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Passivity-preserving model reduction by analytic interpolation ☆

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Abstract

Antoulas and Sorensen have recently proposed a passivity-preserving model-reduction method of linear systems based on Krylov projections. The idea is to approximate a positive-real rational transfer function with one of lower degree. The method is based on an observation by Antoulas (in the single-input/single-output case) that if the approximant is preserving a subset of the spectral zeros and takes the same values as the original transfer function in the mirror points of the preserved spectral zeros, then the approximant is also positive real. However, this turns out to be a special solution in the theory of analytic interpolation with degree constraint developed by Byrnes, Georgiou and Lindquist, namely the maximum-entropy (central) solution. By tuning the interpolation points and the spectral zeros, as prescribed by this theory, one is able to obtain considerably better reduced-order models. We also show that, in the multi-input/multi-output case, Sorensen's algorithm actually amounts to tangential Nevanlinna–Pick interpolation.
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1. Introduction

Consider a time-invariant linear system

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and the constant matrices A, B, C and D have compatible dimensions with the properties that the eigenvalues of A all lie in the open left half of the complex plane, $\mathbb{C}_-$, (A, B) is reachable, (C, A) is observable and $D + D^\mathsf{T}$ is positive definite. Moreover, suppose that the transfer function

$$G(s) = C(sI - A)^{-1}B + D \tag{1.2}$$

is *positive real*; i.e.,

$$G(\mathrm{i}\omega) + G(-\mathrm{i}\omega)^\mathsf{T} \geqslant 0, \quad \omega \in \mathbb{R}. \tag{1.3}$$

Such systems Σ are *passive*; i.e.,

$$\int_0^T u(t)^\mathsf{T} y(t)\,\mathrm{d}t \geqslant 0$$

for all $T > 0$ and all square-integrable inputs u. In physical terms, such a system produces no energy internally. Passive systems are important in many applications, such as, for example, in VLSI design and stochastic systems theory. In fact,

$$\Phi(s) := G(s) + G(-s)^\mathsf{T} \tag{1.4}$$

can be interpreted as a spectral density, and there are rational functions $W(s)$, called *spectral factors*, such that

$$G(s) + G(-s)^\mathsf{T} = W(s)W(-s)^\mathsf{T}. \tag{1.5}$$

Often passive systems are too large for analysis and synthesis.

The problem considered in this paper is to find a passive reduced-order system

$$\hat{\Sigma} \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t) + Du(t), \end{cases} \tag{1.6}$$

where $\hat{x}(t) \in \mathbb{R}^k$, $k < n$, and $\hat{y}(t) \in \mathbb{R}^m$; i.e., the transfer function

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D \tag{1.7}$$

has a lower degree but has retained the positive-real property.

Such model reduction is often performed by some projection method that determines matrices $U, V \in \mathbb{R}^{n \times k}$ such that $U^\mathsf{T} V = I_k$ and

$$\hat{A} = U^\mathsf{T} AV, \quad \hat{B} = U^\mathsf{T} B, \quad \hat{C} = CV. \tag{1.8}$$

The most popular such model reduction methods preserving positive-realness is *stochastically balanced truncation* (or *positive-real balanced truncation*), originally proposed by Desai and Pal [17,18] in the context of stochastic realization theory [2,20,32]. Some early contributions to this topic include [28,36,40]. For an explanation in terms of stochastic realization theory, see [33]. Stochastically balanced model reduction has the advantage that it comes with easily computed bounds; see, e.g., [28,40].

In this paper, we shall consider another class of model reduction procedures based on interpolation, in which the transfer function $\hat{G}$ of the reduced-order system satisfies the interpolation conditions

$$\hat{G}(s_j) = G(s_j), \quad j = 1, 2, \dots, k \tag{1.9}$$

for some suitable points $s_1, s_2, \dots, s_k$ in the open right half $\mathbb{C}_+$ of the complex plane. In the scalar case, $m = 1$, Antoulas [4] has recently observed that, if the interpolation points $s_1, s_2, \dots, s_k$ are mirror images of some spectral zeros of G, i.e., zeros of (1.4), and these zeros are also the spectral zeros of $\hat{G}$, then $\hat{G}$ is positive real. Sorensen [38] has developed an efficient algorithm based on Antoulas' idea [4] that does not explicitly use spectral zeros but also works in the case $m > 1$. We shall demonstrate that Sorensen's solution amounts to tangential interpolation rather than matricial interpolation involving the condition (1.9).

