Programming II (ID1019) 2018-03-13
08:00-12:00

Name: ______________________________

Instructions

• All answers should be written in these pages, use the space allocated after each question to write down your answer.
• Answers should be written in English.
• You should hand in the whole exam.
• No additional pages should be handed in.

Grade

The exam is divided into a number of questions where some are a bit harder than others. The harder questions are marked with a star [p*], and will give you points for the higher grades. The exam is thus divided into basic points and points for higher grades. First of all make sure that you pass the basic points before engaging with the higher points.

• E: 12 basic points
• D: 15 basic points
• C: 18 basic points
• B: 20 basic points and 8 higher points
• A: 20 basic points and 10 higher points

The limits could be adjusted to lower values but not raised.
1 Lambda calculus [2p]

Evaluate the following lambda expressions:

- \((\lambda x \rightarrow x + 5)4\)  
  \textbf{Answer}: 9

- \((\lambda x \rightarrow (\lambda y \rightarrow x + 2 \ast y)3)5\)  
  \textbf{Answer}: 11

- \((\lambda x \rightarrow (x)5)(\lambda z \rightarrow z + z)\)  
  \textbf{Answer}: 10

2 Operational semantics [2p]

Given the rules for the operational semantics in the appendix, show step by step which rules are used and evaluate the following expressions:

\[
\frac{y/a \in \{y/a\}}{E\{y/a\}(y) \rightarrow a}
\]

\[
E\{\}(:a) \rightarrow a \quad S(\{y/b\}, y) \rightarrow \{} \quad P(\{}y, a\} \rightarrow \{y/a\} \quad E\{y/a\}(y) \rightarrow a
\]

\[
E\{\}(y = :a; y) \rightarrow a
\]

\[
E\{\}(y = :b; y = :a; y) \rightarrow a
\]

\[
\frac{y/a \text{ in } \{y/a\}}{E\{\}} \rightarrow \}
\]

\[
P(\{}y, a\} \rightarrow \{y/a\} \quad P(\{}y/a\}(y, b) \rightarrow \text{fail}
\]

\[
P(\{}(y, y), \{a, b\}\} \rightarrow \text{fail}
\]

\[
E\{\}(\{a, b\}) \rightarrow \{y, y\} \quad S(\{}y, y\}) \rightarrow \{\} \quad P(\{}y, y\}, \{a, b\}\} \rightarrow \text{fail}
\]

\[
E\{\}(\{y, y\} = \{a, b\}; y) \rightarrow \perp
\]
3 Pattern matching [2 p]

Given the expressions below, what is the resulting environment in the cases where it succeeds?

a: \([x, y \mid z] = [1, 2, 3]\)

Answer: \(x = 1, y = 2, z = [3]\)

b: \([x, y \mid z] = [1, [2, 3]]\)

Answer: \(x = 1, y = [2,3] z = []\)

c: \([x, y \mid z] = [1 \mid [2, 3]]\)

Answer: \(x = 1, y = 2, z = [3]\)

d: \([x, y \mid z] = [1 \mid [2, 3] \mid [4]]\)

Answer: syntax error

e: \([x, y \mid z] = [1, 2, 3, 4]\)

Answer: \(x = 1, y = 2, z = [3, 4]\)

Answer: The syntax error in the fourth example was not intended so the question is removed.
4 Recursion

a binary tree [2 p]

Implement a function, sum/1, that takes a binary tree and returns the sum of all values in the tree. The tree is represented as follows:

@type tree :: {:node, integer(), tree(), tree()} | nil

Answer:
def sum(nil) do 0 end
def sum({:node, v, left, right}) do v + sum(left) + sum(right) end

tail recursion [2 p*]

The regular definition of append/2 is not tail recursive. Implement the function reverse/1 as a tail recursive function and use this to implement append/2 in a tail recursive way.

Answer:
def reverse(a) do reverse(a, []) end
def reverse([], b) do b end
def reverse([h|t], b) do reverse(t, [h|b]) end
def append(a, b) do reverse(reverse(a), b) end

5 Time complexity

mirror a tree [2 p]

If we have the below definition of a function that mirrors a tree, what is the assymptoptic time complexity of teh function?
def mirror(nil) do nil end
def mirror({:node, left, right}) do {:node, mirror(right), mirror(left)} end
Answer: The time complexity is $O(n)$ where $n$ is the number of nodes in the tree.

