoski, [2], has given an expression for κ when $n = 3$ and K a regular r-polygon. We have given the whole probability distribution of X for $n = 3$ and $n = 4$ and K a parallelogram, [8], and for K a triangle, [9].

R. E. Miles, [7], generalizes the asymptotic estimates for K a circle to higher dimensions. Using the formula of Reńyi and Sulanke, [11], C. Buchta and M. Reitzner, [4], deduce a formula for κ for $n \geq 4$ in a tetrahedron A. It is this κ that we compute for $n = 4$ by a different

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method in this paper. The same κ was also obtained by D. Mannion, [6]. Our method is most like the one by Mannion. The value for a random tetrahedron in a mother tetrahedron is

$$\kappa = \frac{13}{720} - \frac{\pi^2}{15015} \approx .017398.$$

2. Notation and formulation.

As A, we shall use the tetrahedron that has its vertices in $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. We use a constant probability density in A for generating 4 random points in A. The points will be denoted P_k and have coordinates (x_k, y_k, z_k) for $1 \leq k \leq 4$. Let T be the tetrahedron spanned by the 4 points. We shall determine the expectation κ of the random variable $X = volume(T)/volume(A)$.

The generated T spans a tetrahedron with sides parallell to the sides of A. We shall call this spanned tetrahedron B the 'big' tetrahedron. The random variable X, that we study will be written as the product of the two random variables

$$U = volume(T)/volume(B) \text{ and } V = volume(B)/volume(A).$$

Roughly speaking, U describes the shape of T and V its size. We shall show that U and V are independent so that we can combine the expectations of U and V to get $E(X) = E(U) \cdot E(V)$.

3. The four geometrical cases for calculating $E(U)$.

The way B is spanned by the four points gives rise to four cases:

(1) One point in a vertex, one on the opposite side and two interior points,
(2) Two points on 'opposite' edges and two interior,
(3) One point on an edge, two in 'opposite' sides and one interior,
(4) One point in each side.

These four cases are pictured in Figures 1 to 4.

In Figure 1, Case 1, we have without loss of generality (WLOG) chosen P_1 to be the point that sits in a vertex of B, and this vertex is chosen to be the one nearest to the origin The point P_4 sits on the opposite side and P_2 and P_3 are interior. In the other cases, the point numbering and their positions have, WLOG, been chosen in a similar way.

We shall show that the four cases occur with the probabilities $p_1 = \frac{4}{55}$, $p_2 = \frac{6}{55}$, $p_3 = \frac{36}{55}$, and $p_4 = \frac{9}{55}$, respectively. Compare table 1.

3.1. **Calculation of p_1.** This case occurs when P_1 sits in a vertex, chosen to be the origin, P_4 sits on the opposite side, and P_2 and P_3 are

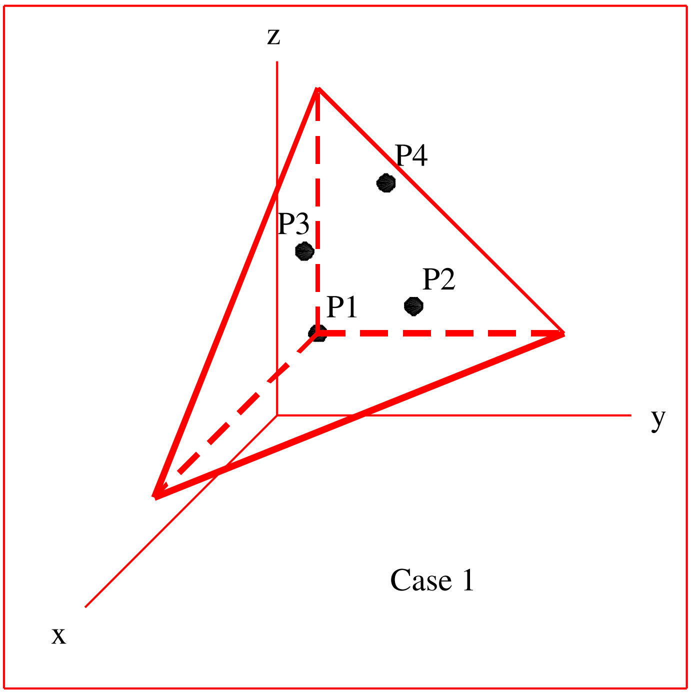

FIGURE 1. The 'big' tetrahedron in Case 1. P_1 in a vertex, P_4 on the opposite side, P_2 and P_3 interior.

interior. This is described by the inequalities

$$x_1 \leq x_j, y_1 \leq y_j, z_1 \leq z_j \text{ for } 2 \leq j \leq 4 \text{ and}$$
$$x_k + y_k + z_k \leq x_4 + y_4 + z_4 \text{ for } 1 \leq k \leq .3$$

Start by calculating the conditional probability f_{14} that P_2 and P_3 sit in the tetrahedron defined by P_1 and P_4. The factor 36 is $(1/6)^{-2}$ where $1/6$ is the volume of the tetrahedron integrated over

$$f_{14} = 36 \int_{x_1}^{x_4+y_4+z_4-y_1-z_1} dx_2 \int_{y_1}^{x_4+y_4+z_4-x_2-z_1} dy_2 \int_{z_1}^{x_4+y_4+z_4-x_2-y_2} dz_2$$
$$\int_{x_1}^{x_4+y_4+z_4-y_1-z_1} dx_3 \int_{y_1}^{x_4+y_4+z_4-x_3-z_1} dy_3 \int_{z_1}^{x_4+y_4+z_4-x_3-y_3} 1\, dz_3$$

To get the probability, denoted $p_1^\star$, that the points sit as described by the inequalities above, f_{14} shall be integrated over all possible positions of P_1 and P_4

case	P_1	P_2	P_3	P_4	p_j	$E(U_j)$	$E(U_j{}^2)$
1	3	0	0	1	$\frac{4}{55}$	$\frac{3}{64}$	$\frac{1}{200}$
2	2	0	0	2	$\frac{6}{55}$	$\frac{7}{144} - \frac{\pi^2}{2310}$	$\frac{1}{225}$
3	2	1	0	1	$\frac{36}{55}$	$\frac{11}{216} - \frac{\pi^2}{3465}$	$\frac{1}{216}$
4	1	1	1	1	$\frac{9}{55}$	$\frac{23}{486} + \frac{2\pi^2}{6237}$	$\frac{1}{216}$

TABLE 1. Giving, for each of the 4 cases, the number of faces determined by each P_k, their probability to occur, the expectation of U_j and the second moment of U_j.

$$\begin{aligned} p_1^\star =&36\int_0^1 dx_1 \int_0^{1-x_1} dy_1 \int_0^{1-x_1-y_1} dz_1 \\ &\int_{x_1}^{1-y_1-z_1} dx_4 \int_{y_1}^{1-x_4-z_1} dy_4 \int_{z_1}^{1-x_4-y_4} f_{14}\, dz_4 = \frac{1}{660}. \end{aligned}$$

To get the probability p_1 for Case 1, $p_1^\star$ shall be multiplied by 4! which is the number of ways the points can be numbered, though divided by 2! because the points P_2 and P_3 enter the calculations in exactly the same way. It shall also be multiplied by 4, which is the number of vertices that P_1 can sit in. We get

$$p_1 = \frac{24}{2} \cdot 4 \cdot \frac{1}{660} = \frac{4}{55}$$

.

3.2. **Calculation of p_2.** This case occurs when P_1 sits on an edge, which is chosen to be the vertical one, P_4 sits on the opposite edge, and P_2 and P_3 are interior. This is described by the inequalities

$$\begin{aligned} &x_1 \le x_j, y_1 \le y_j, z_4 \le z_j \text{ for all } j \text{ and} \\ &x_k + y_k + z_k \le x_4 + y_4 + z_4 \text{ for all } k. \end{aligned}$$

The calculation of f_{14} is essentially the same as for p_1, but the lower bound z_1 is replaced by z_4. The integration of f_{14} is done in another order

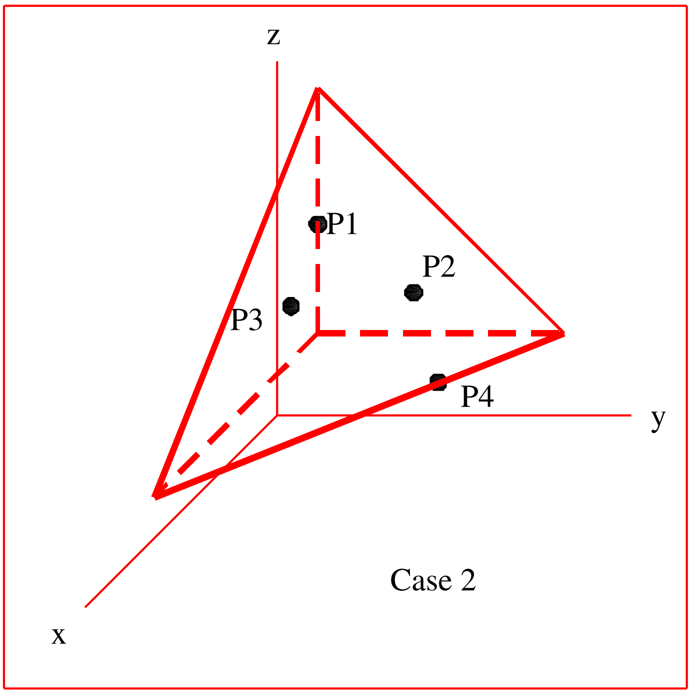

FIGURE 2. The 'big' tetrahedron in Case 2. P_1 on the vertical edge, P_4 on the opposite edge, P_2 and P_3 interior.

$$p_2^\star = 36\int_0^1 dx_1 \int_0^{1-x_1} dy_1 \int_0^{1-x_1-y_1} dz_4 \int_{x_1}^{1-y_1-z_4} dx_4 \int_{y_1}^{1-x_4-z_4} dy_4 \int_{z_4}^{x_4+y_4+z_4-x_1-y_1} f_{14}\, dz_1 = \frac{1}{330}.$$