However, Antoulas' observation does not come as great surprise to us, since the concept of spectral zeros is a key ingredient in a theory of analytic interpolation developed over the last decades by Byrnes, Georgiou, Lindquist and their coworkers [6–16,19,22–27,29,30,35]. Indeed, given $k + 1$ interpolation points and corresponding interpolation values, the class of all analytic interpolants of McMillan degree at most k are completely parameterized by the stable spectral zeros. Moreover, given a specific choice of such spectral zeros, there is a pair of dual convex optimization problems determining the unique corresponding interpolant. We shall demonstrate that Antoulas' solution is essentially the central solution or the maximum entropy solution in this theory. This opens up the questions of whether the full power of the theory of analytic interpolation with degree constraint can be used to obtain better approximations. We shall provide numerical examples showing that this is indeed the case. Model reduction from selected spectral zeros has previously been performed in [16] in the context of covariance extension.

2. Spectral zeros

The quadruple (A, B, C, D) in (1.2) is often called a minimal realization of $G(s)$, and we shall denote it

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

In particular, the *zeros* of $G(s)$ are precisely the complex numbers λ for which

$$\begin{pmatrix} A - \lambda I & B \\ C & D \end{pmatrix}$$

is singular.

In the context of passive systems, we are interested in the *spectral zeros* of $G(s)$; i.e., the zeros of (1.4). Since

$$G(-s)^{\mathsf{T}} \sim \left[\begin{array}{c|c} -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ \hline B^{\mathsf{T}} & D^{\mathsf{T}} \end{array}\right]$$

it readily follows that

$$G(s) + G(-s)^{\mathsf{T}} \sim \left[\begin{array}{cc|c} A & & B \\ & -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ \hline C & B^{\mathsf{T}} & D + D^{\mathsf{T}} \end{array}\right],$$

i.e., the spectral zeros are the λ for which the matrix $\mathscr{A} - \lambda\mathscr{E}$ is singular, where

$$\mathscr{A} := \begin{bmatrix} A & & B \\ & -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ C & B^{\mathsf{T}} & D + D^{\mathsf{T}} \end{bmatrix}, \quad \mathscr{E} = \begin{bmatrix} I_n & & \\ & I_n & \\ & & 0_m \end{bmatrix}. \tag{2.1}$$

Consequently, the spectral zeros are the generalized eigenvalues of $(\mathscr{A}, \mathscr{E})$.

For simplicity of presentation, from now on, we make the same assumption as in [4,38], namely that *the spectral zeros are distinct*.

3. The Antoulas–Sorensen approach

The starting point in Sorensen's algorithm is a partial real Schur decomposition

$$\mathscr{A}Q = \mathscr{E}QR \tag{3.1}$$

for the pair $(\mathscr{A}, \mathscr{E})$, where $Q^{\mathsf{T}}Q = I_k$ and R is real and quasi-upper triangular. Clearly, the eigenvalues of R are generalized eigenvalues of $(\mathscr{A}, \mathscr{E})$; i.e., (selected) spectral zeros, and we obtain one decomposition (3.1) for each choice of k spectral zeros. Setting $Q^{\mathsf{T}} = (X^{\mathsf{T}}, Y^{\mathsf{T}}, Z^{\mathsf{T}})$, we have

$$\begin{bmatrix} A & & B \\ & -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ C & B^{\mathsf{T}} & D + D^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} R. \tag{3.2}$$

Eliminating Z in these block equations yields

$$\mathscr{H} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R, \tag{3.3}$$

where

$$\mathscr{H} := \begin{bmatrix} A - B(D + D^{\mathsf{T}})^{-1}C & -B(D + D^{\mathsf{T}})^{-1}B^{\mathsf{T}} \\ -C^{\mathsf{T}}(D + D^{\mathsf{T}})^{-1}C & -[A - B(D + D^{\mathsf{T}})^{-1}C)]^{\mathsf{T}} \end{bmatrix} \tag{3.4}$$

It is straight-forward to check that $\mathscr{H}$ is a Hamiltonian matrix; i.e.,

$$(J\mathscr{H})^{\mathsf{T}} = J\mathscr{H}, \quad \text{where } J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{3.5}$$

and consequently it follows from (3.3) that

$$(X^{\mathsf{T}}Y - Y^{\mathsf{T}}X)R = [X^{\mathsf{T}} \quad Y^{\mathsf{T}}]J\mathscr{H} \begin{bmatrix} X \\ Y \end{bmatrix} = R^{\mathsf{T}}(Y^{\mathsf{T}}X - X^{\mathsf{T}}Y). \tag{3.6}$$

To clarify the connections between observability, reachability and nondegeneracy of solutions – a topic to which we shall return below – we now restate Lemmas 1 and 2 in [38] in a slightly more general form and with a more streamlined proof, based on standard constructions; see, e.g., [31].