A queue [2 p*]

Assume that we represent a queue with the help of two lists and have the below implementation of enqueue/2 and dequeue/1. What is the amortized time complexity for adding and then removing an element from a queue?

```emacs
def enqueue({:queue, head, tail}, elem) do
  {:queue, head, [elem|tail]}
end

def dequeue({:queue, [], []}) do :fail end
def dequeue({:queue, [elem|head], tail}) do
  {:ok, elem, {:queue, head, tail}}
end
def dequeue({:queue, [], tail}) do
def {queue, reverse(tail), []})
end
```

Answer: The amortized time complexity is $O(1)$ since we can add an element in constant time, move it to the front list in constant time (reverse/1 is $O(n)$ but we can amortize this cost over the $n$ elements) and constant time to remove the element from the front list.
6 Process description

TRB: two-four-six ... [2 p]

Given the below state diagram, implement a process with the specified behaviour.

Answer:

def start() do
    spawn(fn() -> closed() end)
end

def closed() do
    receive do
        2 -> two()
        _ -> closed()
    end
end

def two() do
    receive do
        2 -> two()
        4 -> four()
        _ -> closed()
    end
end

def four() do
    receive do
        2 -> two()
        6 -> six()
        _ -> closed()
    end
end

def six() do
    receive do
        2 -> two()
        _ -> closed()
    end
end
**tic-tac-toe [2 p]**

Assume that we have the following definition of `first/1`, `second/1` and `third/1`.

```eex
def first(p) do
  receive do
    :tic ->
      second(p, [:tic])
    :tac ->
      second(p, [:tac])
  end
end

def second(p, all) do
  receive do
    :tic -> third(p, [:tic|all])
    :tac -> third(p, [:tac|all])
    :toe -> third(p, [:toe|all])
  end
end

def third(p, all) do
  receive do
    x -> send(p, {:ok, [x|all]})
  end
end
```

What is the result when we evaluate the call `test/0`?

```eex
def test() do
  self = self()
  p = spawn(fn() -> first(self) end)
  send(p, :toe)
  send(p, :tac)
  send(p, :tic)
  receive do
    {:ok, res} -> res
  end
end
```

**Answer:** `[:tic, :toe, :tac]`
parallel sum [2 p*]

Implement a function `sum/1` that takes a binary tree with numbers in the leaves, and sums all numbers of the tree in parallel.

Answer:

```erlang
def sum({:leaf, N}) do n end
def sum({:node, Left, Right}) do
  self = self()
  spawn(fn() -> n = sum(Left); send(self, n) end)
  spawn(fn() -> n = sum(Right); send(self, n) end)
  receive do
    N1 ->
      receive do
        N2 ->
          N1 + N2
      end
    end
  end
end
```

8
7 Programming

A heap is a tree structure where the largest element is in the root of the tree and where the left and right branch are heaps.

a heap [2 p]

Define a data structure that is suitable to represent a heap and implement a function \texttt{new/0} that returns a heap. Assume that we only should handle integers.

- \begin{verbatim}
  \texttt{@spec new() :: heap()}
\end{verbatim}

Answer:

\begin{verbatim}
  @type heap() :: nil | {:heap, integer(), heap(), heap()}
  def new() do
    nil
  end
\end{verbatim}

add/2 [2 p]

Implement the function \texttt{add/2} that adds an integer to a heap.

- \begin{verbatim}
  \texttt{@spec add(heap(), integer()) :: heap()}
\end{verbatim}

To keep the heap balanced you should switch the left and right branches that is, when you add an element to a branch you add it to the right branch but make the result the left branch of the heap.