Like in the calculation of p_1, the points P_2 and P_3 enter the calculations in exactly the same way. The two opposite edges on which P_1 and P_4 sit can be chosen in 3 ways. We get

$$p_2 = \frac{24}{2}\cdot 3\cdot \frac{1}{330} = \frac{6}{55}.$$

3.3. **Calculation of p_3.** This case occurs when P_1 sits on an edge, which is chosen to be the intersection of the horisontal and the slanting side, P_2 and P_4 sit in the faces not determined by P_1, and P_3 is interior. This is described by the inequalities

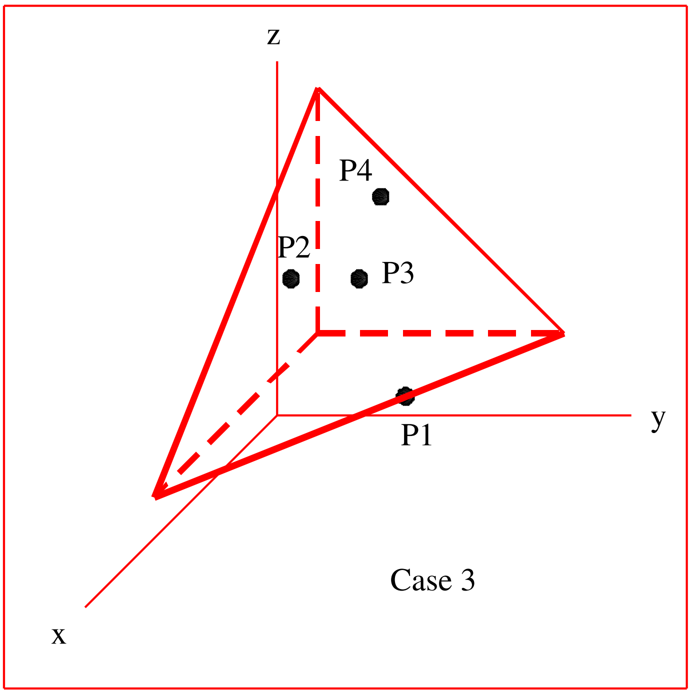

FIGURE 3. The 'big' tetrahedron in Case 3. P_1 on the edge $x_1 + y_1 = 1$, P_2 and P_4 on the non-adjacent sides and P_3 interior.

$$x_4 \leq x_j, y_2 \leq y_j, z_1 \leq z_j \text{ for all } j \text{ and}$$
$$x_k + y_k + z_k \leq x_1 + y_1 + z_1 \text{ for all } k.$$

The integration is not split into two parts as above

$$p_3^\star = 36^2 \int_0^1 dx_4 \int_0^{1-x_4} dy_2 \int_0^{1-x_4-y_2} dz_1 \int_{x_4}^{1-y_2-z_1} dx_1 \int_{y_2}^{1-x_1-z_1} dy_1$$
$$\int_{x_4}^{x_1+y_1-y_2} dx_2 \int_{z_1}^{x_1+y_1+z_1-x_2-y_2} dz_2 \int_{y_2}^{x_1+y_1-x_4} dy_4$$
$$\int_{z_1}^{x_1+y_1+z_1-x_4-y_4} dz_4 \int_{x_4}^{x_1+y_1-y_2} dx_3$$
$$\int_{y_2}^{x_1+y_1-x_3} dy_3 \int_{z_1}^{x_1+y_1+z_1-x_3-y_3} dz_3 = \frac{1}{220}.$$

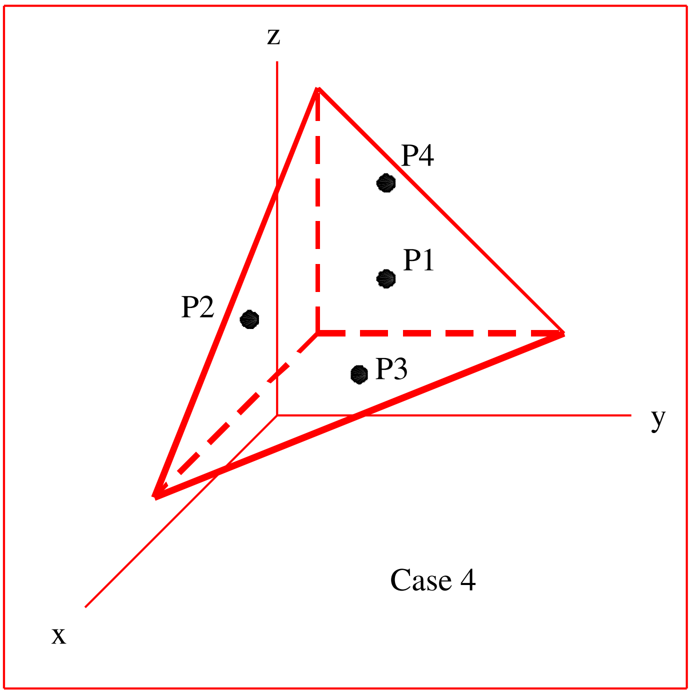

FIGURE 4. The 'big' tetrahedron in Case 4. One point on each side.

In this calculation, each point is treated in a particular way. The edge on which P_1 sits can be chosen in 6 ways. We get

$$p_2 = 24 \cdot 6 \cdot \frac{1}{220} = \frac{36}{55}.$$

3.4. **Calculation of p_4.** This case occurs when there is one point on each face. This is described by the inequalities

$$x_1 \le x_j, y_2 \le y_j, z_3 \le z_j \text{ for all } j \text{ and}$$
$$x_k + y_k + z_k \le x_4 + y_4 + z_4 \text{ for all } k.$$

(1)
$$p_4^\star = 36^2 \int_0^1 dx_1 \int_0^{1-x_1} dy_2 \int_0^{1-x_1-y_2} dz_3 \int_{x_1}^{1-y_2-z_3} dx_4 \int_{y_2}^{1-x_4-z_3} dy_4$$
$$\int_{z_3}^{1-x_4-y_4} dz_4 \int_{y_2}^{x_4+y_4+z_4-x_1-z_3} dy_1 \int_{z_3}^{x_4+y_4+z_4-x_1-y_1} dz_1$$
$$\int_{x_1}^{x_4+y_4+z_4-y_2-z_3} dx_2 \int_{z_3}^{x_4+y_4+z_4-x_2-y_2} dz_2$$
$$\int_{x_1}^{x_4+y_4+z_4-y_2-z_3} dx_3 \int_{y_2}^{x_4+y_4+z_4-x_3-z_3} 1\, dy_3 = \frac{3}{440}.$$

In this calculation, each point is treated in a particular way. We get

$$p_4 = 24 \cdot \frac{3}{440} = \frac{9}{55}.$$

4. The expectation of U in each of the four cases.

When calculating the expectation of U, we enlarge the 'big' tetrahedron B so that its vertices become $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This doesn't affect the ratio U. We will continue to call the points P_k even though the problem has been translated and rescaled. The transformed tetrahedron T is spanned by the three vectors

$$P_2 - P_1,\ P_3 - P_1, \text{ and } P_4 - P_1.$$

The side spanned by $P_2 - P_1$ and $P_3 - P_1$ has the normal $n = (P_2 - P_1) \times (P_3 - P_1)$. The volume fraction U is the absolute value of the scalar product

$$D = n \cdot (P_4 - P_1). \tag{2}$$

The complexity of the calculation stems from this absolute value. We have to identify the sets where D is positive and negative. Let B_+ and B_- be these subsets of the enlarged B. We have $B = B_+ + B_-$. To get the expectation of U, we are going to integrate D over the whole of B and subtract twice its integral over B_-.

4.1. **Calculation of $E(U_1)$.** In this case P_1 is the origin, P_2, P_3 are interior in B and P_4 will sit on the side defined by $x_4 + y_4 + z_4 = 1$. Substituting $z_4 = 1 - x_4 - y_4$ in (2), we get

$$D = (n_1 - n_3)x_4 + (n_2 - n_3)y_4 + n_3.$$

In Case 1, we have all components of $P_2 - P_1$ and $P_3 - P_1$ positive, implying that the n_k cannot all have the same sign. To determine where D is positive, we assume, WLOG, that

$$n_1 \geq n_2 \geq 0.$$

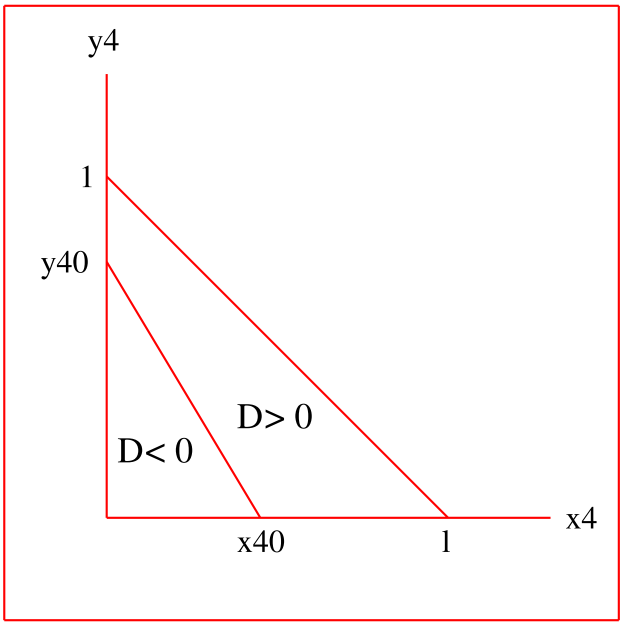

FIGURE 5. The areas to integrate over in x_4y_4-space in Case 1.

This implies that $n_3 \leq 0$. With these inequalities, we single out one of twelve cases which all have the same probability of occurring and all have the same expectation of $E(U_1)$.