Lemma 1. *Suppose that $k \leqslant n$ and that R has no pair (λ, μ) of eigenvalues such that $\lambda = -\bar{\mu}$. Then X has full rank if (A, B) is reachable, and Y has full rank if (C, A) is observable. Moreover $X^{\mathsf{T}}Y = Y^{\mathsf{T}}X$.*

Proof. We begin with the last statement (Lemma 1 in [38]). Since the linear map $L(P) := PR + R^{\mathsf{T}}P$, sending symmetric $k \times k$ matrices P to symmetric $k \times k$ matrices $PR + R^{\mathsf{T}}P$, is regular if and only if R has no pair (λ, μ) of eigenvalues such that $\lambda = -\bar{\mu}$ [21, p. 225], it follows from (3.6) that $P := X^{\mathsf{T}}Y - Y^{\mathsf{T}}X = 0$, as claimed. Then it also follows from (3.6) that

$$M(X,Y) := [-Y^{\mathsf T} \quad X^{\mathsf T}]\mathscr{H}\begin{bmatrix} X \\ Y \end{bmatrix} = 0. \tag{3.7}$$

Next, from (3.7) we have that, for any $a \in \ker Y$,

$$a^{\mathsf T} M(X,Y)a = a^{\mathsf T} X^{\mathsf T} C^{\mathsf T} (D + D^{\mathsf T})^{-1} C X a = 0;$$

that is, $Xa \in \ker C$ for all $a \in \ker Y$. Moreover, by (3.3),

$$\mathscr{H}\begin{bmatrix} Xa \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} Ra,$$

from which we have $AXa = XRa$ and $YRa = 0$. Consequently, $Ra \in \ker Y$ and $AX \ker Y \subset X \ker Y$, which, in turn, implies that

$$A^j X \ker Y \subset X \ker Y \subset \ker C, \quad j = 0, 1, 2, \ldots.$$

Consequently, $\cap_{j=0}^{\infty} CA^j Xa = 0$ for all $a \in \ker Y$. Therefore, if (C, A) is observable, we must have $Xa = 0$. However, if $Xa = Ya = 0$, then, from the last block equation of (3.2), $Za = 0$; i.e., $Qa = 0$. Hence, since $Q := (X^{\mathsf T}, Y^{\mathsf T}, Z^{\mathsf T})^{\mathsf T}$ has full rank, we must have $a = 0$. This establishes that Y has full rank.

To show that X has full rank if (A, B) is reachable, we take $a \in \ker X$, $a^{\mathsf T} M(X,Y)a = 0$, and proceed as above to show that a must be zero if (A, B) is reachable. □

In particular the requirements on R in Lemma 1 are satisfied if all eigenvalues of R are located in $\mathbb{C}_-$ or in $\mathbb{C}_+$.

It is now instructive to observe the connections to stochastically balanced truncation. Taking $k = n$ and the spectral zeros (eigenvalues of R) to be those in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively, we obtain the solutions X_-, Y_- and X_+, Y_+, respectively, of (3.3). It is well known that $P_- := X_- Y_-^{-1}$ and $P_+ := X_+ Y_+^{-1}$ are the minimal and maximal solutions respectively (in the ordering of positive definite matrices) of the algebraic Riccati equation

$$AP + PA^{\mathsf T} - (B + PC^{\mathsf T})(D + D^{\mathsf T})^{-1}(B + PC^{\mathsf T})^{\mathsf T} = 0, \tag{3.8}$$

standard in stochastic realization theory [2,20,32]. Stochastic balancing amounts to finding a linear regular transformation T such that

$$TP_- T^{\mathsf T} = \Sigma = T^{-\mathsf T} P_+^{-1} T^{-1},$$

where Σ is the diagonal matrix with the diagonal elements being the singular values of $P_- P_+^{-1} = Y_-^{-\mathsf T} X_-^{\mathsf T} X_+ Y_+^{-1}$, ordered by size. Positive-real balanced truncation is then a projection method where the matrices (1.8) are chosen as

$$U^{\mathsf T} = [I_k \quad 0]T, \quad V = T^{-1}\begin{bmatrix} I_k \\ 0 \end{bmatrix}.$$

