THE AREA OF A RANDOM CONVEX POLYGON.

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ABSTRACT. We consider the area of the convex hull of n points with random positions in a square. We give the distribution function of the area for three and four random points. We also present some results on the number of vertices of the convex hull. Results from Monte Carlo tests with large n are presented and compared with asymptotic estimates.

1. INTRODUCTION

Random objects are used when testing motion algorithms in computer vision. The typical object is the convex hull of a set of points in a square image. The performance of such an algorithm depends on several parameters and one of them is the size of the object. This raised our interest in the area of the polygon spanned by n randomly generated points in a square.

We shall denote the square by K and the convex hull of the n points by T and shall consider the random variable $X = area(T)/area(K)$. It is well known that an affine transformation will preserve the ratio X. This follows from the fact that the area scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation. This means that our results hold when K is a parallelogram.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [10]. J. J. Sylvester considered the problem of a random triangle T in an arbitrary convex set K and posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown, [2] that $\kappa = \frac{11}{144}$ for K a square.

A. Reńyi and R. Sulanke, [8] and [9], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. R. E. Miles [7] generalizes these asymptotic estimates for K a

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circle to higher dimensions. C. Buchta and M. Reitzner, [4], has given values of κ (generalized to three dimensions) for $n \geq 4$ points in a tetrahedron. H. A. Alikoski [2] has given expressions for κ when $n = 3$ and K a regular r-polygon. Here we deduce the whole distribution of X for $n = 3$ and $n = 4$ when K is a square. From these distributions we calculate some probabilities for the number of vertices of random convex polygons. We also give some asymptotic estimates in the spirit of Reńyi and R. Sulanke. All calculated quantities of this paper have been confirmed by Monte Carlo tests.

2. Notation and formulation.

As K, we will take the unit square $(0 \leq x \leq 1, 0 \leq y \leq 1)$. We use a constant probability density in K for generating n random points in K. The coordinates of the points will be denoted (x_k, y_k) for $1 \leq k \leq n$. Each x_k and y_k is evenly distributed in $(0, 1)$ and they are independent. Let T be the convex hull of the n points. We shall determine the probability distribution of the random variable $X = area(T)/area(K)$ when $n = 3$ and 4.

The generated T spans a rectangle with sides parallel to the sides of K. We will denote this spanned rectangle B and call it the 'big' rectangle. The random variable X, that we study will be written as the product of two random variables

$$V = area(B)/area(K) \text{ and } W = area(T)/area(B).$$

Roughly speaking, W describes the shape of T and V its size. We shall show in section 3.3 that V and W are independent. We shall determine the distributions of V and W and combine them to get the distribution of $X = VW$.

3. The convex hull of three points.

For $n = 3$, the convex hull is with probability one a triangle having maximal area $= 1/2$.

Without loss of generality (WLOG), we can name (=number) the points after the size of their x-coordinates, so that

$$x_1 \leq x_2 \leq x_3.$$

Then, the numbering of the corresponding y-coordinates will be a permutation of 1, 2, and 3. There are $3! = 6$ such permutations which are equally probable to occur. Potentially, these 6 permutations correspond to different geometrical configurations. However, a reversal of the y-index sequence corresponds to turning the triangle upside down, which doesn't affect its area. This leaves us with $3!/2 = 3$ sets of permutations. Each set has a characteristic geometry and all sets have the same probability of occurring. Such sets will be called cases. As we shall see, two of these cases have the same kind of geometry, so we

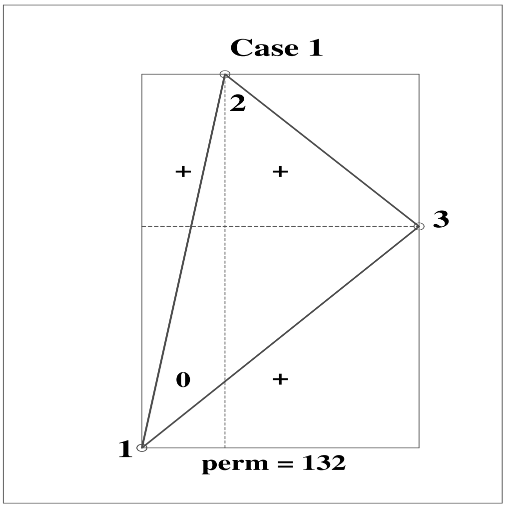

FIGURE 1. Triangle with point 2 on the boundary of the 'big' subrectangle that it spanns (Case 1) . The sum of the areas of the small (dashed) subrectangles marked with a + equals twice the triangle area.

are left with two cases needing consideration and these are depicted in Figures 1 and 2.

Figure 1 depicts the y-permutation {1,3,2}. Changing to {3,1,2} will flip the figure left-right, which preserves the geometry. This means that the case depicted in Figure 1 occurs twice as often as that in Figure 2, which corresponds to the permutation is {1,2,3}

3.1. **The distribution of V.** The horisontal side of the 'big' rectangle has length $x_3 - x_1$. The vertical side has length $y_{max} - y_{min}$.

The distribution function for the k-th ordered variable among n variables with density = 1 in (0,1) is

$$F_{n,k}(x) = \frac{n!}{(k-1)!(n-k)!} \int_0^x t^{k-1}(1-t)^{n-k}\, dt. \tag{1}$$

For $n = 3$ the distribution functions for the smallest, middle, and largest variables are

$$F_{3,1}(x) = 1 - (1-x)^3, \tag{2}$$

$$F_{3,2}(x) = 6 \int_0^x t(1-t)\, dt = 3x^2 - 2x^3, \tag{3}$$

$$F_{3,3}(x) = x^3. \tag{4}$$

The lengths of the intervals $(0, x_1), (x_1, x_2), (x_2, x_3),$ and$(x_3, 1)$ all have the same probability distribution, namely $F_{3,1}(x)$. The lengths

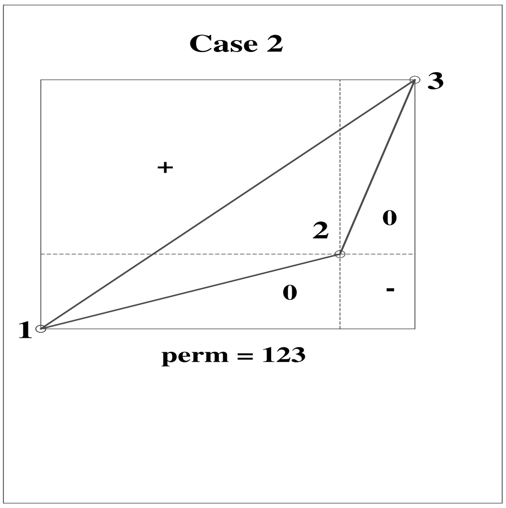

FIGURE 2. Triangle with point 2 in the interior of the 'big' subrectangle (Case 2). The difference of the areas of the small (dashed) subrectangles marked with a + and - equals twice the triangle area.

of the intervals $(0, x_2), (x_1, x_3)$, and $(x_2, 1)$ all have the same probability distribution, namely $F_{3,2}(x)$. This implies that the sides of the 'big' rectangle in Figures 1 and 2 have the probability distributions $F_{3,2}(x)$ and $F_{3,2}(y)$ and these distributions are independent because the x-coordinates are independent of the y-coordinates. Let $G_3(v)$ be the distribution function for the area of the 'big' rectangle. We have $G_3(v) =$Prob$(area = xy \leq v) = 1-$Prob$(xy > v)$. We get [1]

$$G_3(v) = 1 - \int_v^1 (1 - F_{3,2}(v/x))\, dF_{3,2}(x) \tag{5}$$
$$= 28v^3 - 27v^2 - 6v^2(3 + 2v)\log(v), \quad 0 \leq v \leq 1.$$

3.2. **The distribution of W.** We shall study the fraction W that the triangle area takes up of its surcumscribed 'big' rectangle B.

To avoid the factor 1/2 in the area formula for a triangle in numerous places below, we will work with twice the fraction $U = 2W$ and with twice the fraction $Y = 2X$. Let $H_{3,1}(u) =$ Prob$(2W = U \leq u)$ in Case 1 depicted in Figure 1. Let $H_{3,2}(u)$ be the corresponding probability for Case 2 of Figure 2.

When the 'big' rectangle is fixed, we have a constant conditional density distribution in B. In Case 1, x_2 has a constant density in (x_1, x_3) and y_3 a constant density in (y_1, y_2). In Case 2. (x_2, y_2) has a constant density in the 'big' rectangle. For the calculation of the

[1] We are indebted to Maple for helping us calculate the integrals of this paper.

(double) fraction U, we dilate the 'big' rectangle so that it fills the unit square. This doesn't affect the fraction.

The middle coordinates (x_2, y_3) in Figure 1 and (x_2, y_2) in Figure 2 determine a splitting of the 'big' rectangle into four subrectangles. It takes some time to realize that the area of the triangle in Figure 1 is half the sum of the areas of the three subrectangles marked with a +. In Figure 2, the area of the triangle is half the difference of the two subrectangles marked with + and -.