For fixed P_2 and P_3, Figure 5 shows the areas in x_4y_4-space where $D \geq 0$ and $D \leq 0$. The line separating these areas intersects the axes in the points

$$x_{40} = \frac{-n_3}{n_1 - n_3} \text{ and } y_{40} = \frac{-n_3}{n_2 - n_3}, \text{ where } 0 \leq x_{40} \leq y_{40} \leq 1.$$

For fixed P_2 and P_3, we get the average over P_4 as

(3)
$$e_{P_4}(P_2, P_3) = \int_0^1 dy_4 \int_0^{1-y_4} D\, dx_4 - 2 \int_0^{y_{40}} dy_4 \int_0^{x_{40}(1-y_4/y_{40})} D\, dx_4.$$

Next, $e_{P_4}(P_2, P_3)$ shall be averaged over P_2 and P_3. We start with x_2 and y_2 and keep z_2 and P_3 fixed. The area to integrate over is determined by the inequalities $n_1 \geq n_2 \geq 0$ and is shown in Figure 6. We get

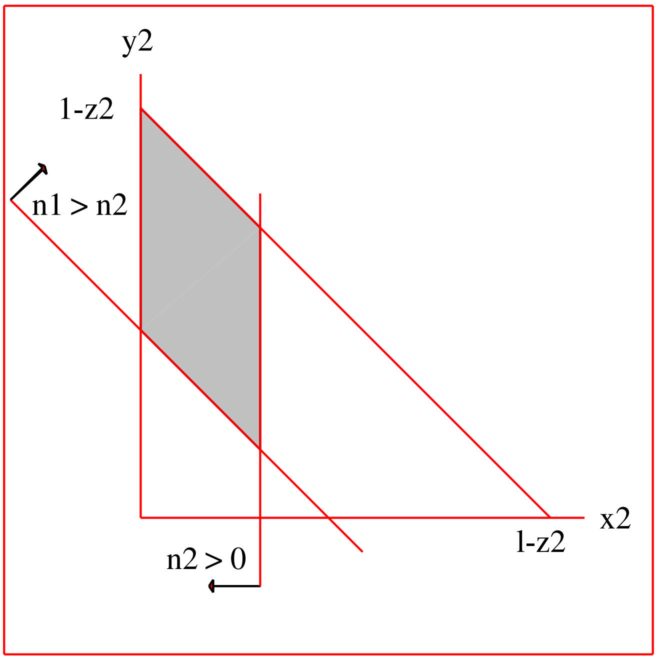

FIGURE 6. The area to integrate over in x_2y_2-space in Case 1.

$$e_{P_4,x_2,y_2}(z_2,P_3) = \int_0^{y_2x_3/z_3} dx_2 \int_{z_2(x_3+y_3)/z_3-x_2}^{1-x_2-z_2} e_{P_4}(P_2,P_3)\,dy_2. \tag{4}$$

The area in Figure 6 becomes zero when $z_2 \geq z_3/(x_3+y_3+z_3)$, so the next integration is

$$e_{P_4,P_2}(P_3) = \int_0^{z_3/(x_3+y_3+z_3)} e_{P_4,x_2,y_2}(z_2,P_3)\,dz_2. \tag{5}$$

At last, we shall integrate P_3 over the whole tetrahedron, giving

$$e_{P_4,P_2,P_3} = \int_0^1 dx_3 \int_0^{1-x_3} dy_3 \int_0^{1-x_3-y_3} e_{P_4,P_2}(P_3)\,dz_3 = \frac{1}{18432}. \tag{6}$$

To get the expectation of U_1, this number shall be multiplied by 2, which is the inverse of the area integrated over in P_4-space, twice by 6, which is the inverse of the volume integrated over in P_2- and P_3-space, and by 12, which is number of cases that are equivalent to the one chosen by assuming $n_1 \geq n_2 \geq 0$. We get

$$E(U_1) = 2 \cdot 6^2 \cdot 12 \cdot \frac{1}{18432} = \frac{3}{64}.$$

We are indebted to Maple for helping us do the integrations. We have deliberately omitted writing out the results of the integrations in equations (3) - (5) because the expressions are very long. The Maple integration in (5) gives 2216 terms before the boundaries are inserted. Our computer, dedicated to numerical computations, uses 70 seconds to compute $E(U_1)$. This first case is the simplest one of the four. Mannion, [6], finds the average in this case without integration by using an average for triangles. We have chosen to describe the integration to fascilitate the understanding of the coming cases. The Maple worksheet for the calculation is given in Appendix A.

4.2. **Calculation of** $E(U_2)$**.** In this case P_1 sits on the vertical axis, P_4 on the opposite edge, and P_2 and P_3 are interior. Cf. Figure 7. In this case, we define

$$n = (P_3 - P_1) \times (P_4 - P_1)$$

and

$$D = n \cdot (P_2 - P_1) = n_1 \cdot x_2 + n_2 \cdot y_2 + n_3 \cdot (z_2 - z_1).$$

The plane separating positive and negative D goes though P_1, P_3, and P_4. Figure 7 shows this plane in P_2-space. When drawing this Figure and in the calculations, we have, WLOG, singled out one of four cases which all have the same probability of occurring and all have the same expectation of $E(U_2)$ by assuming $n_1 \geq n_2$ and $n_3 \geq 0$.

The separating plane intersects the x_2-axis in the point $x_{20} = n_3 z_1/n_1$. It intersects the edge $y_2 + z_2 = 1$ in the point $y_{20} = n_3(1 - z_1)/(n_3 - n_2)$, $z_{20} = 1 - y_{20}$. One can show that the inequalities $n_1 \geq n_2$ and $n_3 \geq 0$ imply $0 \leq x_{40} \leq 1$ and $0 \leq y_{40} \leq 1$. D is positive above this plane. As before, we are going to integrate D over the whole tetrahedron and subtract twice its integral over the volume below the two shaded surfaces in Figure 7. We are not going to carry out the integration over the whole tetrahedron. In fact, this integral is zero as can be expected from the symmetric character of of the point distributions.

The integral of D over the part marked a in Figure 7 is

(7) $$e_{a,P_2}(P_3, z_1, x_4) = \int_0^{x_4} dx_2 \int_{y_{20}+(1-x_4-y20)x_2/x_4}^{1-x_2} dy_2 \int_0^{1-x_2-y_2} D\, dz_2.$$

The integral of D over the part marked b in Figure 7 is

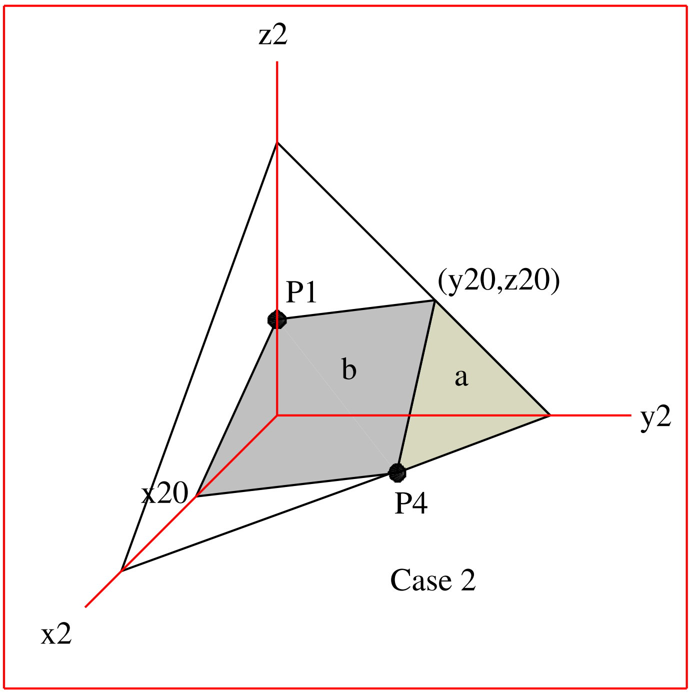

FIGURE 7. The volume to integrate over in P_2-space in Case 2.

(8) $$e_{b,P_2}(P_3, z_1, x_4) = \int_0^{x_4} dx_2 \int_0^{y_{20}+(1-x_4-y20)x_2/x_4} dy_2 \int_0^{z_1-(n_1x_2+n_2y_2)/n_3} D\,dz_2 + \int_0^{1-x_4} dy_2 \int_{x_4}^{x_{20}+(x_4-x_{20})y_2/(1-x_4)} dx_2 \int_0^{z_1-(n_1x_2+n_2y_2)/n_3} D\,dz_2.$$

The lower limit of the second x_2−integral may be bigger than the upper limit so that this part gives a negative contribution. The sum of (7) and (8) shall be integrated over all positions of P_3 complying with $n_1 \geq n_2$ and $n_3 \geq 0$. First, $n_1 \geq n_2$ is equivalent to $z_3 \leq z_1(1-x_3-y_3)$ and we have

(9) $$e_{P_2,z_3}(x_3, y_3, z_1, x_4) = \int_0^{z_1(1-x_3-y_3)} (e_{a,P_2}(P_3, z_1, x_4) + e_{b,P_2}(P_3, z_1, x_4))\,dz_3.$$

Then, $n_3 \geq 0$ is equivalent to $y_3 x_4 \leq x_3(1-x_4)$, giving

$$(10)\quad e_{P_2,P_3}(z_1,x_4) = \int_0^{x_4} dx_3 \int_0^{x_3(1-x_4)/x_4} e_{P_2,z_3}(x_3,y_3,z_1,x_4)\,dy_3 + \int_{x_4}^1 dx_3 \int_0^{1-x_3} e_{P_2,z_3}(x_3,y_3,z_1,x_4)\,dy_3.$$

At last,

$$(11)\quad e_{P_1,P_2,P_3,P_4} = \int_0^1 dz_1 \int_0^1 e_{P_2,P_3}(z_1,x_4)\,dx_4 = -\frac{7}{41472} + \frac{\pi^2}{665280}.$$

Remembering that the integral of D over the whole tetrahedron is zero, we get the expectation of U_2, by multiplying this number by -2, because we shall subtract twice the integral over negative D , twice by 6, which is the the inverse of the volume integrated over in P_2- and P_3-space, and by 4, which is number of cases that are equivalent to the one chosen by assuming $n_1 \geq n_2$ and $n_3 \geq 0$. We get

$$E(U_2) = -2 \cdot 6^2 \cdot 4 \cdot e_{P_1,P_2,P_3,P_4} = \frac{7}{144} - \frac{\pi^2}{2310}.$$

4.3. **Calculation of $E(U_3)$.** In this case, we put P_1 on the edge $x_1 + y_1 = 1$, $z_1 = 0$, P_2 in the side $y_2 = 0$, P_4 in the side $x_4 = 0$, and let P_3 be interior. Cf. Figure 3. We define

$$n = (P_4 - P_1) \times (P_2 - P_1)$$

and have

$$D = n \cdot (P_3 - P_1) = n_1 \cdot (x_3 - x_1) + n_2 \cdot (y_3 - 1 + x_1) + n_3 \cdot z_3.$$

The plane separating positive and negative D goes though P_1, P_2, and P_4. Figure 8 shows this plane in P_3-space. When drawing this Figure and in the calculations, we have, WLOG, singled out one of two cases which both have the same probability of occurring and both have the same expectation of $E(U_3)$ by assuming $n_1 \geq n_2$. This, in its turn, implies $n_3 \geq 0$ and $n_3 \geq n_2$.