In Sorensen's algorithm one takes instead the partial real Schur decomposition (3.1) corresponding to k selected spectral zeros in $\mathbb{C}_-$ and performs singular value decomposition on $X^{\mathsf T} Y$. More precisely, this amounts to determining unitary $k \times k$ matrices Q_x and Q_y such that $Q_x \Sigma^2 Q_y^{\mathsf T} = X^{\mathsf T} Y$ is the singular value decomposition of $X^{\mathsf T} Y$, and setting

$$V := XQ_x\Sigma^{-1}, \quad U := YQ_y\Sigma^{-1}. \tag{3.9}$$

In [38] Sorensen proves, using the Positive Real Lemma, that the reduced-order transfer function

$$\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & D \end{array}\right] \tag{3.10}$$

obtained by taking V, U defined by (3.9) in (1.8) is positive real. We shall now prove that, in addition, a tangential interpolation condition with interpolation points at the selected spectral zeros also holds, (as we shall see, only partly) in harmony with Antoulas' result [4].

Proposition 2. *Given a minimal realization* (1.2) *of* G, *let* $\{s_j\}_{j=1}^k \subset \mathbb{C}_+$ *be* k *arbitrary* (*distinct*) *spectral zeros of* G, *and let* (3.9) *be the projection matrices determined by Sorensen's algorithm. Then the reduced-order* $\hat{G}$ *obtained by the projection* (1.8) *satisfies the right tangential interpolation conditions*

$$\hat{G}(s_j)z_j = G(s_j)z_j, \quad j = 1, \dots, k, \tag{3.11}$$

where $z_j := Zr_j \neq 0$ *for* $k = 1, 2, \dots, k$, *and* r_j *is the right eigenvector of* R *corresponding to the eigenvalue* s_j.

In addition, $\hat{G}$ *satisfies the left tangential interpolation condition*

$$z_j^{\mathsf{T}}\hat{G}(-s_j) = z_j^{\mathsf{T}}G(-s_j) \tag{3.12}$$

for each $j = 1, 2, \dots, k$ *such that* $(-s_jI_k - \hat{A})$ *is invertible. In particular*, *if* (3.10) *is a minimal realization*, (3.12) *holds for all* $j = 1, 2, \dots, k$.

Proof. We begin by verifying the interpolation condition (3.11). To this end, we use (1.8) and (3.9) to obtain

$$\begin{aligned} \hat{G}(s_j)z_j &= \hat{C}(s_jI_k - \hat{A})^{-1}\hat{B}z_j + Dz_j \\ &= CX(Y^{\mathsf{T}}(s_jI_n - A)X)^{-1}Y^{\mathsf{T}}Bz_j + Dz_j \end{aligned} \tag{3.13}$$

for $j = 1, 2, \dots, k$. In fact, since $\hat{G}$ is positive real, $\hat{A}$ has all its eigenvalues in $\mathbb{C}_-$, and therefore $s_jI_k - \hat{A}$ is invertible. Moreover, since $Y^{\mathsf{T}}X = Q_y\Sigma^2Q_x$ and $Q_y^{\mathsf{T}}Q_y = I_k = Q_xQ_x^{\mathsf{T}}$,

$$s_jI_k - \hat{A} = \Sigma^{-1}Q_y^{\mathsf{T}}Y^{\mathsf{T}}(s_jI_n - A)XQ_x^{\mathsf{T}}\Sigma^{-1},$$

and hence $Y^{\mathsf{T}}(s_jI_n - A)X$ is also invertible. From the first block equation in (3.2) we obtain, for $j = 1, 2, \dots, k$,

$$(s_jI_n - A)Xr_j = Bz_j, \tag{3.14}$$

where r_j the right eigenvector of R corresponding to the eigenvalue s_j and $z_j := Zr_j$. Since $s_j \in \mathbb{C}_+$ and A has all its eigenvalues in the left half plane, $(s_jI_n - A)$ is invertible, and hence

$$Xr_j = (s_jI_n - A)^{-1}Bz_j. \tag{3.15}$$

Moreover, $Y^{\mathsf{T}}(s_jI_n - A)Xr_j = Y^{\mathsf{T}}Bz_j$; i.e.,

$$(Y^{\mathsf{T}}(s_jI_n - A)X)^{-1}Y^{\mathsf{T}}Bz_j = r_j,$$

which inserted into (3.13) yields

$$\hat{G}(s_j)z_j = CXr_j + Dz_j, \quad j = 1, 2, \dots, k.$$

In view if of (3.15), this is the same as (3.11). Clearly, $z_j \neq 0$, because otherwise, by (3.15), $Xr_j = 0$, which would contradict the fact that X has full rank (Lemma 1).