In Case 1, we have $U = 1-x_2y_3$, so we have $U \le u$ when $x_2y_3 \ge 1-u$. We get

(6)
$$H_{3,1}(u) = \int_{1-u}^{1} dx_2 \int_{(1-u)/x_2}^{1} dy_3 = u + (1-u)\log(1-u), \quad 0 \le u \le 1.$$

In Case 2, we have $U = |x_2(1-y_2) - (1-x_2)y_2| = |x_2 - y_2|$. Assuming that point 2 is below the diagonal, we have $U = x_2 - y_2$. This assumption is accounted for by the factor 2 in front of the following integral

$$(7) \qquad H_{3,2}(u) = 1 - 2\int_{u}^{1} dx_2 \int_{0}^{x_2-u} dy_2 = 2u - u^2, \quad 0 \le u \le 1.$$

Since Case 1 occurs twice as often as Case 2, we get the distribution function for U irrespective of case as

(8)
$$H_3(u) = \frac{2}{3}H_{3,1}(u) + \frac{1}{3}H_{3,2}(u) = \frac{1}{3}(4u - u^2 + 2(1-u)\log(1-u)), \quad 0 \le u \le 1.$$

3.3. V **and** W **are independent.** The variables V and W are independent because they depend on different x- and y- values. In Figure 1, the size of the 'big' rectangle, which determines V, is a function of x_1, y_1, y_2, and x_3, The sizes of the subrectangles, which determine W, are functions of x_2 and y_3. In Figure 2, V is a function of x_1, y_1, x_3, and y_3, while W is a function of x_2 and y_2.

This arguement would not hold with a rotated coordinate system. The deeper reason why V and W are independent is that W depends on variations along the sides of B while V depends on variations orthogonal to the sides.

3.4. **The distribution of** X **for three generated points.** Let $F_3(x)$ be the distribution function for X. We have $X \le x$ when $Y = 2X = 2WV = UV \le y$

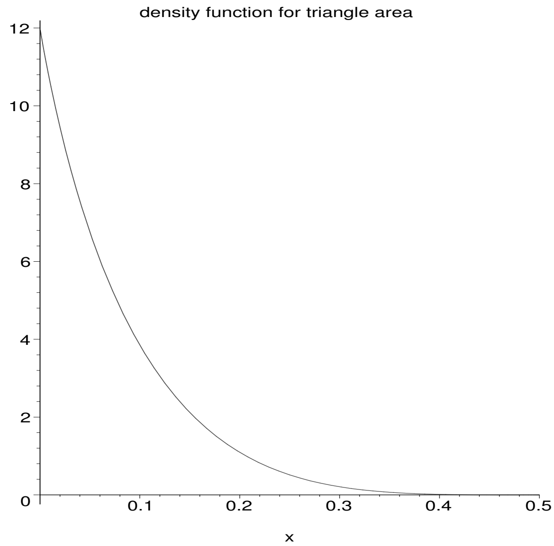

FIGURE 3. Density function for the area of an arbitrary triangle.

$$
\begin{aligned}
F_3(y/2) &= \int_0^1 G_3(y/u)\, dH_3(u) = \\
&= [G_3(y/u)H_3(u)]_0^1 - \int_y^1 H_3(u)\frac{d}{du}G_3(y/u)\, du = \\
&= G_3(y) - \int_y^1 H_3(u)\frac{d}{du}G_3(y/u)\, du, \quad 0 \le y \le 1.
\end{aligned} \tag{9}
$$

The partial intergration in (9) is used to avoid integrating to the lower bound $u = 0$. To write the result, we need the ν function

$$
\nu(x)) = -\int_0^x \frac{\log|1-t|}{t}\, dt. \tag{10}
$$

This function is the real part of the dilogarithm function $\mathrm{Li}_2(x)$ discussed by Euler in 1768 and named by Hill, [5]. $\nu(x)$ is well defined on the whole real axis. Some properties of $\nu(x)$ are given in Appendix A.

We will not carry out the integration (9) in detail, but will just give the result

$$
\begin{aligned}
F_3(x) &= \frac{4x}{3}(10-17x) - \frac{16x^3}{3}(17 - 3\log(2x))\log(2x) \\
&+ \frac{2}{3}(1 - 16x - 68x^2)(1-2x)\log(1-2x) + 16x^2(3+2x)(\nu(2x) - \frac{\pi^2}{6}) \\
&\qquad 0 \le x \le 1/2.
\end{aligned} \tag{11}
$$

The density function dF_3/dx is shown in Figure 3.

The first moments and the standard deviation of the triangle area are

$$\alpha_1 = \int_0^{\frac{1}{2}} x\, dF_3(x) = \frac{11}{144} \approx .076389, \tag{12}$$

$$\alpha_2 = \int_0^{\frac{1}{2}} x^2\, dF_3(x) = \frac{1}{96}, \tag{13}$$

$$\sigma = \sqrt{\alpha_2 - {\alpha_1}^2} = \frac{\sqrt{95}}{144} \approx .067686. \tag{14}$$

4. The area of the convex hull of four points.

The convex hull of four points can be either a triangle or a quadrangle. The triangle case occurs if one point is generated inside the triangle spanned by the other three points. The probability for this to happen equals four times the expected size of the triangle $= 4\alpha_1 = \frac{11}{36}$.

To find the distribution function for the area of the convex hull of four points, we shall go about in the same way as above and number the points so that

$$x_1 \le x_2 \le x_3 \le x_4.$$

The corresponding y_k can be permuted in $4! = 24$ ways. We form 12 sets each consisting of a permutation and its reversed permutation. All these sets are equally probable to occur. The permutations, but not the reversed ones, are listed in Table 1. The 13 cases in the table each correspond to a geometrical configuration. Cases 1 - 5 are triangles, cases 6 - 13 are quadrangles.

4.1. **The distribution of V.** Like the case with three points, the generated convex hull spans a 'big' rectangle with sides $x_4 - x_1$ and $y_{max} - y_{min}$. The distributions of these sidelengths are $F_{4,3}(x)$ and $F_{4,3}(y)$ where

$$F_{4,3}(x) = 4x^3 - 3x^4. \tag{15}$$

In analogy with (5), we get the distribution function for the 'big' rectangle

$$G_4(v) = 1 - \int_v^1 (1 - F_{4,3}(v/x))\, dF_{4,3}(x) = \tag{16}$$
$$= 81v^4 - 80v^3 - 12v^3(4+3v)\log(v), \quad 0 \le v \le 1.$$

4.2. **The distribution of U.** We will have to determine a $H_{4,k}$ for each of the 13 cases of Table 1. We must also determine the probabilities for these cases to occur, which are given in Table 1. Each of these cases requires the evalution of an integral in four-space over a

		case												
#	permutation	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1234				$\frac{1}{3}$								$\frac{1}{3}$	$\frac{1}{3}$
2	1243	$\frac{1}{3}$								$\frac{2}{3}$				
3	1423		$\frac{1}{3}$								$\frac{2}{3}$			
4	4123			$\frac{1}{2}$					$\frac{1}{2}$					
5	4132		$\frac{1}{3}$								$\frac{2}{3}$			
6	1432			$\frac{1}{2}$					$\frac{1}{2}$					
7	1342		$\frac{1}{3}$								$\frac{2}{3}$			
8	1324					$\frac{1}{3}$						$\frac{2}{3}$		
9	3124		$\frac{1}{3}$								$\frac{2}{3}$			
10	3142						1							
11	3412							1						
12	4312	$\frac{1}{3}$								$\frac{2}{3}$				
Σ		$\frac{2}{3}$	$\frac{4}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1	1	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$E\{U\}$		$\frac{2}{3}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{10}{9}$	$\frac{8}{9}$	$\frac{28}{27}$	$\frac{2}{3}$	1	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{9}$

TABLE 1. Probabilities of geometrical cases for each permutation. Cases 1 - 5 are triangles. Cases 6 - 13 are quadrangles. The Σ-row is the sum of the probabilities above. The $E\{U\}$-row shows the expectation of U for each case.

set bounded by linear and nonlinear inequalities. Having limited geometrical intuition in four-space, each integral evaluation has been a challenge. We are not going to describe all these calculations in detail here but shall carry out the calculations for only one case in the text. Two more cases are done in appendicies B and C. The author can provide the interested reader with Maple files describing the remaining cases.

Consider the possible geometric configurations that can occur for the y-permutation 1234. These are shown in Figure 4.

In these figures, we don't show the whole unit square but only the 'big' rectangle. This rectangle has been enlarged by an affine transformation so that it fills a unit square. This doesn't affect the area fraction that we are studying.

The triangle Case 4 depicted in Figure 4 is in fact the subcase of Case 4 having point 2 as a vertex below the diagonal and point 3 as an interior point. Case 4 also includes the situations with point 2 above the diagonal and point 3 a vertex below and above the diagonal.