The separating plane intersects the x_3-axis in the point $x_{30} = x_1 + n_2(1-x_1)/n_1$. It intersects the edge $y_3 + z_3 = 1$ in the point $y_{30} = (n_3 - n_1 x_1 - n_2(1-x_1))/(n_3 - n_2)$, $z_{30} = 1 - y_{30}$. One can show that the inequalitiy $n_1 \geq n_2$ implies $0 \leq x_{30} \leq 1$ and $0 \leq y_{30} \leq 1$. Like in Case 2, the integral over the whole tetrahedron is zero. We shall integrate over the volume below the separating plane where $D \leq 0$.

The integral of D over the part marked a in Figure 8 is

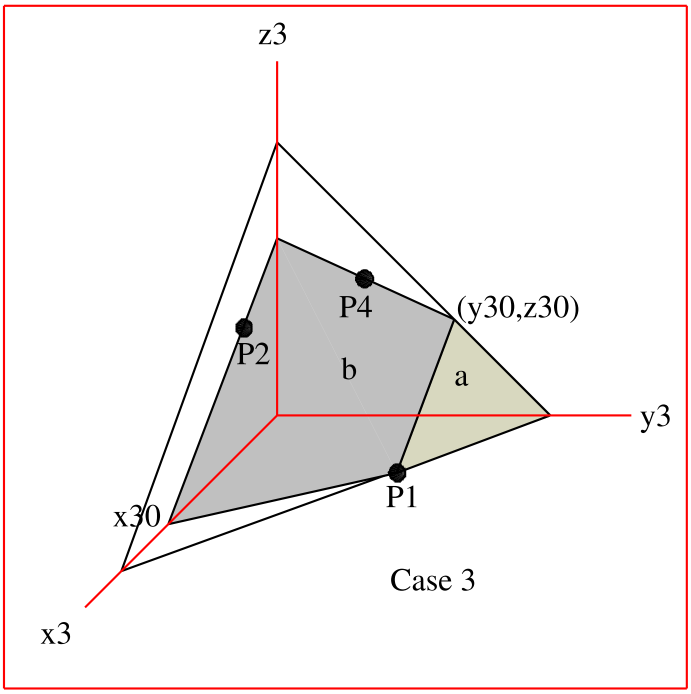

FIGURE 8. The volume to integrate over in P_3-space in Case 3.

$$(12)\quad e_{a,P_3}(P_2,P_4,x_1)$$
$$= \int_0^{x_1} dx_3 \int_{y_{30}+(1-x_1-y30)x_3/x_1}^{1-x_3} dy_3 \int_0^{1-x_3-y_3} D\,dz_3.$$

The integral of D over the part marked b in Figure 8 is

$$(13)\quad e_{b,P_3}(P_2,P_4,x_1)$$
$$= \int_0^{x_1} dx_3 \int_0^{y_{30}+(1-x_1-y30)x_3/x_1} dy_3 \int_0^{(n_1(x_1-x_3)+n_2(1-x_1-y_3))/n_3} D\,dz_3$$
$$+\int_0^{1-x_1} dy_3 \int_{x_1}^{x_{30}+(x_1-x_{30})y_3/(1-x_1)} dx_3 \int_0^{(n_1(x_1-x_3)+n_2(1-x_1-y_3))/n_3} D\,dz_3.$$

The lower limit of the second x_3-integral may be bigger than the upper limit so that this part gives a negative contribution. The sum of (12) and (13) shall be integrated over all positions of P_4 complying with $n_1 \geq n_2$. First, $n_1 \geq n_2$ is equivalent to $z_4(1-x_2) \geq z_2(1-y_4)$ and we have

$$(14)\quad e_{P_3,z_4}(y_4,P_2,x_1) = \int_{z_2(1-y_4)/(1-x_2)}^{(1-y_4)} \left(e_{a,P_2}(P_2,P_4,x_1) + e_{b,P_2}(P_2,P_4,x_1)\right) dz_4.$$

Then,

$$(15)\quad e_{P_1,P_2,P_3,P_4} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{1-x_2} dz_2 \int_0^1 e_{P_3,z_4}(y_4,P_2,x_1)\, dy_4 = -\frac{11}{20736} + \frac{\pi^2}{332640}.$$

Remembering that the integral of D over the whole tetrahedron is zero, we get the expectation of U_3, by multiplying this number by -2, because we shall subtract twice the integral over negative D , once by 6, which is the inverse of the volume integrated over in P_3-space, twice by 2, which is the the inverse of the area integrated over in P_2- and P_4-space, and by 2, which is number of cases that are equivalent to the one chosen by assuming $n_1 \geq n_2$. We get

$$E(U_3) = -2 \cdot 6 \cdot 2^2 \cdot 2 \cdot e_{P_1,P_2,P_3,P_4} = \frac{11}{216} - \frac{\pi^2}{3465}.$$

The calculation of $E(U_3)$ takes 160 secs. and requires 225 MB of RAM-memory. When integrating, Maple expands expressions into many terms. The dy_4-integral in (15) has over 22000 terms. The Maple worksheet for the calculation is in Appendix C.

4.4. **Calculation of $E(U_4)$.** In this case, there is one point P_k in each face of the tetrahedron. Cf. Figure 4.

We define

$$n = (P_1 - P_3) \times (P_2 - P_3)$$

and have

$$D = n \cdot (P_4 - P_3) = n_1 \cdot (x_4 - x_3) + n_2 \cdot (y_4 - y_3) + n_3 \cdot (1 - x_4 - y_4).$$

We have $D \geq 0$ when

$$(n_1 - n_3)x_4 + (n_2 - n_3)y_4 \geq n_1 x_3 + n_2 y_3 - n_3$$

.

WLOG, we assume $n_1 \geq n_3$ and $n_2 \geq n_3$. If $n_1x_3 + n_2y_3 - n_3 \geq 0$, we have the same Figure in x_4y_4-space as in Case 1, i.e. Figure 5. Here the expressions for x_{40} and y_{40} are

$$x_{40} = \frac{n_1x_3 + n_2y_3 - n_3}{n_1 - n_3} \text{ and } y_{40} = \frac{n_1x_3 + n_2y_3 - n_3}{n_2 - n_3}$$

Writing out the expressions, one can show that $|x_{40}| \leq 1$ and $|y_{40}| \leq 1$.

The area where $D \leq 0$ exists only when the numerator $n_1 x_3 + n_2 y_3 - n_3 \geq 0$, so we shall only integrate over the area where this the case. This inequality reduces to

$$\frac{x_3}{x_{30}} + \frac{y_3}{y_{30}} \leq 1 \text{ where } x_{30} = \frac{x_2}{1-z_2} \text{ and } y_{30} = \frac{y_1}{1-z_1}.$$

Unlike Cases 2 and 3, the integral over the whole tetrahedron is not zero.

$$(16) \quad E_T(D) = 16 \int_0^1 dz_1 \int_0^{1-z_1} dy_1 \int_0^1 dz_2 \int_0^{1-z_2} dx_2 \int_0^1 dx_3 \int_0^{1-x_3} dy_3 \int_0^1 dx_4 \int_0^{(1-x_4)} D\, dy_4 = \frac{1}{27}.$$

The integration over negative D reads

$$(17) \quad e_{P_1,P_2,P_3,P_4} = \int_0^1 dz_1 \int_0^{1-z_1} dy_1 \int_0^1 dz_2 \int_0^{1-z_2} dx_2 \int_0^{x_{30}} dx_3 \int_0^{y_{30}(1-x_3/x_{30})} dy_3 \int_0^{x_{40}} dx_4 \int_0^{y_{40}(1-x_4/x_{40})} D\, dy_4 = \frac{5}{46656} - \frac{\pi^2}{299376}.$$

When calculating the expectation of U_4, we shall multiply e_{P_1,P_2,P_3,P_4} by -2, because we shall subtract twice the integral over negative D , four times by 2, which is the the inverse of the area integrated over in each space, and by 3, which is number of ways to choose the n_i that is smaller than the other two. We get

$$E(U_4) = \frac{1}{27} - 2 \cdot 2^4 \cdot 3 \cdot e_{P_1,P_2,P_3,P_4} = \frac{23}{486} + \frac{2\pi^2}{6237}.$$

The computation of $E(U_4)$ takes 180 secs. The increasing computation time from case to case reflects the increasing complexity of the calculations. To coach the the calculation of $E(U_4)$ through Maple, we had e.g. to split up the integration of z_2 into eleven terms and give each term a special consideration. See Appendix D.