For the second statement, first note that A^{T} cannot have an eigenvalue in $-s_j$, because then there would be a cancellation in the spectral factor W in (1.5) so that $\deg W < n$. This would imply that $\deg G < n$, which contradicts the assumption that (1.2) is a minimal realization. Hence $(-s_j I_n - A^{\mathsf{T}})$ is invertible. For the same reason, $(-s_j I_n - \hat{A}^{\mathsf{T}})$ is invertible if (3.10) is minimal. Given that $(-s_j I_n - \hat{A}^{\mathsf{T}})$ is invertible, the second statement can be proven analogously as above. To this end, we form

$$z_j^{\mathsf{T}}\hat{G}(-s_j) = z_j^{\mathsf{T}}CX(Y^{\mathsf{T}}(-s_j I_n - A)X)^{-1}Y^{\mathsf{T}}B + z_j^{\mathsf{T}}D. \tag{3.16}$$

From the second block equation in (3.2) we have

$$(-s_j I_n - A^{\mathsf{T}})Yr_j = C^{\mathsf{T}}z_j. \tag{3.17}$$

Since $(-s_j I_n - A^{\mathsf{T}})$ is invertible,

$$Yr_j = (-s_j I_n - A^{\mathsf{T}})^{-1}C^{\mathsf{T}}z_j. \tag{3.18}$$

Moreover, $X^{\mathsf{T}}(-s_j I_n - A^{\mathsf{T}})Yr_j = X^{\mathsf{T}}C^{\mathsf{T}}z_j$; i.e.,

$$z_j^{\mathsf{T}}CX(Y^{\mathsf{T}}(-s_j I_n - A)X)^{-1} = r_j^{\mathsf{T}},$$

which inserted into (3.16), together with (3.18), yields (3.12). □

Consequently, in the scalar case, the Antoulas–Sorensen reduced-order transfer function $\hat{G}$ interpolates in the *unstable* spectral zeros. If the reduced-order realization (3.10) is minimal, it also interpolates in the stable spectral zeros. The minimality of (3.10) is important, so we pause to consider some consequences of this.

A basic question, raised by Antoulas in [4], is when a rational function G satisfying both the interpolation conditions

$$G(s_j) = w_j, \quad j = 1, 2, \ldots, k$$

and the corresponding “mirror-image” interpolation conditions

$$G(-\bar{s}_j) = -\bar{w}_j, \quad j = 1, 2, \ldots, k$$

is positive real. In [4] it is claimed that all minimum-degree interpolants are positive real [4, Lemma 3.1]. This is not correct. A simple first-order counterexample is obtained by taking $(s_1, w_1) = (1, 1)$. A function satisfying both $G(1) = 1$ and $G(-1) = -1$ cannot be of degree zero, so the claim in [4, Lemma 3.1] is that any degree-one function satisfying both $G(1) = 1$ and $G(-1) = -1$ is positive real. A counterexample is $G(s) = (1 - 2s)/(s - 2)$, which is not even analytic in $\mathbb{C}_+$, let alone positive real.

There could be a mistake in transferring the statement of Lemma 3.1 in [4] from that in Theorem 4.2 in the previous paper [5], co-authored by the same author, where there is one less mirror-image interpolation condition. This is more natural, since, in general, $2k - 1$ linear equations are required to determine a rational function of degree $k - 1$.

However, the statement of Theorem 4.2 in [5] is also incorrect. In fact, transferred into the setting of positive real functions in the right half plane, this theorem implies the following: given (s_j, w_j), $j = 1, \ldots, k$, such that the Pick matrix

$$\widetilde{P} := \left[\frac{w_i + \bar{w}_j}{s_i + \bar{s}_j}\right]_{i,j=1}^{k}$$

is positive definite, there exists a unique rational function f of degree less or equal to $k - 1$ such that $f(s_j) = w_j$, $j = 1, \ldots, k$ and $f(-\bar{s}_j) = -\bar{w}_j$, $j = 1, \ldots, k - 1$, and this rational function is positive real.

It is true that there exists a positive real function f of degree $\leqslant k-1$ which satisfies $f(s_j) = w_j$, $j = 1, \dots, k$. If $\widetilde{P}$ is positive definite, all such solutions are parameterized by Theorem 4. However, there does not necessarily exist an interpolant of degree at most $k-1$ which also satisfies the mirror interpolation conditions. In fact, the following is a simple counterexample. If $w_j = 1$ for $j = 1, \dots, k$, there is a unique function of degree at most $k-1$ satisfying $G(s_j) = w_j$ for $j = 1, 2, \dots, k$, namely $G \equiv 1$. However, this function does not satisfy the mirror conditions $G(-\bar{s}_j) = -1$, $j = 1, 2, \dots, k-1$. Hence any function satisfying the $2k-1$ interpolation conditions must have a degree no smaller than k.