Answer:

\begin{verbatim}
def add(nil, v) do
  {:heap, v, nil, nil}
end
def add({:heap, k, left, right}, v) when k > v do
  {:heap, k, add(right, v), left}
end
def add({:heap, k, left, right}, v) do
  {:heap, v, add(right, k), left}
end
\end{verbatim}
### pop/1 [2 p]

Implement the function `pop/1` that removes the highest element in a heap and returns either: `:fail`, if the heap is empty, or `{:ok, integer(), heap()}`

- @spec pop(heap()) :fail | {:ok, integer(), heap()}

**Answer:**

```elixir
def pop(nil) do :fail end
def pop({:heap, k, left, nil}) do {:ok, k, left} end
def pop({:heap, k, nil, right}) do {:ok, k, right} end
def pop({:heap, k, left, right}) do {:heap, l, _, _} = left
{:heap, r, _, _} = right
if l < r do
  {:ok, _, rest} = pop(right)
  {:ok, k, {:heap, r, left, rest}}
else
  {:ok, _, rest} = pop(left)
  {:ok, k, {:heap, l, rest, right}}
end
end
```

### swap/2 [2 p]

Implement the function `swap/2` that takes a heap and a number and returns `{:ok, integer(), heap()}` where the number is the highest number and the heap the remaining heap. The function should have the same meaning as first `add/2` a number to a heap and then `pop/1` the highest but we should do this in one function, not call the two functions.

- @spec swap(heap(), integer()) {:ok, integer(), heap()}

**Answer:**

```elixir
def swap(nil, v) do {:ok, v, nil} end
```
```elixir
def swap({:heap, k, left, right}, v) when k > v do
  {:ok, v, left} = swap(left, v)
  {:ok, v, right} = swap(right, v)
  {:ok, k, {:heap, v, left, right}}
end

def swap(heap, v) do
  {:ok, v, heap}
end

a general heap [2 p]

The heap we now have will have the largest element in the root and be limited to integers (or what is comparable with <). Implement a function add/3 that takes a heap, an element and a function that can be used to compare two elements. The function add/3 should as before add an element to a heap but now used the provided function when doing the comparison.

• @type cmp() :: (any(), any()) -> bool()
• @spec add(heap(), any(), cmp()) :: heap()

Answer:

def add(nil, v, _) do
  {:heap, v, nil, nil}
end
def add({:heap, k, left, right}, v, cmp) do
  if cmp.(v, k) do
    {:heap, k, add(right, v, cmp), left}
  else
    {:heap, v, add(right, k, cmp), left}
  end
end

Specify a type cheap() that holds a function for comparison and a heap. Implement a function new/1 that takes a function and returns a structure of the type cheap().

• @spec new(cmp()) :: cheap()

Answer:

@type cheap() :: {:cheap, cmp(), heap()}
def new(f) do {:cheap, f, nil} end
```
Implement a function \texttt{add/2} that takes a structure of type \texttt{cheap()}, that calls \texttt{add/3} with the correct arguments and returns a structure of the same form.

\begin{itemize}
  \item \texttt{@spec add(cheap(), any()) :: cheap()}
\end{itemize}

**Answer:**

\begin{verbatim}
def add({:cheap, cmp, heap}, v) do
  {:cheap, cmp, add(heap, v, cmp)}
end
\end{verbatim}

### The middle number

You should implement a process that holds a state consisting of a set of numbers. The set is initially empty but the process should then accept the following messages:

\begin{itemize}
  \item {:add, integer()} : the number should be added to the set
  \item {:get, pid()} : The process should reply with either :\texttt{fail}, if the set is empty, or {:\texttt{ok, integer()} } where the integer is the middle number in the set (if an even number any one of the two middle numbers) that is also removed from the set.
\end{itemize}

To complicate matters both operations should be done in $O(lg(n))$ time, where $n$ is the number of elements in the set. You are not allowed to use any libraries to store the set of elements but should use the implementation of a heap in the previous questions. The process will apart from keeping track of the middle element have two heaps, one for smaller elements and one for larger.

### State diagram \([2^*]\)

Start by describing a state diagram. The process should have three states: \texttt{empty} when the set is empty, \texttt{left} when we have a middle element and possibly one more element to the “left” (that are less) then we have to the “right” and, \texttt{right} when we have a middle element and possibly one more elements to the “right” than we have to the “left”.

**Answer:**

### No elements \([2^*]\)

Implement how we start the process and its behavior in its empty state.
Assume that we have the following functions defined in a module `Heap`.