4.5. **The expectation of U.** Having calculated the probabilities for all four cases and the expectation of U_j in each case as they are given in table 1, we can combine them to get the expectation of U.

$$(18) \quad E(U) = \sum_{j=1}^{4} p_j\, E(U_j) = \frac{1}{55}(3\,\frac{3}{64} + 6\,(\frac{7}{144} - \frac{\pi^2}{2310}) + 36\,(\frac{11}{216} - \frac{\pi^2}{3465}) + 9\,(\frac{23}{486} + \frac{3\pi^2}{6237})) = \frac{1183}{23760} - \frac{\pi^2}{5445}.$$

5. The volume V of the 'big' tetrahedron B.

We shall start by showing that the 'shape' variable U is independent of the 'size' variable V. The argument is that the position of a point P_k is determined by three independent orthogonal coordinates. The coordinates can be chosen so that one or more are orthogonal to the face(s) of B that they determine while the other(s) are in the face. This is easiest to see in Case 4. Here, x_1 is orthogonal to the face $x_1 = 0$ and y_1 and z_1 are coordinates in this face. In the same way, y_2 and z_3 are orthogonal to faces while x_2, z_2, x_3, and y_3 are coordinates in faces. For P_4, we make a coordinate transformation by replacing x_4 by $t = x_4 + y_4 + z_4$. Then, t is orthogonal to the slanting face and y_4 and z_4 are variables in the face. The functional determinant of this transformation is 1. In this way, the twelve coordinates of the four points are split up into four independent ones that determine V and eight independent ones that determine U. It is easy to see how this split can be done in the other cases. Since U and V are independent, V has the same distribution in all four cases. We shall calculate $E(V)$ for Case 4. Then, the side of the 'big' tetrahedron B is

$$s = x_4 + y_4 + z_4 - x_1 - y_2 - z_3 = t - x_1 - y_2 - z_3.$$

The volume ratio $V = s^3$. We get the expectation of V by doing the integration in (1) though with 1 replaced by s^3 and dividing the result by the result in (1). We get

$$E(V) = \frac{33}{91}. \tag{19}$$

Note that the first eight (U-) integrations are the same in (1) as it stands and with 1 replaced by s^3. They will result in λs^8 and λs^{11}, respectively, where λ is the product of the volumes integrated over. The remaining four (V-) integrations will bring forth the factors $\frac{1}{9 \cdot 10 \cdot 11 \cdot 12}$ and $\frac{1}{12 \cdot 13 \cdot 14 \cdot 15}$, respectively. The ratio between these numbers is 33/91.

6. The expectation of X.

Having calculated the expectations of the independent variables U and V, we get the expectation of the ratio X =volume(T)/volume(A), where A is a given tetrahedron and T is a random tetrahedron inside A as

$$\kappa = E(X) = E(UV) = E(U) \cdot E(V) \tag{20}$$

$$= \left(\frac{1183}{23760} - \frac{\pi^2}{5445}\right) \cdot \frac{33}{91} = \frac{13}{720} - \frac{\pi^2}{15015}. \tag{21}$$

7. Where does the π^2–term come from?

V. Klee studied the value of κ in the 1960:th and was convinced that it should be a rational number, [5]. First, he conjectured the value $\frac{1}{60}$. Monte Carlo tests gave that the true value is closer to $\frac{1}{57}$. The belief in rational numbers rests on the fact that the corresponding κ in two dimensions is rational for several convex polygonal mother sets A, e.g. for a triangle and a square, see [3], [2]. Another argument for rational numbers is that the volume is (the absolute value of) a polynomial of the coordinates of the random points and integration of polynomials results in integer factors in the denominator. The distribution functions of X for a triangle, [9], and a square, [8] have π^2–terms. However, these terms disappear when the moments are calculated. So where do the π^2–terms come from and why are they still present in the first moment in three dimensions?

Because of the absolute value, the integration of the volume polynomial goes to boundaries containing rational functions of the coordinates, like x_{30} and x_{40} in (17). When these are integrated in the next step, the log-function appears. Subsequent integrations will result in functions of the following form

$$\nu(x) = \int_x^1 \frac{\log(t)}{t-1}\,dt + \log(x)\log|1-x| = \int_1^x \frac{\log|1-s|}{s}\,ds. \tag{22}$$

One can deduce the following series expansion

$$\nu(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{x^k}{k^2}\ ,\ |x| \le 1. \tag{23}$$

See also [1], page 1004, and [8] about the ν–function. It follows from the definition in (22) that the ν–function is well defined on the whole real axis. By the definition in (22), it takes the values $\nu(1) = 0$. The value $\nu(0) = \pi^2/6$ is obtained by summing the series in (23) for $x = 1$. The π^2 term will enter the expression for $E(U_j)$ when the lower boundary value 0 is inserted in the integrals. For instance, the y_1–integration in (17) will have a term of the form

$$p(y_1, z_1)\,\nu(1 - \frac{1}{y_1 + z_1}),$$

where $p(y_1, z_1)$ is a polynomial in y_1 and z_1. Here, $p(y_1, z_1)$ is not zero for $y_1 = 1 - z_1$ and we get the π^2-term. The following z_1–integration will produce terms of the form $p(z_1)\,\nu(z_1)$ and $p(z_1)\,\nu(1 - z_1)$ resulting in additional π^2-terms.

The appearance of terms of the form (22) is hard to predict. First, the variable to be integrated shall appear once in the denominator to produce a logarithm and then appear once more in the denominator and there must be no canceling polynomial in front of it. The increased number of succesive integrations in three dimensions compared to two dimensions is an explanation for the π^2-terms in the three-dimensional moments.

The integration of $x^m\nu(x)$, where $m \geq 0$ is an integer, brings forth a term of the form $x^{(m+1)}\nu(x)$ plus logarithmic terms. This explains why the π^2-terms, which are present in the distribution function in two dimensions, disappear when the moments are calculated.

The integration of $x^m\nu(x)$, where $m < 0$ is an integer, brings forth so called polylog functions. Such functions are bound to appear in the distribution functions of the U_j of this paper.

8. The second moment.

The second and other even moments of X are easy to calculate, since the trouble with the absolute value sign isn't present. It has been given earlier by, among others, [4] and [6]. Here, we just change $|U_j|$ to ${U_j}^2$ in the Maple programs and integrate over the whole tetrahedron to get the second moments given in the last column of Table 1. Combining the second moments in Table 1 with their weights, we get

(24)
$$E(U^2) = \sum_{j=1}^{4} p_j\, E({U_j}^2) = \frac{1}{55}(\frac{4}{225} + \frac{6}{200} + \frac{36}{216} + \frac{9}{216}) = \frac{51}{11000}.$$

The second moment of V can be calculated by the argument at the end of section 5 as the ratio between $\frac{1}{15\cdot16\cdot17\cdot18}$ and $\frac{1}{9\cdot10\cdot11\cdot12}$, giving
$$E(V^2) = \frac{11}{68}.$$

Since U and V are independent, we have
$$E(X^2) = E(U^2)E(V^2) = \frac{51}{11000}\cdot\frac{11}{68} = \frac{3}{4000}.$$

We get
$$\sigma_X = \sqrt{E(X^2) - E(X)^2} \approx .0211495.$$

9. The density function for X.

To give idea of the distribution we are working with, we present its density obtained from Monte Carlo tests in Figure 9.

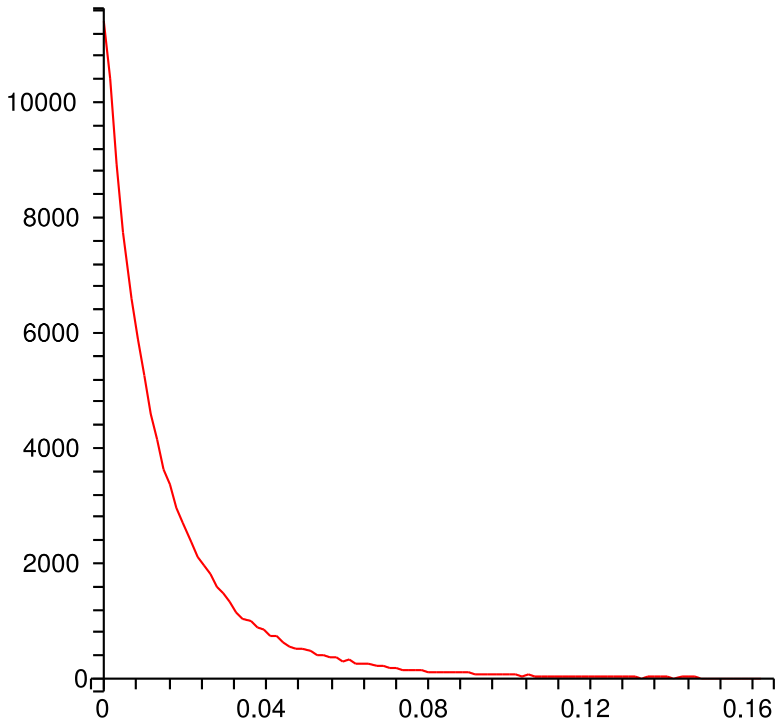

FIGURE 9. Density function for X from Monte Carlo tests. We have $E(X) \approx .017398$ and $\sigma_X \approx .021149$.

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Appendix A. Maple sheet for calculation of $E(U_1)$.