The following numerical example further elucidates this point.

Example 3. Consider the second-order positive real transfer function

$$G(s) = \frac{6s^2 + 22s + 9}{6s^2 + 15s + 16}, \tag{3.19}$$

for which

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{8}{3} & -\frac{5}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{7}{6} & \frac{7}{6} \end{bmatrix}, \quad D = 1$$

is a minimal realization, and $s = \pm 1$ and $s = \pm 2$ are the spectral zeros.

First, we compute the first-order transfer function $\hat{G}$ with the stable spectral zero at $s_1 = -2$. The corresponding solution of (3.3) is

$$X = \begin{bmatrix} -0.0266 \\ -0.0533 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.6887 \\ 0.6524 \end{bmatrix}, \quad R = 2,$$

which, by (3.9), yields

$$V = \begin{bmatrix} -0.1156 \\ -0.2312 \end{bmatrix}, \quad U = \begin{bmatrix} -2.9890 \\ -2.8316 \end{bmatrix}.$$

Then from (1.8), we obtain

$$(\hat{A}, \hat{B}, \hat{C}, D) = (-1.8182, -2.8316, -0.1348, 1),$$

which clearly is both observable and reachable and hence minimal. The reduced-degree function $\hat{G}$ is positive real, and both the interpolation conditions $\hat{G}(-2) = G(-2)$ and $\hat{G}(2) = G(2)$ hold.

Next, we compute the first-order transfer function $\hat{G}$ with the stable spectral zero at $s_1 = -1$. Then

$$X = \begin{bmatrix} -0.0343 \\ -0.0343 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.8801 \\ 0.4224 \end{bmatrix}, \quad R = 1,$$

and

$$V = \begin{bmatrix} -0.1622 \\ -0.1622 \end{bmatrix}, \quad U = \begin{bmatrix} -4.1667 \\ -2.0000 \end{bmatrix}.$$

This yields

$$(\hat{A}, \hat{B}, \hat{C}, D) = (-1, -2, 0, 1),$$

which is not observable and hence not minimal. The reduced-degree transfer function is

$$\hat{G} \equiv 1,$$

which is clearly positive real, and satisfies $\hat{G}(1) = G(1) = 1$, but *not* $\hat{G}(-1) = G(-1)$.

4. Analytic interpolation with degree constraint

In [38] Sorensen verifies the stability and passivity of the reduced-order model of Section 3 using the Positive Real Lemma. However, the reason why this particular choice of interpolation points lead to a positive real reduced-order model is no coincidence. In fact, in the next section we shall demonstrate that the Antoulas–Sorensen solution can be interpreted in the context of the theory of analytic interpolation with degree constraint developed by Byrnes, Georgiou and Lindquist.

To this end, we now restate some basic results from this theory in the continuous-time setting. For consistency with the setting in [4] we confine the initial analysis real, scalar interpolants, although, strictly speaking, this is not necessary. Given a set of self-conjugate pairs of complex numbers

$$\{(s_j, w_j) : s_j \in \mathbb{C}_+\}_{j=0}^{k}, \quad \begin{array}{l} s_i \neq s_j \text{ if } i \neq j,\ s_0 \text{ real}, \\ w_i = \bar{w}_j \text{ if } s_i = \bar{s}_j, \end{array} \tag{4.1}$$

find all functions f with real coefficients that satisfy the following three conditions:

(1) *Positive real property*: the function f is analytic in $\mathbb{C}_+$, and

$$\operatorname{Re} f(s) \geqslant 0, \quad \forall s \in \mathbb{C}_+. \tag{4.2}$$

(2) *Interpolation conditions*:

$$f(s_j) = w_j, \quad j = 0, 1, \ldots, k. \tag{4.3}$$

(3) *Degree constraint*: f is real rational and

$$\deg f \leqslant k. \tag{4.4}$$

A necessary and sufficient condition for the existence of f satisfying these three conditions is the positive semidefiniteness of the Pick matrix

$$P := \left[\frac{w_i + \bar{w}_j}{s_i + \bar{s}_j}\right]_{i,j=0}^{k}. \tag{4.5}$$

Theorem 4. *Suppose that the Pick matrix* (4.5) *constructed from the interpolation data* (4.1) *is positive definite. Let* $\{\lambda_j\}_{j=1}^{k} \subset \mathbb{C}_-$ *be an arbitrary self-conjugate set of k spectral zeros, and define* $\sigma(s) := \prod_{j=1}^{k}(s - \lambda_j)$. *Then, there exists a unique (modulo sign) pair of real Hurwitz polynomials* (α, β) *of degree k such that*

(i) $f := \beta/\alpha$ *is positive real,*
(ii) $f(s_j) = w_j$, $j = 0, 1, \ldots, k$, *and*
(iii) $\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \sigma(s)\sigma(-s)$.