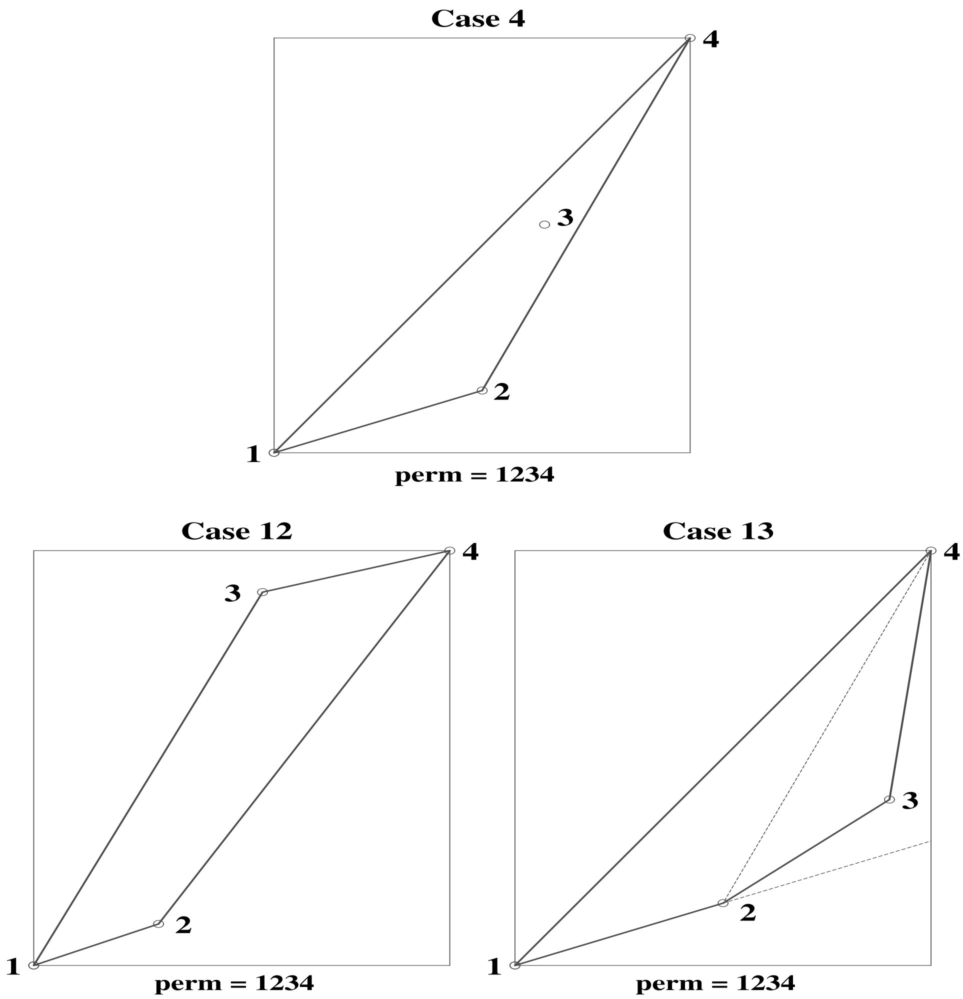

FIGURE 4. Cases 4, 12 , and 13.

These four subcases all have the same probability and the same type of geometry. We shall do the case in Figure 4.

For fixed values of x_2 and y_2, the conditional probability density for point 3 is constant and equals $1/(1-x_2)(1-y_2)$ in the rectangle $x_2 \leq x_3 \leq 1$, $y_2 \leq y_3 \leq 1$. We have the conditional probability that point 3 is interior in the triangle spanned by 1, 2, and 4 :

$$p(x_2, y_2) = \int_{x_2}^{1} \frac{dx_3}{1-x_2} \int_{1-(1-y_2)(1-x_3)/(1-x_2)}^{x_3} \frac{dy_3}{1-y_2}. \tag{17}$$

This probability shall be integrated over all positions of point 2. The density for point 2 is the same as for the smaller of two ordered points in the unit interval, i.e. $2(1-x_2)dx_2 \times 2(1-y_2)dy_2$. Notice that the factors $1-x_2$ and $1-y_2$ of the density cancel the same factors in (17). We get

(18)
$$\begin{aligned}
&\text{Prob(triangle under diagonal with vertex in point 2)} = \\
&= \int_0^1 2(1-x_2)\,dx_2 \int_0^{x_2} 2(1-y_2)p(x_2,y_2)\,dy_2 = \\
&= 4\int_0^1 dx_2 \int_0^{x_2} dy_2 \int_{x_2}^1 dx_3 \int_{1-(1-y_2)(1-x_3)/(1-x_2)}^{x_3} dy_3 = \\
&= \frac{1}{12}.
\end{aligned}$$

The total probability for Case 4 is $\frac{4}{12} = \frac{1}{3}$, which is inserted in Table 1.

Case 13 is a quadrangle with points 2 and 3 on the same side of the diagonal, see Figure 4. In analogy with (18), we get

(19)
$$\begin{aligned}
&\text{Prob(quadrangle under diagonal)} = \\
&= 4\int_0^1 dx_2 \int_0^{x_2} dy_2 \int_{x_2}^1 dx_3 \int_{y_2x_3/x_2}^{1-(1-y_2)(1-x_3)/(1-x_2)} dy_3 = \\
&= \frac{1}{6}.
\end{aligned}$$

Case 13 also includes a quadrangle above the diagonal. This gives the probability $\frac{2}{6} = \frac{1}{3}$ given in Table 1.

Case 12 is a quadrangle with one vertex above and one below the diagonal, see Figure 4. It has the remaining probability $= \frac{1}{3}$.

Returning to the distribution function for Case 4, we note that the triangle is the same as in Figure 2. When point 2 is below the diagonal, we have the same formula as in Case 2 for $n = 3$, so that $U \le u$ reduces to $x_2 - y_2 \le u$. Since this inequality is linear, we happen to have one of the simplest cases. The integral below is for the complementary event.

(20)
$$\begin{aligned}
H_{4,4}(u) &= 1 - 48\int_u^1 dx_2 \int_0^{x_2-u} dy_2 \int_{x_2}^1 dx_3 \int_{1-(1-y_2)(1-x_3)/(1-x_2)}^{x_3} dy_3 = \\
&= 3u^4 - 8u^3 + 6u^2, \quad 0 \le u \le 1.
\end{aligned}$$

The factor 48 is $4 \cdot (1/12)^{-1}$, where $1/12$ is the probability calculated in (18).

$H_{4,8}(u)$ and $H_{4,13}(u)$ are calculated in appendices B and C. We picture all the cases and list their $H_{4,k}(u)$ in Appendix D. The total fraction probability distribution function denoted H_4 is obtained by weighting the $H_{4,k}(u)$ together with $1/12$ times the weights in the Σ-row of Table 1. It is

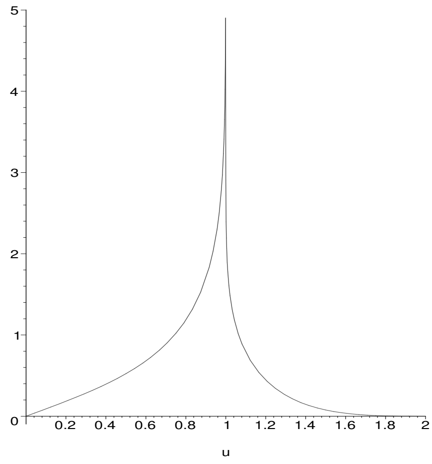

FIGURE 5. Density function dH_4/du for twice the fraction that the four point object takes up of the 'big' rectangle. The singularity at $u = 1$ stems from Cases 3, 6, and 7.

(21)
$$H_4(u) = \begin{cases} \frac{u}{72}(24 + 84u - 16u^2 + u^3) + \\ +\frac{1}{3}(1 - u^2)\log(1 - u) - \frac{\pi^2 u^2}{18}, \\ \qquad\qquad 0 \le u \le 1, \\ \frac{1}{72}(-56 + 212u - 54u^2 - 8u^3 - u^4) + \\ +\frac{1}{6}(9u - 7)(u - 1)\log(u - 1) + \frac{2}{3}u^2\nu(u) - \frac{\pi^2 u^2}{6}, \\ \qquad\qquad 1 \le u \le 2. \end{cases}$$

The density dH_4/du is shown in Figure 5.

4.3. **The distribution of X for four generated points.** In analogy with (9), we can now form

$$\begin{aligned} F_4(y/2) &= \int_0^2 G_4(y/u)\, dH_4(u) = \\ &= [G_4(y/u)H_4(u)]_0^2 - \int_y^2 H_4(u)\frac{d}{du}G_4(y/u)\, du = \\ &= G_4(y/2) - \int_y^2 H_4(u)\frac{d}{du}G_4(y/u)\, du, \quad 0 \le y \le 2. \end{aligned} \tag{22}$$

Substituting back $y = 2x$, we have the distribution function for the area spanned by four points

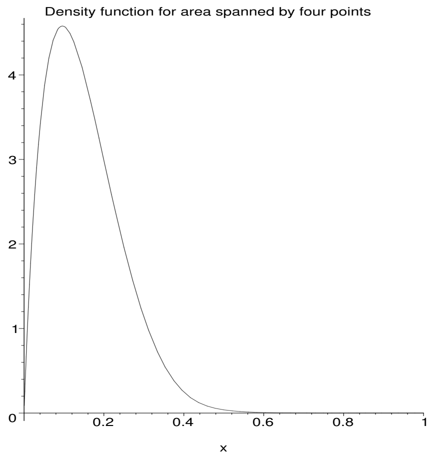

FIGURE 6. Density function dF_4/dx for area spanned by four points. Only .002 of the total mass is above $x = 1/2$.