Calculation of the average of the fraction U in case I for terahedron in terahedron

```
> restart;
```

assumption: $0 < n2 < n1$

```
> n1:=y2*z3-z2*y3;
> n2:=z2*x3-x2*z3;
> n3:=x2*y3-y2*x3;
```

The volume is $|U|$. Integrate over positive and negative parts separately

```
> U:=n3+x4*(n1-n3)+y4*(n2-n3);
> t1:=int(U,x4=0..1-y4);
> t2:=int(t1,y4=0..1);
> t3:=int(U,x4);
> t3u:=simplify(subs(x4=-(n3+(n2-n3)*y4)/(n1-n3),t3));
> t3l:=subs(x4=0,t3);
> t4:=int(t3u-t3l,y4);
> t4u:=subs(y4=-n3/(n2-n3),t4);
> t4l:=subs(y4=0,t4);
```

Ep4 is volume integrated over P_4

```
> Ep4:=simplify(t2-2*(t4u-t4l));
```

Start integrating over P_2

```
> s1:=int(Ep4,y2);
> s1l:=limit(s1,y2=z2*(x3+y3)/z3-x2);
> s1u:=limit(s1,y2=1-z2-x2);
```

s1ul is integral over y2

```
> s1ul :=s1u-s1l;
> s2:=int(s1ul,x2);
> s2l:=limit(s2,x2=0);
> s2u:=limit(s2,x2=z2*x3/z3);
```

s2ul is integral ove x2 and y2

```
> s2ul:=simplify(s2u-s2l);
> s3:=int(s2ul,z2);
> s3l:=simplify(subs(z2=0,s3));
```

```
> s3u:=limit(s3,z2=z3/(x3+y3+z3));
```
s3ul is integral over x4,y4,x2,y2, and z2
```
> s3ul:=simplify(subs(ln(-x3)=ln(x3),s3u-s3l),size);
> u1:=int(s3ul,y3);
> u1l:=subs(y3=0,u1);
> u1u:=subs(y3=1-x3-z3,u1);
```
u1ul is integral over y3
```
> u1ul:=simplify(u1u-u1l,size);
> u2:=int(u1ul,x3);
> u2l:=limit(u2,x3=0);
> u2u:=subs(x3=1-z3,u2);
```
u2ul is integral over x3 and y3
```
> u2ul:=simplify(u2u-u2l,size);
> u3:=map(int,u2ul,z3);
> u3a:=subs(ln(-1+z3)=ln(1-z3),u3);
> u3l:=limit(u3a,z3=0);
> u3u:=limit(u3a,z3=1);
> Eu1:=12*2*6^2*(u3u-u3l);
```

Appendix B. Maple sheet for calculation of $E(U_2)$.

Calculation of Eu2. Here x4 is denoted x and z1 is denoted z.
```
> restart;
> with (LinearAlgebra):
> v2:=<x, 1-x, -z>;
> v1:=<x3, y3, z3-z>;
> n:=v1 &x v2;
> simplify(n[1]-n[2]);
```
Assume n3>0 and n1 > n2 .
```
> k:=n[3]*z;
> x20:=k/n[1];
> eta0:=k/n[2];
> z30:=solve(x20=1,z3);
> pl:=n[1]*xi+n[2]*eta+n[3]*zeta;
> ss:=simplify(solve(pl=k,xi+eta+zeta=1,xi,zeta));
```
This is xi1 and y20
```
> tt:=simplify(subs(eta=0,ss),size);
> xi1:=-(z-1)*((x3+y3)*x-x3)/((y3+x3+z3-z)*x-y3*z-z3-x3+z);
> ss:=simplify(solve(pl=k,xi+eta+zeta=1,eta,zeta));
```

```
> tt:=simplify(subs(xi=0,ss),size);
> y20:=-((x3+y3)*x-x3)*(z-1)/((y3+x3+z3-z)*x+x3*(z-1));
> U:=n[1]*x2+n[2]*y2+n[3]*(z2-z);
```

Start integrating U over whole tetrahedron

```
> s1:=simplify(int(U,z2=0..1-x2-y2),size);
> s2:=simplify(int(s1,y2=0..1-x2));
```

s3 is integral over whole P2-space

```
> s3:=simplify(int(s2,x2=0..1));
> s4:=simplify(int(s3,z3=0..z*(1-x3-y3)),size);
> s5:=simplify(int(s4,x3=x/(1-x)*y3..1-y3),size);
> s6:=simplify(int(s5,y3=0..1-x),size);
> s7:=int(s6,z=0..1);
> s8:=int(s7,x=0..1);
```

Integrate y2 and x2 in case a

```
> a1:=simplify(int(s1,y2=y20+(1-x-y20)*x2/x..1-x2));
> a2:=simplify(int(a1,x2=0..x));
> b0:=simplify(int(U,z2=0..(k-n[1]*x2-n[2]*y2)/n[3]));
```

Integrate y2 in case b

```
> b1:=simplify(int(b0,y2));
> b1l:=subs(y2=0,b1);
> b1u:=simplify(subs(y2=y20+(1-x-y20)*x2/x,b1));
> b1ul:=b1u-b1l;
```

Integrate x2 in case b

```
> b2:=simplify(int(b1ul,x2=0..x));
```

Integrate x2 and y2 for extra part of domain in case b

```
> bb1:=simplify(int(b0,x2=x20+(x-x20)*y2/(1-x)..x));
> bb2:=simplify(int(bb1,y2=0..1-x));
```

Sum all parts and integrate with respect to P3

```
> a3:=simplify(int(a2+b2-bb2,z3));
> a3l:=subs(z3=0,a3);
> a3u:=subs(z3=z*(1-x3-y3),a3);
> a3ul:=simplify(a3u-a3l);
> a4:=int(a3ul,y3);
> a4l:=subs(y3=0,a4);
> a4l1:=simplify(a4l);
> a4u:=limit(a4,y3=x3*(1-x)/x);
> a5:=int(a4u,x3);
> a5l:=limit(a5,x3=0);
> a5u:=limit(a5,x3=x);
```

```
> a5ul:=a5u-a5l;
> b4u:=limit(a4,y3=1-x3);
> b5:=int(b4u,x3);
> b5u:=limit(b5,x3=1);
> b5l:=limit(b5,x3=x);
> c5:=int(a4l1,x3);
> c5l:=limit(c5,x3=0);
> c5u:=limit(c5,x3=1);
```

Combine P3-integrals

```
> temp:=simplify(a5ul+b5u-b5l-c5u+c5l);
> temp1:=simplify(subs(ln(-z*x)=ln(z)+ln(x),
  ln(-(z-1)*(-1+x))=ln(1-z)+ln(1-x), temp),size);
```

Integrate with respect to z from 0 to 1

```
> a6:=simplify(int(temp1,z),size);
> a6l:=limit(a6,z=0);
> a6u:=limit(a6,z=1);
> a6ul:=simplify(a6u-a6l);
> a6ul1:=simplify(subs(dilog((-1+x)/x)
        =-dilog(x/(-1+x))-(ln(x)-ln(1-x)-I*Pi)^2/2,a6ul));
> a6ul3:=simplify(subs(ln((-1+x)/x)=ln(1-x)-ln(x),
         ln(x/(-1+x))=ln(x)-ln(1-x), ln(x^2)=2*ln(x),
         ln(-x^2)=2*ln(x),ln(-(x-1)^2)=2*ln(1-x),
         ln(-x*(x-1))=ln(x)+ln(1-x),ln(x-1)=ln(1-x),
         ln((-1+x)^2)=2*ln(1-x),ln(x*(-1+x))=ln(x)+ln(1-x),
         a6ul1),size);
```

Integrate with respect to x from 0 to 1

```
> a7:=int(a6ul3,x);
> a71:=simplify(subs(ln(-x)=ln(x),ln(-1+x)=ln(1-x),a7));
> a7l:=subs(x=0,a71);
> a7u:=limit(a71,x=1);
> a7ul:=simplify(a7u-a7l);
> Eu2:=-8*6^2*a7ul;
```

Appendix C. Maple sheet for calculation of $E(U_3)$.

Calculation of Eu3

```
> restart;
> with (LinearAlgebra):
> v1:=<-x1, y4-1+x1, z4>;
> v2:=<x2-x1, x1-1, z2>;
> n:=v1 &x v2;
```

```
> simplify(n[1]-n[2]);
```

. Assume n1 > n2 .which implies z4*(1-x2) > z2*(1-y4)
The plane is pl=k

```
> k:=simplify(x1*n[1]+(1-x1)*n[2]);
> pl:=simplify(n[1]*xi+n[2]*eta+n[3]*zeta);
> ss:=subs(eta=0,zeta=0,pl-k);
> x30:=simplify(solve(ss=0,xi));
```

Find y30

```
> tt:=simplify(subs(xi=0,pl=k),size);
> solve(tt,eta+zeta=1,eta,zeta);
> y30 := -(x2*y4+z4*x2+x2*x1-y4*x1-x2+x1*z2*y4-x1*z4*x2)/
         (-x2*y4-z4*x2-x2*x1+y4*x1+z4*x1+x2-x1*z2);
> U:=simplify(n[1]*(x3-x1)+n[2]*(y3-1+x1)+n[3]*z3);
```

Start integrating U over whole tetrahedron

```
> s1:=simplify(int(U,z3=0..1-x3-y3),size);
> s2:=simplify(int(s1,y3=0..1-x3));
```

s3 is integral over whole P3-space

```
> s3:=simplify(int(s2,x3=0..1));
> s4:=simplify(int(s3,z4= z2*(1-y4)/(1-x2)..1-y4),size);
> s5:=simplify(int(s4,y4=0..1),size);
> s6:=simplify(int(s5,z2=0..1-x2));
> s7:=simplify(int(s6,x2=0..1));
> s8:=int(s7,x1=0..1);
```

Integrate y3 and x3 in case a

```
> a1:=simplify(int(s1,y3=y30+(1-x1-y30)*x3/x1..1-x3));
> a2:=simplify(int(a1,x3=0..x1));
```