Conversely, any pair of real polynomials (α, β) *of degree k satisfying* (i) *and* (ii) *determines, via* (iii), *a unique (modulo sign) Hurwitz polynomial* σ *of degree* k, *and the map* $\sigma \mapsto (\alpha, \beta)$ *is a diffeomorphism. Moreover, setting*

$$\Psi(\mathrm{i}\omega) := \left|\frac{\sigma(\mathrm{i}\omega)}{\tau(\mathrm{i}\omega)}\right|^2, \quad \textit{where } \tau(s) := \prod_{j=1}^{k}(s + s_j) \tag{4.6}$$

and

$$\mathbb{I}_\Psi(f) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\mathrm{i}\omega)\log[f(\mathrm{i}\omega)+f(-\mathrm{i}\omega)]\frac{\mathrm{d}\omega}{\omega^2+s_0^2}, \tag{4.7}$$

the optimization problem

$$\max \mathbb{I}_\Psi(f) \quad \textit{subject to } f(s_j)=w_j, \quad j=0,1,\ldots,k, \tag{4.8}$$

where the maximization is over all positive real functions, has a unique solution f that is precisely the unique f satisfying the conditions (i), (ii) *and* (iii). *Finally, if (α,β) is the corresponding pair of polynomials,*

$$Q(\mathrm{i}\omega) := |a(\mathrm{i}\omega)|^2, \quad \textit{where } a(s)=\frac{\alpha(s)}{\tau(s)}, \tag{4.9}$$

is the unique solution to the convex optimization problem

$$\min_{Q\in\mathcal{Q}} \mathbb{J}_\Psi(Q), \tag{4.10}$$

where

$$\mathbb{J}_\Psi(Q) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\{[w(\mathrm{i}\omega)+w(-\mathrm{i}\omega)]Q(\mathrm{i}\omega)-\Psi(\mathrm{i}\omega)\log Q(\mathrm{i}\omega)\}\frac{\mathrm{d}\omega}{\omega^2+s_0^2}, \tag{4.11}$$

$\mathcal{Q}$ is the class of all rational functions of the form (4.9) *with α free to vary over all stable polynomials of degree at most k, and w is any proper, stable* (*not necessarily positive real*) *real, rational function satisfying the interpolation condition* (ii).

The statements of the theorem have been proven in the discrete time setting in various places: the first part in [8,25] (also, see [12,10], and, as for existence only, the early work [22,23,24]), the diffeomorphism result in [15], and the optimization results (in various versions) in [13,11,8,14,10]. Transferring this results to the continuous-time setting is quite straight-forward.

Proof. The results can be transferred from the discrete-time setting to the continuous-time setting via the Möbius transformation

$$s\in\mathbb{C}_+ \mapsto z=\frac{s_0-s}{s_0+s}\in\mathbb{D} \tag{4.12}$$

and its inverse

$$z\in\mathbb{D}\mapsto s=s_0\frac{1-z}{1+z}\in\mathbb{C}_+. \tag{4.13}$$

In particular, the unit circle $\{z=\mathrm{e}^{\mathrm{i}\theta}|\theta\in[-\pi,\pi]\}$ is mapped to the imaginary axis $\{s=\mathrm{i}\omega|\omega\in(-\infty,\infty)\}$, and

$$\mathrm{d}\theta=\frac{-2s_0}{\omega^2+s_0^2}\mathrm{d}\omega. \tag{4.14}$$

Transforming (4.6) via the Möbius transformation (4.13), we obtain

$$\hat{\Psi}(z) := \Psi\left(s_0\frac{1-z}{1+z}\right)=\frac{\hat{\sigma}(z)\hat{\sigma}(z^{-1})}{\hat{\tau}(z)\hat{\tau}(z^{-1})},$$

where $\hat{\Psi}$ is also a rational function with numerator polynomial $\hat{\sigma}$ and denominator polynomial $\hat{\tau}$ that are both Hurwitz polynomials. Moreover, define the sequence

$$z_j := \frac{s_0 - s_j}{s_0 + s_j}, \quad j = 0, 1, \ldots, k.$$

Then, it was shown in [8,10] that there is a unique (modulo sign) pair of Schur polynomials $(\hat{\alpha}, \hat{\beta})$ of degree k such that