- \( \text{@spec new(cmp()) :: cheap()} \)

**Answer:**

```elixir
defmodule Middle do
  def start() do
    spawn(fn() -> empty() end)
  end

  def empty() do
    receive do
      {:add, v} ->
        left = Heap.new(fn(x,y) -> x < y end)
        right = Heap.new(fn(x,y) -> x > y end)
        left(v, left, right)
      {:get, pid} ->
        send(pid, :fail)
        empty()
    end
  end
end
```
maybe more to the left \([4^*]\)

In its “left” state the process has a middle element, a number of elements that are less (to the left) and as many or one less that are greater (to the right). Implement the behaviour of the process in its left state.

Assume that we have the following functions defined in a module Heap.

- \(@\text{spec} \text{add}(\text{cheap}(), \text{any}()) \rightarrow \text{cheap}()\)
- \(@\text{spec} \text{pop}(\text{cheap}()) \rightarrow \text{fail} | \{\text{ok}, \text{any}(), \text{cheap}()\}\)
- \(@\text{spec} \text{swap}(\text{cheap}(), \text{any}()) \rightarrow \{\text{ok}, \text{any}(), \text{cheap}()\}\)

Answer:

```erlang
def left(m, left, right) do
  receive do
    {:add, v} when v < m ->
      {ok, k, swaped} = Heap.swap(left, v)
      right(k, swaped, Heap.add(right, m))
    {:add, v} ->
      right(m, left, Heap.add(right, v))
    {:get, pid} ->
      send(pid, {ok, m})
      case Heap.pop(left) do
        {ok, k, rest} ->
          right(k, rest, right)
        :fail
          empty()
        end
      end
  end
end
```

You don’t have to implement the “right” state since this state will be identical to the left state apart from doing the opposit.
Appendix - operational semantics

pattern matching

\[
\frac{a \equiv s}{P_\sigma(a, s) \to \sigma} \quad \quad \frac{a \not\equiv s}{P_\sigma(a, s) \to \text{fail}} \\
\frac{v/t \not\in \sigma}{P_\sigma(v, s) \to \{v/s\} \cup \sigma} \quad \quad \frac{v/t \in \sigma \quad t \not\equiv s}{P_\sigma(v, s) \to \text{fail}} \\
\frac{v/s \in \sigma}{P_\sigma(v, s) \to \sigma} \quad \quad \frac{P_\sigma(\_s)}{P_\sigma(\_s) \to \sigma}
\]

\[
P_\sigma(p_1, s_1) \to \sigma' \land P_\sigma'(p_2, s_2) \to \theta \\
\frac{P_\sigma(\{p_1, p_2\}, \{s_1, s_2\}) \to \theta}{P_\sigma(p_1, s_1) \to \text{fail} \quad P_\sigma(p_1, s_1) \to \sigma' \land P_\sigma'(p_2, s_2) \to \text{fail}}
\]

scoping

\[
\sigma' = \sigma \setminus \{v/t \mid v/t \in \sigma \land v \ in \ p\} \\
S(\sigma, p) \to \sigma'
\]

expressions

\[
\frac{a \equiv s}{E_\sigma(a) \to s} \quad \quad \frac{v/s \in \sigma}{E_\sigma(v) \to s} \quad \quad \frac{v/s \not\in \sigma}{E_\sigma(v) \to \bot} \\
\frac{E_\sigma(e_1) \to s_1}{E_\sigma(e_1, e_2) \to \{s_1, s_2\}} \quad \quad \frac{E_\sigma(e_2) \to s_2}{E_\sigma(e_1, e_2) \to \bot} \\
\frac{E_\sigma(e) \to t}{S(\sigma, p) \to \sigma'} \quad \quad \frac{P_\sigma'(p, t) \to \theta}{E_\sigma(p = e; \text{sequence}) \to s} \\
\frac{E_\sigma(e) \to \bot}{E_\sigma(p = e; \text{sequence}) \to \bot}
\]

\[
\frac{E_\sigma(e) \to t}{S(\sigma, p) \to \sigma'} \quad \quad \frac{P_\sigma'(p, t) \to \text{fail}}{E_\sigma(p = e; \text{sequence}) \to \bot} \\
\frac{E_\sigma(e) \to \bot}{E_\sigma(p = e; \text{sequence}) \to \bot}
\]

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