Integrate in z3, y3, and x3 case b

```
> zz3:=(k-n[1]*x3-n[2]*y3)/n[3];
> b0:=simplify(int(U,z3=0..zz3));
> b1:=simplify(int(b0,y3=0..y30+(1-x1-y30)*x3/x1));
> b2:=simplify(int(b1,x3=0..x1));
> c1:=simplify(int(b0,x3=x1..x30+(x1-x30)/(1-x1)*y3));
> c2:=simplify(int(c1,y3=0..1-x1));
> abc2:=a2+b2+c2;
> a3:=simplify(int(abc2,z4));
> a3l:=subs(z4=z2/(1-x2)*(1-y4),a3);
> a3u:=subs(z4=1-y4,a3);
> a3ul:=simplify(a3u-a3l,ln,size);
```

```
> a3ul1:=simplify(subs(ln((z2-1+x2)*(-x1*x2+y4*x1+x2-y4*x2)/
         (-1+x2))=ln(1-x2-z2)+ln(-x1*x2+y4*x1+x2-y4*x2)-ln(1-x2),
        ln(-x1*(z2-1+x2))=ln(x1)+ln(1-x2-z2),
        ln((y4*x2-y4*x1-x2+x1*x2)*(z2-1+x2)/(-1+x2))
        =ln(1-x2-z2)+ln(y4*x2-y4*x1-x2+x1*x2)-ln(1-x2),
         ln(z2*(y4*x2-y4*x1-x2+x1*x2)/(-1+x2))
        =ln(z2)+ln(y4*x2-y4*x1-x2+x1*x2)-ln(1-x2),a3ul),size);
> a4:=int(a3ul1,y4);
> a4l:=subs(y4=0,a4);
> a4l1:=simplify(subs(ln(x2*(-x1+x1*x2+1-x2+x1*z2-z2)
       /(-1+x2)) =ln(x2)-ln(1-x2)+ln(1-x1)+ln(1-x2-z2),a4l));
> a4l2:=simplify(subs(ln(-x2*(-1+x2+z2)*(x1-1)/(-1+x2))
       =ln(x2)+ln(1-x2-z2)+ln(1-x1)-ln(1-x2),a4l1),size);
> a4u:=subs(y4=1,a4);
> a4ul:=simplify(subs(ln(-x1*(-1+x2+z2))=ln(x1)+ln(1-x2-z2),
       ln(-(x1-2*x1*x2+x1*x2^2+x1*z2*x2-x2*z2)/(-1+x2))
       =ln(x1-2*x1*x2+x1*x2^2+x1*x2*z2-x2*z2)-ln(1-x2),
       a4u-a4l2));
> a4ul1:=simplify(subs(ln(-x1*(-1+x2+z2))=ln(x1)+ln(1-x2-z2),
        ln(-(x1-2*x1*x2+x1*x2^2+x1*z2*x2-x2*z2)/(-1+x2))
        =ln(x1-2*x1*x2+x1*x2^2+x1*x2*z2-x2*z2)-ln(1-x2),
        ln(-z2*x2*(x1-1)/(-1+x2))=ln(z2)+ln(x2)+ln(1-x1)
        -ln(1-x2), ln(-x1*(-1+x2))=ln(x1)+ln(1-x2),
        ln(-x2*(x1-1))=ln(x2)+ln(1-x1),ln(-x1*z2)
        =ln(x1)+ln(z2), ln(-(z2-1)*(x1-1))=ln(1-z2)+ln(1-x1),
        a4ul));
> a5:=simplify(int(a4ul1,z2),size);
> a51:=collect(a5,ln(z2));
> nops(a51);
> a511:=op(1,a51);
> a512:=op(2,a51);
> a511l:=limit(a511,z2=0);
> a512l:=simplify(subs(z2=0,a512));
> a5l:=a511l+a512l;
> a5u:=simplify(subs(z2=1-x2,a5));
> a5ul:=a5u-a5l;
> a5ul1:=simplify(subs(ln(-(-1+x2)*(-x2+x1))
        =ln(1-x2)+ln(x1-x2), ln(x1*(-1+x2)^2)
        =ln(x1)+2*ln(1-x2),ln(-x2*(x1-1))=ln(x2)+ln(1-x1),
        ln(-(-1+x2)*x1)=ln(1-x2)+ln(x1),ln(x1-1)=ln(1-x1),
        ln((-x2+x1)/x1)=ln(x2-x1)-ln(x1),
        ln((-x2+x1)/(x1-1))=ln(x2-x1)-ln(1-x1),
        ln(-x2+x1)=ln(x2-x1),a5ul));
> a5ult:=collect(a5ul1,ln(1-x2));
> nops(a5ult);
> a5ult1:=op(1,a5ult);
```

```
> a5ult2:=op(2,a5ult);
> a61:=int(a5ult1,x2);
> a5ult3:=collect(a5ult2,ln(x2));
> nops(a5ult3);
> a5ult4:=op(1,a5ult3);
> a62:=int(a5ult4,x2);
> a5ult5:=op(2,a5ult3);
> a63:=int(a5ult5,x2);
> a61l:=simplify(subs(x2=0,a61));
> a62l:=limit(a62,x2=0);
> a63l:=simplify(subs(x2=0,a63));
> a61u:=limit(a61,x2=1);
> a62u:=simplify(subs(x2=1,a62));
> a63u:=simplify(subs(x2=1,a63));
> a6ul:=simplify(a61u+a62u+a63u-a61l-a62l-a63l);
> a6ul1:=simplify(subs(ln(-x1)=ln(x1),ln(-x1^2)=2*ln(x1),
        ln(-(x1-1)^2)=2*ln(1-x1),ln(-x1*(x1-1))
        =ln(x1)+ln(1-x1), ln(x1-1)=ln(1-x1),
        ln((x1-1)/x1)=ln(1-x1)-ln(x1),
        ln(x1/(x1-1))=ln(x1)-ln(1-x1),a6ul));
> a6ul2:=simplify(subs(dilog((x1-1)/x1)
       =-dilog(x1/(x1-1))-(ln(x1)-ln(1-x1)-I*Pi)^2/2,a6ul1));
> a7:=int(a6ul2,x1);
> a71:=simplify(subs(ln(x1-1)=ln(1-x1),ln(1/(x1-1))
      =-ln(1-x1), ln(x1/(x1-1))=ln(x1)-ln(1-x1),
      ln((x1-1)/x1)=ln(1-x1)-ln(x1),a7));
> a72:=simplify(subs(ln(x1/(x1-1))=ln(x1)-ln(1-x1),
       ln(-1/x1)=-ln(x1),ln((x1-1)/x1)=ln(1-x1)-ln(x1),a71));
> a7l:=limit(a72,x1=0);
> a7u:=limit(a72,x1=1);
> a7ul:=simplify(a7u-a7l);
> Eu3:=-8*6*2*(a7ul);
```

Appendix D. Maple sheet for calculation of $\mathrm{E}(U_4)$.

Calculation of Eu4

```
> restart;
> with (LinearAlgebra):
> q1:=<-x3, y1-y3, z1>;
> q2:=<x2-x3, -y3, z2>;
```

```
> n:=q1 &x q2;
```

Notice one positive and one negative solution above

```
> U:=simplify(n[1]*(x4-x3)+n[2]*(y4-y3)+n[3]*(1-x4-y4));
```

Assume $n1 > n3$ and $n2 > n3$ which implies $N1 > 0$ and $N2 > 0$.
$x40 = T/N1$ and $y40 = T/N2$

```
> N1:=simplify(n[1]-n[3]);
> N2:=simplify(n[2]-n[3]);
> x40:=simplify((n[1]*x3+n[2]*y3-n[3])/(n[1]-n[3]));
> y40:=simplify((n[1]*x3+n[2]*y3-n[3])/(n[2]-n[3]));
> T:=collect(op(2,x40),{x3,y3});
> simplify(subs(x4=x40*(1-y4/y40),U));
```

$T < N1$ when $x40 < 1$. $x40 > 0$ when $T > 0$

```
> simplify(T-N1,size);
```

$T < N2$ when $y40 < 1$. $y40 > 0$ when $T > 0$

```
> simplify(T-N2,size);
> simplify(n[1]-n[3],size);
> simplify(n[2]-n[3],size);
> y30:=y1/(1-z1);
> x30:=x2/(1-z2);
```

Integrate over whole tetrahedron

```
> s1:=simplify(int(U,x4=0..1-y4),size);
> s2:=simplify(int(s1,y4=0..1),size);
> s3:=int(s2,y3=0..1-x3);
> s4:=int(s3,x3=0..1);
> s5:=int(s4,z2=0..1-x2);
> s6:=int(s5,x2=0..1);
```

This is average over whole 8-dimensional space without $n1 > n3$ and $n2 > n3$

```
> Uaverage:=s6*8;
```