(23)
$$F_4(x) = \begin{cases} \frac{x}{9}(6 + 1302x - 856x^2 + 1574x^3)- \\ -\frac{8}{3}x^2(x+1)(x+3)\pi^2- \\ -\frac{8}{3}x^4(235 - 43\log(2x))\log(2x)+ \\ +\frac{1}{3}(1 + 2x - 140x^2 - 792x^3)(1-2x)\log(1-2x)- \\ +64x^3(3x+4)(\nu(2x) - \frac{\pi^2}{6}), \\ \hfill 0 \le x \le 1/2 \\ -\frac{1}{9}(7 - 149x - 1365x^2 - 76x^3 + 1574x^4)+ \\ -\frac{1}{6}(7 - 114x - 660x^2 - 296x^3)(2x-1)\log(2x-1)- \\ +32x^2(x+1)(x+3)(\nu(2x) - \frac{\pi^2}{4}), \\ \hfill 1/2 \le x \le 1. \end{cases}$$

The density function dF_4/dx is shown in Figure 6.

The first moments and the standard deviation of the four point area are

(24)
$$\begin{aligned} \alpha_1 &= \int_0^1 x\,dF_4(x) = \frac{11}{72} \approx .15278, \\ \alpha_2 &= \int_0^1 x^2\,dF_4(x) = \frac{859}{27000}, \\ \alpha_3 &= \int_0^1 x^3\,dF_4(x) = \frac{73}{9000}, \\ \sigma &= \sqrt{\alpha_2 - {\alpha_1}^2} = \frac{17\sqrt{95}}{1800} \approx .09205. \end{aligned}$$

5. The number of vertices of the convex hull.

Our detailed study of the convex hull of 3 and 4 points permits us to calculate some expectations for the number of vertices of a random convex polygon in a square.

For $k \leq n$, define

$$q_n(k) = \text{ Prob(} n \text{ points generate a convex polygon with } k \text{ vertices).}$$

Of course, $q_3(3) = 1$. We noted in the beginning of section 4 that $q_4(3) = \frac{11}{36}$ implying $q_4(4) = \frac{25}{36}$. These two probabilities can also be deducted from the Σ-row of Table 1 as follows. The sum of the elements of the first five cases, which are triangle cases, divided by 12 equals $q_4(3)$.

The knowledge of $F_3(x)$ permits us to calculate $q_n(3)$ for all $n \geq 3$. In fact, we get a triangle if points 4 through n are generated inside the triangle generated by the first three points. The probability $\pi(m)$ that m points fall inside an area of size x distributed according to $F_3(x)$ is

$$\pi(m) = \int_0^{1/2} x^m \, dF_3(x).$$

Taking into account the number of ways the three points forming the triangle can be choosen, we get

$$q_n(3) = \binom{n}{3} \pi(n-3) = \binom{n}{3} \int_0^{1/2} x^{n-3} \, dF_3(x). \tag{25}$$

Some values are

$$q_4(3) = \frac{11}{36} \approx .3056, \; q_5(3) = \frac{5}{48} \approx .1042, \; q_6(3) = \frac{137}{3600} \approx .0381,$$

$$q_7(3) = \frac{7}{480} \approx .0146, \quad q_8(3) = \frac{363}{62720} \approx .0058.$$

For $k = 4$, we need the conditional probability that points 5 through n are generated inside the area generated by the first four points, provided these four points span a quadrangle. Cases 6 - 13 are quadrangles, so summing the $H_{4,k}$ for $6 \leq k \leq 13$ multiplied by the weights in Table 1 will give us the wanted conditional distribution function $H_{4q}(u)$. Combining H_{4q} with G_4 will give the conditional distribution function $F_{4q}(x)$. Including the probability $\frac{25}{36}$ of getting a quadrangle, we get

$$q_n(4) = \frac{25}{36} \binom{n}{4} \int_0^{1} x^{n-4} \, dF_{4q}(x). \tag{26}$$

Some values are

$$q_5(4) = \frac{5}{9}, \quad q_6(4) = \frac{1307}{3600} \approx .3631, \quad q_7(4) = \frac{203}{900} \approx .2256.$$

From the above, we can deduce $q_5(5) = 1 - \frac{5}{9} - \frac{5}{48} = \frac{49}{144} \approx .3402$

These numbers can be compared with those of Table 2, which contains results from 10000 Monte Carlo tests for each n.

6. Asymptotic estimates

Rènyi and Sulanke [9] consider the convex hull H_n of n random points generated inside a convex set K. For large n, we have the following formulas for the expected value $E_{H_n}(K)$ of $area(H_n)/area(K)$. The formulas for K a square and a circle are from [9], while the formula for K a triangle is deduced in Appendix E.

$$\begin{aligned} E_{H_n}(\text{triangle}) &= 1 - 2\frac{\log(n)+\gamma}{n} + \mathrm{O}\left(\frac{1}{n^2}\right), \\ E_{H_n}(\text{square}) &= 1 - \frac{8}{3}\frac{\log(n)}{n} + \mathrm{O}\left(\frac{1}{n}\right), \\ E_{H_n}(\text{circle}) &= 1 - \frac{(24\pi)^{2/3}\Gamma(8/3)}{10n^{2/3}} + \mathrm{O}\left(\frac{1}{n}\right) \approx 1 - \frac{3.3832}{n^{2/3}}, \end{aligned} \tag{27}$$

where $\gamma \approx .5772$ is Euler's constant.

These theoretical expectations are plotted in Figure 7 together with the results from Monte Carlo tests.

Closely related to the above are the formulas for the expected number of vertices E_n of the convex hull of n random points inside various convex polygons K. For K a triangle, a square, and a circle and for large n, we have

(28)
$$\begin{aligned} E_n(\text{triangle}) &= 2(\log(n/2)+\gamma) + \mathrm{o}(1), \\ E_n(\text{square}) &= \frac{8}{3}(\log(n/2)+\gamma) + \mathrm{o}(1), \\ E_n(\text{circle}) &= \frac{4}{3}\left(\frac{3\pi}{2}\right)^{2/3}\Gamma\left(\frac{5}{3}\right)\ n^{1/3} - \frac{4}{15}\left(\frac{3\pi}{2}\right)^{4/3}\Gamma\left(\frac{7}{3}\right)\ n^{-1/3} + \\ &\quad + \mathrm{O}(n^{-2/3}) \approx 3.3832\ n^{1/3} - 2.5084\ n^{-1/3}. \end{aligned}$$

The formulas for the triangle and the square and the first term for the circle are given in [8]. The second term for the circle is deduced in Appendix F. This term is needed to get the accuracy for the circle comparable to the other two.

The theoretical E_n and the result of the Monte Carlo tests are shown in Figure 8. The formulas in (28) conform so well with the Monte Carlo tests all the way from $n = 3$ so that the curves hardly can be distinguished in Figure 8. We have $E_3(K) = 3$ for all K. The formulas in (28) give $E_3(\text{triangle}) = 3.52$, $E_3(\text{square}) = 2.62$, and $E_3(\text{circle}) = 3.14$.

	$n = \#$ points											
k	4	5	6	7	8	9	10	12	15	20	40	100
3	3068	1060	374	147	64	30	4	4	4			
4	6932	5525	3627	2253	1379	876	562	236	70	27	2	
5		3415	4731	4739	4185	3424	2732	1624	787	311	19	2
6			1268	2528	3407	3805	3944	3534	2448	1340	169	4
7				333	882	1599	2176	3066	3447	2677	715	52
8					83	254	538	1260	2256	2902	1700	237
9						12	42	246	777	1836	2379	645
10							2	27	190	700	2312	1286
11								3	23	175	1621	1920
12									2	30	714	2009
13										2	287	1785
14											58	1084
15											20	602
16											4	267
17												81
18												19
19												6
20												1
E_q	3.69	4.24	4.68	5.06	5.39	5.69	5.95	6.43	6.99	7.72	9.56	11.98
σ_q	.46	.63	.74	.81	.89	.94	.99	1.08	1.18	1.30	1.60	1.95

TABLE 2. Number of vertices k for various number of generated points n. The presented data are the actual outcome of 10000 Monte Carlo tests for each n (= for each column). Notice that the largest number of vertices observed for $n = 100$ is 20. The last two rows give the average and standard deviation for each n.

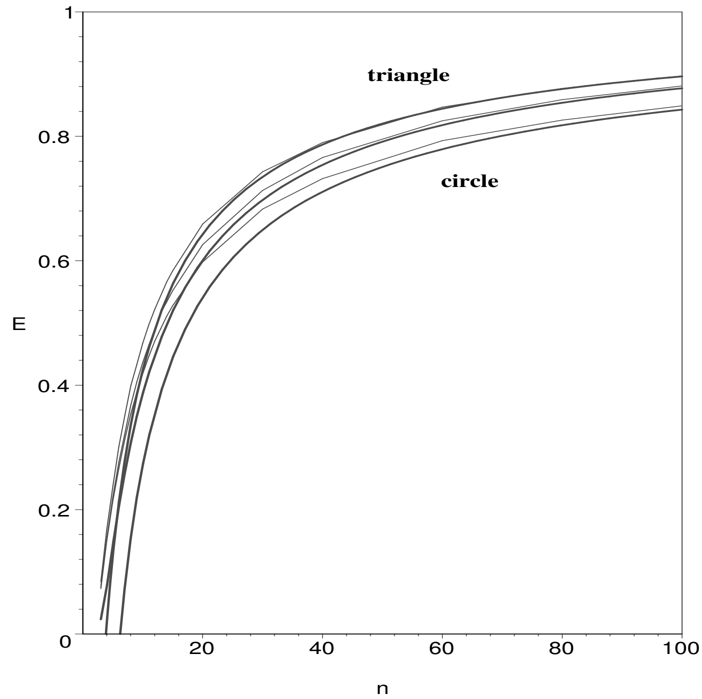

FIGURE 7. The fraction of the area that the convex hull of n random points takes up inside a triangle (top), square (middle), or circle (bottom curves). The thicker lines are theoretical from [9] and Appendix E, the thinner are from Monte Carlo tests.

E_n(square) can be compared with the averages in the last row of Table 2, which holds the numbers

$$E_q(n) = \sum_{k=3}^{n} k q_n(k). \tag{29}$$

7. CONCLUDING COMMENT.

We have not shown any integral calculations in detail. In principle, they are elementary, which doesn't mean that they don't require a substantial effort. As indicated, the calculations have been done in Maple. The calculations would not have been possible without some tool for handling the huge number of terms that come out of the integrations. This doesn't mean that Maple performs the integrations automatically. Generally, we had to split up the integrands in parts and use a particular substitution for each part. Often, we had to do partial integrations manually. Many integrals were improper, calling for a limiting process. We will supply any interested reader with Maple files describing the calculations.

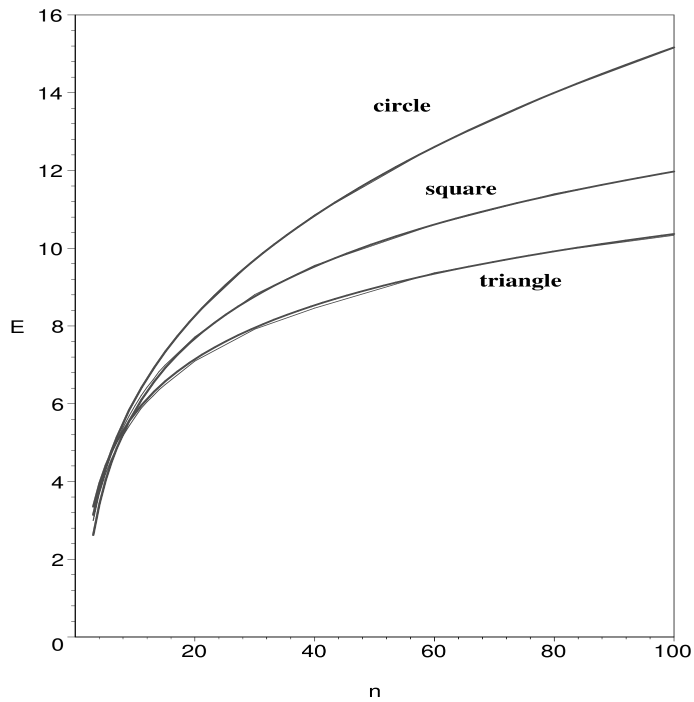

FIGURE 8. Expected number of vertices of the convex hull of n random points inside a triangle, square, or circle. The thicker lines are theoretical, the thinner are from Monte Carlo tests. The difference between theory and tests is hardly discernable.

APPENDIX A

The dilogarithm function $\mathrm{Li}_2(x)$ is defined in [6] for complex x as

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt. \tag{30}$$

When x is real and greater than unity, the logarithm is complex. A branch cut from 1 to ∞ can give it a definite value. In this paper, we are only interested in real x and the real part of Li_2

$$\nu(x) = \mathrm{Re}(\mathrm{Li}_2(x)) = -\int_0^x \frac{\log|1-t|}{t}\, dt. \tag{31}$$

We have the series expansion

$$\nu(x) = \mathrm{Re}(\mathrm{Li}_2(x)) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}\ , \quad |x| \le 1. \tag{32}$$

Although the series is only convergent for $|x| \le 1$, the integrals in (30) and (31) are not restricted to these limits and the ν function is defined and is real on the whole real axis. We use this function for $0 \le x \le 2$.

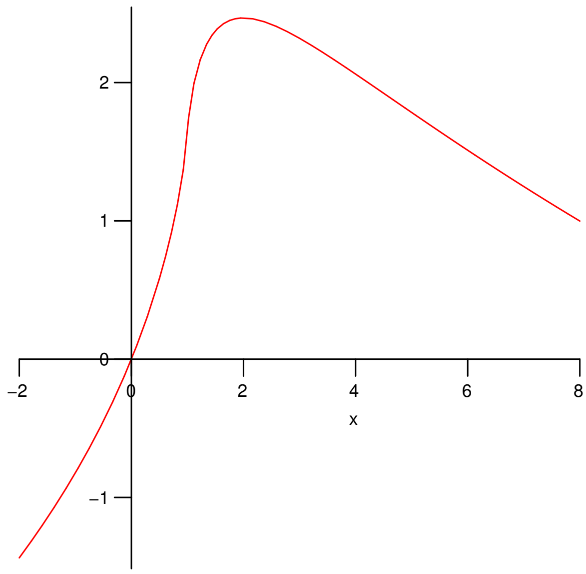

FIGURE 9. The function $\nu(x)$.

The definition of the dilogarithm function has varied a little from author to author. Maple has the function polylog$(2, x)$ which is defined by the series expansion (32) for $|x| \leq 1$ otherwise by analytic continuation. Maple also has a function dilog$(x) = \mathrm{Li}_2(1-x)$ defined on the whole real axis. Maple's dilog function is the same as the dilog function given in [1], page 1004.

$\nu(x)$ is increasing from $\nu(0) = 0$ via $\nu(1) = \pi^2/6$ to $\nu(2) = \pi^2/4$.

The integrals involving $\nu(x)$ needed for calculating the moments of various distributions take rational values like

$$\int_0^1 x\, d\nu(x) = 1, \qquad \int_1^2 x\, d\nu(x) = 1,$$
$$\int_0^1 x^2\, d\nu(x) = \frac{3}{4}, \qquad \int_1^2 x^2\, d\nu(x) = \frac{5}{4},$$
$$\int_0^1 x^3\, d\nu(x) = \frac{11}{18}, \qquad \int_1^2 x^3\, d\nu(x) = \frac{29}{18}.$$

APPENDIX B

We shall show in detail how the distribution function $H_{4,13}(u)$ is calculated in Case 13. The quadrangle of this case is shown in Figure 4.

In analogy with the reasoning for triangle Case 2, we get that the doubled fraction of the big rectangle is

$$U = ((1 - y_2)x_3 - (1 - x_2)y_3). \tag{33}$$

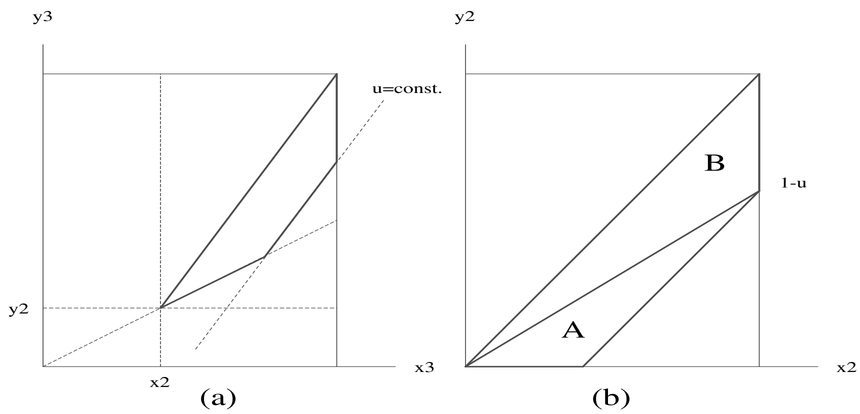

FIGURE 10. (a) Trapezoid to integrate over in x_3y_3-plane. (b) Areas to integrate A and B over in x_2y_2-plane.

We have

$$H_{4,13}(u) = \text{Prob}(U \le u) = \text{Prob}((1-y_2)x_3 - (1-x_2)y_3 \le u). \tag{34}$$

We shall integrate over the set used in evaluating (19) though with the supplementary bound $(1-y_2)x_3 - (1-x_2)y_3 \le u$. When $x_2 - y_2 \le u \le 1 - y_2/x_2$, we shall integrate over the trapezoid in the x_3, y_3-plane marked in Figure 10a.

We get the trapezoid area

$$A(x_2, y_2, u) = \frac{x_2(u - x_2 + y_2)^2}{2(1-x_2)(x_2-y_2)} + \frac{(u-x_2+y_2)(x_2-y_2-ux_2)}{(1-x_2)(x_2-y_2)}. \tag{35}$$

The area is zero for $u \le x_2 - y_2$. For $u \ge 1 - y_2/x_2$, the area is the the same as for $u = 1 - y_2/x_2$, namely

$$B(x_2, y_2, u) = \frac{(x_2-y_2)(1-x_2)}{2x_2}. \tag{36}$$

These two expressions shall be integrated over the triangles marked in Figure 10b.

The obtained distribution function is

$$H_{4,13}(u) = u(6+3u-8u^2) + 6(1-3u)(1-u)\log(1-u) + 12u^2\nu(u) - 2u^2\pi^2, \quad 0 \le u \le 1. \tag{37}$$

The corresponding density function $dH_{4,13}/du$ is shown in Figure 11.