Start with integral over part where $U < 0$ and $n_1 > n_3$ and $n_2 > n_3$

```
> a1:=simplify(int(U,x4=0..x40*(1-y4/y40)),size);
> a2:=simplify(int(a1,y4=0..y40),size);
> a3:=simplify(int(a2,y3),size);
> a3u:=simplify(subs(y3=y30*(1-x3/x30),a3),ln,size);
> a3u1:=simplify(subs(ln(-y1*(x2+z2-1)*(-z1*x2+z1*x3-x3*z2)
        /(-1+z1)/x2)=ln(y1)+ln(1-x2-z2)+ln(-z1*x2+z1*x3-x3*z2)
        -ln(1-z1)-ln(x2),ln((y1+z1-1)*(-z1*x2+z1*x3-x3*z2)
        /(-1+z1))=ln(1-y1-z1)+ln(-z1*x2+z1*x3-x3*z2)
        -ln(1-z1),a3u),size);
```

```
> a3l:=simplify(subs(y3=0,a3),size);
> a3ul:=a3u1-a3l;
> a4:=int(a3ul,x3);
> a4l:=simplify(subs(x3=0,a4),size);
> a4l1:=simplify(subs(ln(-z1*x2)=ln(z1)+ln(x2),
       ln(-z1*x2-y1*x2)=ln(x2)+ln(y1+z1),
       ln(z2*y1+y1*x2)=ln(y1)+ln(x2+z2),  a4l),size);
> a4u:=simplify(subs(x3=x30,a4),ln,size);
> a4u1:=simplify(subs(ln(-x2*z2*(y1+z1-1)/(z2-1))=
       ln(x2)+ln(z2)+ln(1-y1-z1)-ln(1-z2),
       ln(-x2*z2*(-1+z1)/(z2-1))=ln(x2)+ln(z2)+ln(1-z1)
       -ln(1-z2), ln(y1*z2*(x2+z2-1)/(z2-1))
       =ln(y1)+ln(z2)+ln(1-x2-z2)-ln(1-z2),a4u),size);
> a4ul:=simplify(subs(ln(-(z1-1+y1)*z1*x2/(-1+z1))
       =ln(1-y1-z1)+ln(z1)+ln(x2)-ln(1-z1),
       ln(-z2*y1*(z1-1+x2)/(z2-1))=ln(z2)
       +ln(y1)+ln(1-x2-z2)-ln(1-z2),ln(-y1*(z2-1+x2)*z1
       /(-1+z1))=ln(y1)+ln(1-x2-z2)+ln(z1)-ln(1-z1),
       ln(-z2*y1*(z2-1+x2)/(z2-1))=ln(z2)+ln(y1)
       +ln(1-x2-z2)-ln(1-z2),ln(-z2*y1-y1*x2)=ln(y1)
       +ln(x2+z2),a4u1-a4l1),size);
> a4ul1:=collect(a4ul,ln(z2));
> nops(a4ul1);
> t1:=op(1,a4ul1);
> rest:=op(2,a4ul1);
> rest1:=collect(rest,ln(1-x2-z2));
> nops(rest1);
> t2:=op(1,rest1);
> rest:=op(2,rest1);
> rest1:=collect(rest,ln(x2+z2));
> nops(rest1);
> t3:=op(1,rest1);
> rest:=op(2,rest1);
> rest1:=collect(rest,ln(1-z2));
> nops(rest1);
> t4:=op(1,rest1);
> t5:=op(2,rest1);
> it1:=simplify(int(t1,x2),size);
> it2:=simplify(int(t2,x2),size);
> it3:=simplify(int(t3,x2),size);
> it4:=simplify(int(t4,x2),size);
> it5:=simplify(int(t5,x2),size);
```

```
> it1l:=subs(x2=0,it1);
> it2l:=subs(x2=0,it2);
> it3l:=subs(x2=0,it3);
> it4l:=subs(x2=0,it4);
> it5l:=subs(x2=0,it5);
> a5l:=it1l+it2l+it3l+it4l+it5l;
> it1u:=simplify(subs(x2=1-z2,it1),ln,size);
> it2u:=limit(it2,x2=1-z2);
> it3u:=simplify(subs(x2=1-z2,it3),ln,size);
> it4u:=simplify(subs(x2=1-z2,it4),size);
> it5u:=simplify(subs(x2=1-z2,it5),size);
> a5u:=it1u+it2u+it3u+it4u+it5u;
> a5ul:=subs(ln(-1+z1)=ln(1-z1),ln(-(-z1+z2)/z1)
       =ln(z1-z2)-ln(z1),ln((- y1-z1+z2)/(-z1-y1))
       =ln(y1+z1-z2)-ln(y1+z1),ln((y1+z1-z2)/(z1-1+y1))
       =ln(y1+z1-z2)-ln(1-y1-z1),ln((z1-z2)/(-1+z1))
       =ln(z1-z2)-ln(1-z1),a5u-a5l);
> temp:=simplify(a5ul);
> nops(temp);
> temp1:=collect(temp,dilog);
> nops(temp1);
> v1:=op(1,temp1);
> iv1:=int(v1,z2);
> iv1l:=limit(iv1,z2=0);
> iv1u:=simplify(subs(z2=1,iv1));
> iv1ul:=simplify(iv1u-iv1l,size);
> v2:=op(2,temp1);
> iv2:=int(v2,z2);
> iv2l:=simplify(subs(z2=0,iv2),size);
> iv2u:=limit(iv2,z2=1);
> iv2ul:=simplify(iv2u-iv2l,size);
> v3:=op(3,temp1);
> iv3:=int(v3,z2);
> iv3l:=simplify(subs(z2=0,iv3));
> iv3u:=simplify(subs(z2=1,iv3));
> iv3ul:=iv3u-iv3l;
> v4:=op(4,temp1);
> iv4:=int(v4,z2);
> iv4l:=simplify(subs(z2=0,iv4));
```

```
> iv4u:=simplify(subs(z2=1,iv4));
> iv4ul:=simplify(subs(ln((-1+z1)/z1)=ln(1-z1)-ln(z1),
        iv4u-iv4l),size);
> v5:=op(5,temp1);
> iv5:=int(v5,z2);
> iv5l:=simplify(subs(z2=0,iv5));
> iv5u:=simplify(subs(z2=1,iv5));
> iv5ul:=simplify(subs(ln((z1-1+y1)/(y1+z1))
        =ln(1-y1-z1)-ln(y1+z1),iv5u-iv5l),size);
> v6:=op(6,temp1);
> iv6:=int(v6,z2);
> iv6l:=simplify(subs(z2=0,iv6));
> iv6u:=simplify(subs(z2=1,iv6));
> iv6ul:=simplify(iv6u-iv6l,size);
> v7t:=simplify(subs(ln(-y1-z1+z2)=ln(y1+z1-z2),
       ln(-z1+z2)=ln(z1-z2),op(7,temp1)),size);
> v7t1:=collect(v7t,ln(z2));
> nops(v7t1);
> v71:=op(1,v7t1);
> v71a:=collect(v71,ln(z1-z2));
> nops(v71a);
> v711:=op(1,v71a);
> iv711:=int(v711,z2);
> iv711u:=simplify(subs(z2=1,iv711));
> iv711l:=limit(iv711,z2=0);
> iv711ul:=simplify(subs(ln(-z1)=ln(z1),
          ln(-1+z1)=ln(1-z1),iv711u-iv711l),size);
> v712:=op(2,v71a);
> iv712:=int(v712,z2);
> iv712l:=limit(iv712,z2=0);
> iv712u:=simplify(subs(z2=1,iv712));
> iv712ul:=simplify(subs(ln(z1-1+y1)=ln(1-y1-z1),
          ln(-z1-y1)=ln(y1+z1),  iv712u-iv712l));
> v72:=op(2,v7t1);
> v72a:=collect(v72,ln(1-z2));
> nops(v72a);
> v721:=op(1,v72a);
> v721a:=collect(v721,ln(z1-z2+y1));
> nops(v721a);
> v7211:=op(1,v721a);
```

```
> iv7211:=int(v7211,z2);
> iv7211l:=simplify(subs(z2=0,iv7211));
> iv7211u:=limit(iv7211,z2=1);
> iv7211ul:=simplify(iv7211u-iv7211l);
> v7212:=op(2,v721a);
> iv7212:=int(v7212,z2);
> iv7212l:=simplify(subs(z2=0,iv7212));
> iv7212u:=limit(iv7212,z2=1);
> iv7212ul:=simplify(iv7212u-iv7212l);
> v722:=op(2,v72a);
> iv722:=int(v722,z2);
> iv722l:=simplify(subs(z2=0,iv722));
> iv722u:=simplify(subs(z2=1,iv722));
> iv722ul:=simplify(iv722u-iv722l);
> a6ul:=simplify(subs(ln((y1+z1)/(z1-1+y1))=ln(y1+z1)
       -ln(1-y1-z1),ln(z1/(-1+z1))=ln(z1)-ln(1-z1),
       ln((y1+z1-1)/(y1+z1))=ln(1-y1-z1)-ln(y1+z1),
       ln(z1-1)=ln(1-z1),ln(y1+z1-1)=ln(1-y1-z1),
       iv1ul+iv2ul+iv3ul+iv4ul+iv5ul+iv6ul+iv711ul
       +iv712ul+iv7211ul+iv7212ul+iv722ul),size);
> a6ul1:=collect(a6ul,dilog);
> nops(a6ul1);
> p1:=simplify(op(1,a6ul1),size);;
> ip1:=int(p1,y1);
> p2:=op(2,a6ul1);
> ip2:=simplify(int(p2,y1),size);
> p3:=op(3,a6ul1);
> ip3:=int(p3,y1);
> p4:=op(4,a6ul1);
> ip4:=int(p4,y1);
> p5:=simplify(subs(ln(-y1-z1)=ln(y1+z1),
      ln(-z1)=ln(z1),   a6ul1-p1-p2-p3-p4));
> ip5:=int(p5,y1);
> a7:=simplify(subs(ln(z1-1+y1)=ln(1-y1-z1),
     ln(-1/(y1+z1))=-ln(y1+z1),ln(1/(z1-1+y1))
     =-ln(1-y1-z1),ln(1-1/(y1+z1))=ln(1-y1-z1)
     -ln(y1+z1), ln(1+1/(z1-1+y1))=ln(y1+z1)
     -ln(1-y1-z1),ip1+ip2+ip3+ip4+ip5),size);
> a7l:=simplify(subs(y1=0,a7));
> a7u:=limit(a7,y1=1-z1);
> a7ul:=simplify(subs(ln(z1/(-1+z1))=ln(z1)-ln(1-z1),
       a7u-a7l),size);
```

```
> a8:=map(int,expand(a7ul),z1);
> a81:=simplify(subs(ln(1/(-1+z1))=-ln(1-z1),
      ln(-1/z1)=-ln(z1),ln((-1+z1)/z1)=ln(1-z1)-ln(z1),
       ln(z1/(-1+z1))=ln(z1)-ln(1-z1),a8));
> a82:=simplify(subs(ln((-1+z1)/z1)=ln(1-z1)-ln(z1),
       ln(z1/(-1+z1))=ln(z1)-ln(1-z1),a81));
> a83:=simplify(subs(dilog((-1+z1)/z1)=Pi^2/6
      -dilog(1/z1)+(ln(1-z1)-ln(z1))*ln(z1)+I*Pi*ln(z1),
      dilog(z1/(-1+z1))=Pi^2/6-dilog(1/(1-z1))
      +(ln(z1)-ln(1-z1))*ln(1-z1)+I*ln(1-z1),a82));
> a84:=simplify(subs(dilog(-1/(-1+z1))=-dilog(1-z1)
       -(ln(1-z1))^2/2,dilog(1/z1)=-dilog(z1)
      -(ln(z1))^2/2,ln(-1+z1)=ln(1-z1),a83));
> a8l:=limit(a84,z1=0);
> a8u:=limit(a84,z1=1);
> a8ul:=a8u-a8l;
> Eu4:=Uaverage-96*a8ul;
```

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