APPENDIX C

We shall show in detail how to calculate the distribution function $H_{4,8}(u)$ for the area fraction in Case 8. The quadrangle of this case is shown in Figure 15. For ease of computation, we shall carry out the

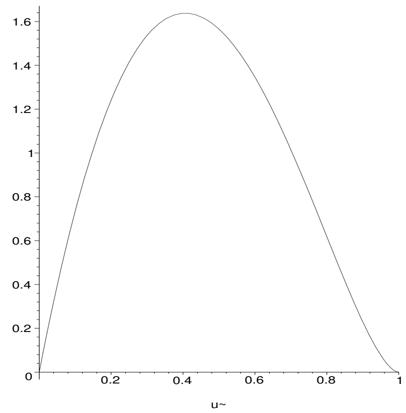

FIGURE 11. The density function $dH_{4,13}/du$.

calculations for this quadrangle turned upside down, namely for the permutation 3214 which is the reverse of the one in Figure 15. Then, the conditions describing Case 8 are

$$\frac{x_2}{x_3} + \frac{y_2}{y_1} \leq 1 \ , \ x_2 \leq x_3 \ , \ y_2 \leq y_1 \ , \ x_3(1-y_2) + y_1(1-x_2) \leq u.$$

These inequalities describe the trapetsoidal domain in the $x_2 y_2$-space marked in Figure 12a. For $(1-x_3)(1-y_1) \geq 1-u$ and $x_3 + y_1 \geq u$, the area of the domain is

$$A(x_3, y_1, u) = \frac{x_3 y_1}{2} - \frac{(x_3 + y_1 - u)^2}{2 x_3 y_1}. \tag{38}$$

The area is zero for $(1-x_3)(1-y_1) \leq 1-u$. For $x_3 + y_1 \leq u$, the area is the the same as for $u = x_3 + y_1$, namely

$$B(x_3, y_1, u) = \frac{x_3 y_1}{2}. \tag{39}$$

For $0 \leq u \leq 1$, these two expressions shall be integrated over the areas marked in Figure 12b. For $1 \leq u \leq 2$ the areas are shown in Figure 12c. We get

$$H_{4,8}(u) = \begin{cases} -6u + 13u^2 - 2(3-5u)(1-u)\log(1-u)- & \\ -4u^2\nu(u), & 0 \leq u \leq 1, \\ -15 + 36u - 14u^2 - 5\pi^2 u^2/3+ & \\ +4(3u-1)(u-1)\log(u-1) + 8u^2\nu(u), & 1 \leq u \leq 2. \end{cases}$$

The corresponding density function $dH_{4,8}/du$ is shown in Figure 13.

APPENDIX D

The $H_{4,k}$-functions for the four point problem are given below. If no specification is given, the functions are defined for $0 \leq u \leq 1$.

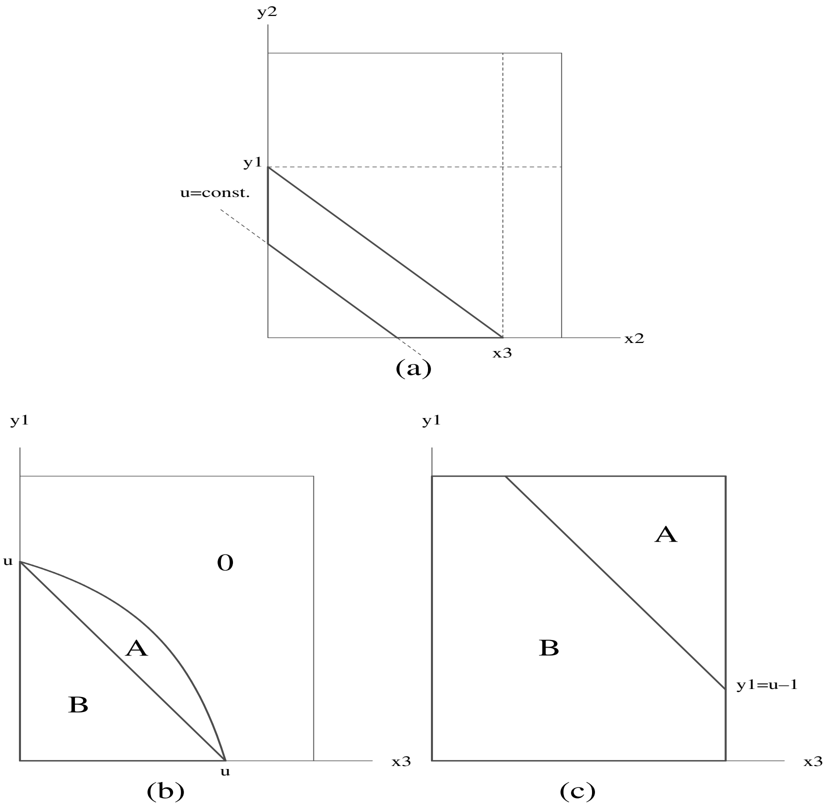

FIGURE 12. Areas to integrate over in Case 8. (a) Trapezoid to integrate over in x_2y_2-plane. (b) Areas to integrate A and B over in x_3y_1-plane when $0 \le u \le 1$. (c) Areas to integrate A and B over in x_3y_1-plane when $1 \le u \le 2$.

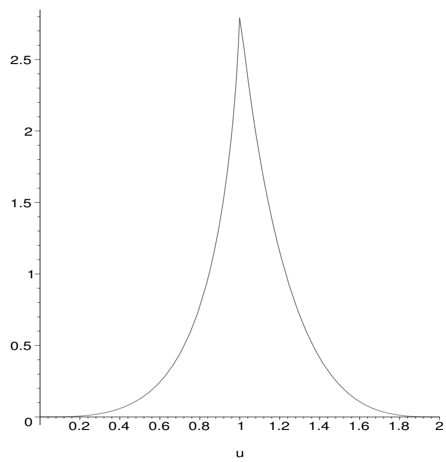

FIGURE 13. Density function $dH_{4,8}/du$ for two times the fraction that the quadrangle takes up of the 'big' rectangle in Case 8. The peak value is $16 - 4\pi^2/3$.

$$H_{4,1} = 1 + (1-u)^2(7 - 4u + 6\log(1-u)).$$
$$H_{4,2} = 1 - (1-u)(1 + 7u - 2u^2 + 6(1-u)\log(1-u)).$$
$$H_{4,3} = u + (5u + 2(3-u)(1-u)\log(1-u)).$$
$$H_{4,4} = 6u^2 - 8u^3 + 3u^4.$$
$$H_{4,5} = 4u^3 - 3u^4.$$
$$H_{4,6} = (u-1)(5u - 9 - 2(u+1)\log(u-1)), \qquad 1 \le u \le 2.$$
$$H_{4,7} = u + (5u + 2(3-u)(1-u)\log(1-u)).$$
$$H_{4,8} = \begin{cases} -6u + 13u^2 - 2(3-5u)(1-u)\log(1-u) - \\ -4u^2\nu(u), & 0 \le u \le 1, \\ -15 + 36u - 14u^2 - 5\pi^2u^2/3 + \\ +4(3u-1)(u-1)\log(u-1) + 8u^2\nu(u), & 1 \le u \le 2. \end{cases}$$
$$H_{4,9} = 6u - 9u^2 + 4u^3 + 6(1-u)^2\log(1-u).$$
$$H_{4,10} = \begin{cases} -3u + 9u^2/2 - u^3 - 3(1-u)^2\log(1-u), & 0 \le u \le 1, \\ -3 + 3u + 3u^2/2 - u^3 + 3(u-1)^2\log(u-1), & 1 \le u \le 2. \end{cases}$$
$$H_{4,11} = \begin{cases} 2u^3 - 5u^4/4, & 0 \le u \le 1, \\ -3 + 8u - 6u^2 + 2u^3 - u^4/4, & 1 \le u \le 2. \end{cases}$$
$$H_{4,12} = 6u^2 - 8u^3 + 3u^4.$$
$$H_{4,13} = u(6 + 3u - 8u^2) + 6(1-3u)(1-u)\log(1-u) + 12u^2\nu(u) - 2\pi^2u^2.$$

The cases are shown in Figures 14 and 15.

Appendix E

We shall deduce the limit formula for large n for the expected area fraction of a triangle

$$E_{H_n}(\text{triangle}) = 1 - 2\frac{\log(n) + \gamma}{n},$$

where γ is Euler's constant.

We shall use the technique of Rényi and Sulanke, [9] and start with their formula

$$E_{H_n}(\text{triangle}) = \frac{\binom{n}{2}}{12F^3}\int_0^{2\pi}\int_0^{p(\phi)}(1 - f/F)^{n-2}l^4p\,dp\,d\phi, \tag{40}$$

where F is the area of the triangle, f is the smaller of the two areas that the triangle is divided into by the line $x\cos(\phi) + y\sin(\phi) = p$ and l is the length of this dividing cut. The origin is supposed to be inside

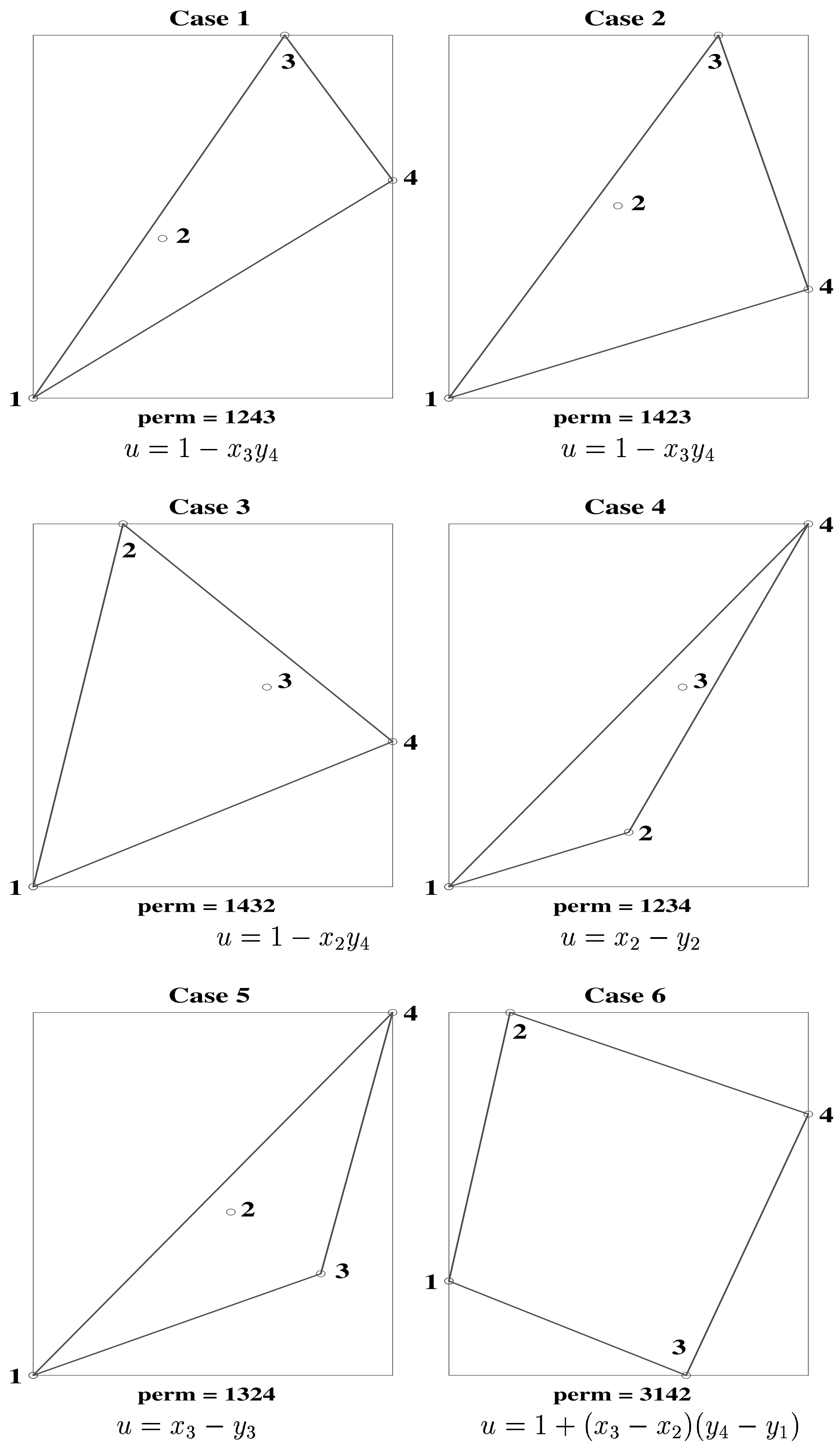

FIGURE 14. Cases 1 - 6.

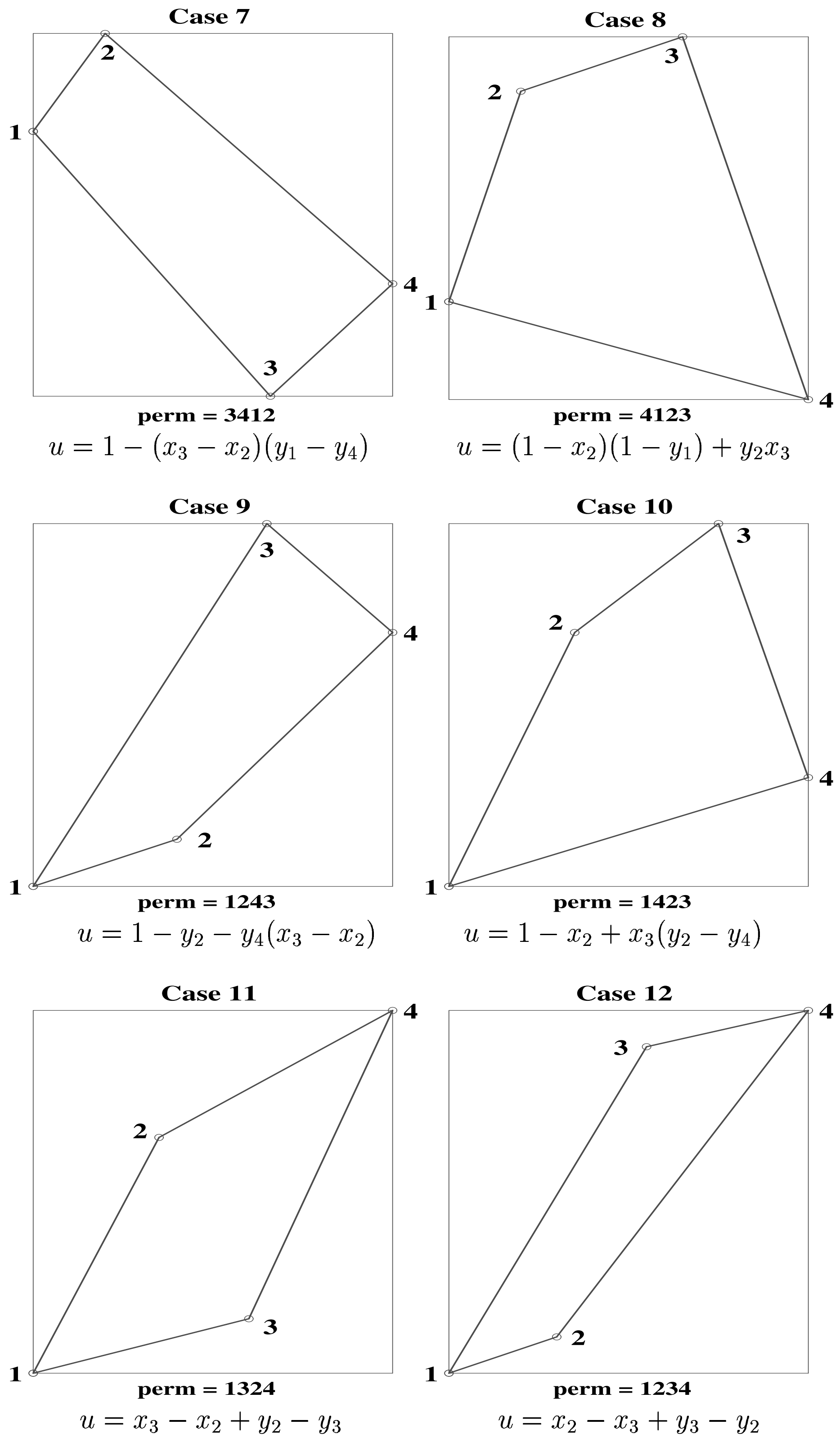

FIGURE 15. Cases 7 - 12. Case 13 is in Figure 4.

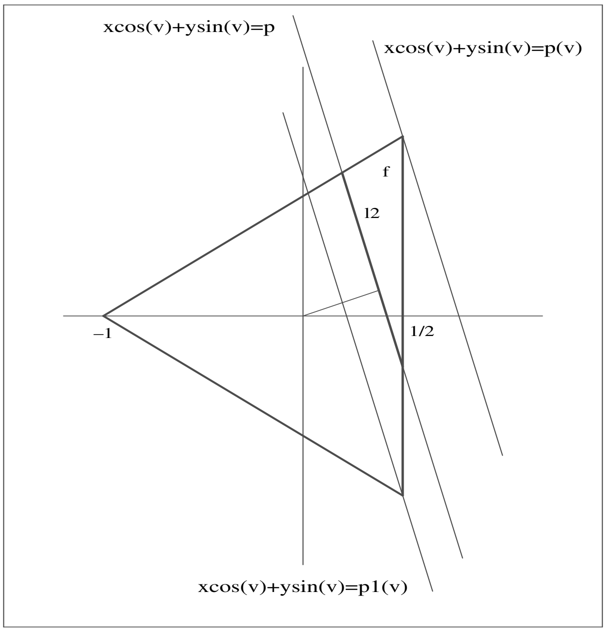

FIGURE 16. Triangle for calculation of E_{H_n}

the triangle and we shall use the equilateral triangle depicted in Figure 16, which has the area $F = 3\sqrt{3}/4$. $p(\phi)$ is the largest p for which the line intersects the triangle. By the symmetry of the triangle, the ϕ-integral can be replaced by 6 times the integral from 0 to $\pi/3$. We have two cases: (i) the line intersects the top and bottom sides of the triangle or (ii) the line intersects the top and right hand sides of the triangle. We have case (i) when

$$0 \leq p \leq p_1(\phi) = \frac{\cos(\phi) - \sqrt{3}\sin(\phi)}{2}.$$

In case (i), we have

$$l_1 = 2\sqrt{3}\frac{\cos(\phi) + p}{4\cos(\phi)^2 - 1} \quad \text{and} \quad r_1 = 1 - \frac{f}{F} = \frac{4}{3}\frac{\cos(\phi)^2 + 2p\cos(\phi) + p^2}{4\cos(\phi)^2 - 1}.$$

We have case (ii) when

$$p_1(\phi) \leq p \leq p(\phi) = \frac{\cos(\phi) + \sqrt{3}\sin(\phi)}{2}.$$

In case (ii), we have

$$l_2 = \frac{p(\phi) - p}{\sin(\phi)\sqrt{1 + 2\cos(\phi)(\cos(\phi) + \sqrt{3}\sin(\phi))}},$$

and

$$r_2 = 1 - \frac{f}{F} = 1 - \frac{2}{3\sqrt{3}}\frac{(p(\phi) - p)^2}{1 - \cos(\phi)(\cos(\phi) - \sqrt{3}\sin(\phi))}.$$

The integrals arising from the two cases will be denoted J_1 and J_2, so that $E_{H_n}(\text{triangle}) = J_1 + J_2$, where

$$J_1 = \frac{16n(n-1)}{81\sqrt{3}} \int_0^{\pi/6} \int_0^{p_1(\phi)} r_1^{n-2} l_1^4 p \, dp \, d\phi$$

and

$$J_2 = \frac{16n(n-1)}{81\sqrt{3}} \int_0^{\pi/3} \int_{\max(p_1(\phi),0)}^{p(\phi)} r_2^{n-2} l_2^4 p \, dp \, d\phi.$$

Here, we integrate over $0 \le \phi \le \pi/6$ in J_1 because $p_1 \le 0$ for $\phi \ge \pi/6$.

For the evaluation of J_1, we substitute p by x by putting $p = x\cos(\phi)$. The integral over p then transforms to

$$\int_0^{(1-\sqrt{3}\tan(\phi))/2} (1+x)^{2n} x \, dx = \left[\frac{(1+x)^{2n+1}}{2(n+1)} \left(x - \frac{1}{2n+1} \right) \right]_0^{(1-\sqrt{3}\tan(\phi))/2}$$

After inserting the bounds, we have a remaining ϕ-integral in which we substitute ϕ by s by putting $\tan(\phi) = \sqrt{3}(1-s)/(1+s)$. After skipping terms of the order 9^{-n}, we get.

$$J_1 \approx \frac{n(n-1)}{3(n+1)(2n+1)} \int_{1/2}^1 ((4n+1)s^n - 2(n+1)s^{n-1}) \, ds =$$

$$= \frac{n(n-1)}{3(n+1)(2n+1)} \left[\frac{(4n+1)s^{n+1}}{n+1} - \frac{2(n+1)s^n}{n} \right]_{1/2}^1 .$$

We get for large n

$$J_1 \approx \frac{1}{3} - \frac{5}{3n} + O\left(\frac{1}{n^2}\right).$$

For the evaluation of J_2, we substitute p by x and ϕ by t by putting

$$p = p(\phi) - \sqrt{\frac{3\sin(\phi)(\sin(\phi) + \sqrt{3}\cos(\phi))x}{2n}} \text{ and } \phi = \arctan(\frac{\sqrt{3}t}{2n-t}).$$

After tedious manipulations, we get

$$J_2 = \frac{n-1}{3n^2} \int_0^n dt \int_0^{x_1(t)} \left(1 - \frac{x}{n}\right)^{n-2} x^{3/2} (nt^{-3/2} + t^{-1/2} - 3x^{1/2}t^{-1}) \, dx,$$

where

$$x_1(t) = \begin{cases} t \,, & 0 \le t \le n/2 \\ (n+t)^2/9t \,, & n/2 \le t \le n \end{cases}.$$

For $n/2 \le t \le n$ we have $4n/9 \le (n+t)^2/9t \le n/2$, meaning that $(1 - \frac{x}{n})^{n-2}$ is smaller that $(5/9)^{n-2}$ for these t. By putting $x_1(t) = n/2$

for $n/2 \le t \le n$ and reversing the order of integration, we have with good approximation for large n

$$J_2 \approx \frac{n-1}{3n^2} \int_0^{n/2} \left(1 - \frac{x}{n}\right)^{n-2} x^{3/2}\, dx \int_x^n (nt^{-3/2} + t^{-1/2} - 3x^{1/2}t^{-1})\, dt =$$

$$= \frac{n-1}{3n^2} \int_0^{n/2} \left(1 - \frac{x}{n}\right)^{n-2} \left[2(n-x)x - 3(\log(n) - \log(x))x^2\right]\, dx.$$

We make an error of at most the order $n^2 2^{-n/2}$ by extending the upper limit of integration to infinity. We have the following expansions

$$\int_0^{\infty} \left(1 - \frac{x}{n}\right)^{n-2} x\, dx = 1 + \frac{1}{n} - \frac{18}{n^2} + O\left(\frac{1}{n^3}\right),$$

$$\int_0^{\infty} \left(1 - \frac{x}{n}\right)^{n-2} x^2\, dx = 2 - \frac{40}{n^2} + O\left(\frac{1}{n^3}\right),$$

$$\int_0^{\infty} \left(1 - \frac{x}{n}\right)^{n-2} x^2 \log(x)\, dx = 3 - 2\gamma + O\left(\frac{1}{n}\right).$$

Using these expansions for the above integral, we get

$$J_2 \approx \frac{2}{3} - \frac{2\log n + 2\gamma + 5/3}{n} + O\left(\frac{1}{n^2}\right).$$

Altogether, we have

$$E_{H_n}(\text{triangle}) = J_1 + J_2 \approx 1 - 2\frac{\log(n) + \gamma}{n} + O\left(\frac{1}{n^2}\right),$$

where γ is Euler's constant.

Appendix F

We shall deduce the formula for the expected number of vertices of the convex hull of n points inside a circle given in (28). We use the method of [8] and are led to the evaluation of the following integral, similar to (40)

$$E_n(\text{circle}) = \frac{\binom{n}{2}}{3F^2} \int_0^{2\pi} \int_0^{p(\phi)} (1 - f/F)^{n-2} l^3\, dp\, d\phi, \tag{41}$$

where F is the area of the circle. We use a circle with radius one centered at the origin. The cutting line is $x = p$. We put $p = \cos(\alpha)$ and get the length of the cut $l = 2\sin(\alpha)$. The area of the circle segment is $f = \alpha - \sin(\alpha)\cos(\alpha)$. We get

$$E_n = \frac{8n(n-1)}{3\pi} \int_0^{\pi/2} \left(1 - \frac{\alpha - \sin(\alpha)\cos(\alpha)}{\pi}\right)^{n-2} \sin(\alpha)^4\, d\alpha.$$

Series expansion of the trigonometric functions gives

$$E_n \approx \frac{8n(n-1)}{3\pi} \int_0^{\pi/2} \left(1 - \frac{1}{\pi}\left(\frac{2}{3}\alpha^3 - \frac{2}{15}\alpha^5 + \mathrm{O}\left(\alpha^7\right)\right)\right)^{n-2} \cdot$$
$$\cdot\, \alpha^4 \left(1 - \frac{\alpha^2}{6}..\right)^4 d\alpha.$$

Use the substitution $\frac{2}{3\pi}\alpha^3 = \frac{x}{n}$ and get

$$E_n \approx 2(n^{1/3} - n^{-2/3})\lambda \int_0^{\pi^2 n/4} \left(1 - \frac{x}{n} + \mu\left(\frac{x}{n}\right)^{5/3} + \mathrm{O}\left(\left(\frac{x}{n}\right)^{7/3}\right)\right)^{n-2} \cdot$$
$$\cdot \left(1 - \lambda\left(\frac{x}{n}\right)^{2/3}\right)^4 x^{2/3}\, dx,$$

where $\lambda = \frac{2}{3}(\frac{3\pi}{2})^{2/3}$ and $\mu = \frac{1}{5}(\frac{3\pi}{2})^{2/3}$.

We have $\left(1 - \frac{x}{n} + \mu\left(\frac{x}{n}\right)^{5/3} + \mathrm{O}\left(\left(\frac{x}{n}\right)^{7/3}\right)\right)^{n-2} =$

$$= e^{-x}\left(1 + \mu x^{5/3} n^{-2/3} - x^2/2n + 2x/n + \mathrm{O}\left(n^{-5/3}\right)\right).$$

Inserting this in the integral, we get

$$E_n \approx n^{1/3}\ 2\lambda \int_0^{\pi^2 n/4} e^{-x} x^{2/3} dx - n^{-1/3}\ 2\lambda^2 \int_0^{\pi^2 n/4} e^{-x} x^{4/3}\, dx +$$
$$+\, n^{-1/3}\ 2\lambda\mu \int_0^{\pi^2 n/4} e^{-x} x^{7/3}\, dx + \mathrm{O}(n^{-2/3}).$$

We make an error of negative exponential order by extending the upper limits of the integrals to infinity and get

$$E_n(\text{circle}) \approx \frac{4}{3}\left(\frac{3\pi}{2}\right)^{2/3} \Gamma\left(\frac{5}{3}\right)\ n^{1/3} - \frac{4}{15}\left(\frac{3\pi}{2}\right)^{4/3} \Gamma\left(\frac{7}{3}\right)\ n^{-1/3} +$$
$$+\, \mathrm{O}\left(n^{-2/3}\right) \approx 3.3832\ n^{1/3} - 2.5084\ n^{-1/3}.$$

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