(i)′ $\hat{f} := \hat{\beta}/\hat{\alpha}$ is positive real,
(ii)′ $\hat{f}(s_j) = w_j$, $j = 0, 1, \ldots, k$, and
(iii)′ $\hat{\alpha}(z)\hat{\beta}(z^{-1}) + \hat{\beta}(z)\hat{\alpha}(z^{-1}) = \hat{\sigma}(z)\hat{\sigma}(z^{-1})$,

and that this $\hat{f}$ is the unique solution of the optimization problem

$$\max \hat{\mathbb{I}}_{\hat{\Psi}}(\hat{f}) \quad \text{subject to } \hat{f}(z_j) = w_j, \quad j = 0, 1, \ldots, k, \tag{4.15}$$

where

$$\hat{\mathbb{I}}_{\hat{\Psi}}(\hat{f}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\Psi}(e^{i\theta}) \log[\hat{f}(e^{i\theta}) + \hat{f}(e^{-i\theta})]d\theta. \tag{4.16}$$

It was also shown in [8,9] that this optimization problem has a dual, namely the problem to minimize the strictly convex functional

$$\hat{\mathbb{J}}_{\hat{\Psi}}(\hat{Q}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \{[\hat{w}(e^{i\theta}) + \hat{w}(e^{-i\theta})]\hat{Q}(e^{i\theta}) - \hat{\Psi}(e^{i\theta}) \log \hat{Q}(e^{i\theta})\}d\theta, \tag{4.17}$$

where $\hat{w}$ is any stable (not necessarily positive real) real function satisfying the interpolation condition (ii)′, over the class of rational functions

$$\hat{\mathscr{Q}} := \left\{ \hat{Q} \,|\, \hat{Q}(e^{i\theta}) = |\hat{a}(e^{i\theta})|^2, \quad \hat{a}(z) = \frac{\hat{\alpha}(z)}{\hat{\tau}(z)}, \quad \hat{\alpha} \text{ Schur polynomial of degree } k \right\}.$$

It was also shown that the optimal $\hat{Q}$ corresponds to $\hat{\alpha}$ in (i)′–(iii)′ above.

Now, via the Möbius transformation (4.12), (i)–(iii) is seen to be equivalent to (i)′–(iii)′, and, also appealing to (4.14), the two discrete-time optimization problems are seen to be equivalent to the two continuous-time ones in the statement of the theorem. □

This theorem yields a complete smooth parameterization of the whole class of positive real interpolants of degree at most n, where tuning can be done via the k spectral zeros. For each choice of spectral zeros, the interpolant f can be obtained via convex optimization [8] or non-linear equations [6]. In particular, if we choose the k spectral zeros at the mirror images of the interpolation points, as suggested by Antoulas and Sorensen,

$$\lambda_j = -\bar{s}_j, \quad j = 1, \ldots, k, \tag{4.18}$$

then $\Psi \equiv 1$, and the interpolant maximizes the entropy gain

$$\mathbb{I}_1(f) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \log[f(i\omega) + f(-i\omega)]\frac{d\omega}{\omega^2 + s_0^2}. \tag{4.19}$$

This is the *central* or *maximum entropy solution*, the determination of which can be reduced to a system of linear equations; also see [34].

Corollary 5. *Let P be the Pick matrix* (4.5), *where s_0 is real*, *let $\tau(s)$ be the Hurwitz polynomial defined in* (4.6), *and set*

$$\Pi(s) := \left[1, \frac{s+s_0}{s+s_1}, \frac{s+s_0}{s+s_2}, \ldots, \frac{s+s_0}{s+s_k}\right].$$

Moreover, suppose that $P > 0$. *Then the maximum entropy solution is*

$$f(s) = \frac{\Pi(s)b}{\Pi(s)a},$$

where $a := (a_0, a_1, \ldots, a_k)^{\mathsf{T}}$ *is given by*

$$a = \frac{1}{\sqrt{2s_0\Pi(s_0)P^{-1}\Pi(s_0)^*}} P^{-1}\Pi(s_0)^*, \tag{4.20}$$

and $b := (b_0, b_1, \ldots, b_k)^{\mathsf{T}}$ *is uniquely determined via the linear system of equations*

$$a(s)b(-s) + a(-s)b(s) = 1 \tag{4.21}$$

with $a(s) := \Pi(s)a$ *and* $b(s) := \Pi(s)b$. *Equivalently*,

$$f(s) = \frac{\beta(s)}{\alpha(s)},$$

where $\alpha(s)$ *and* $\beta(s)$ *are real Hurwitz polynomials such that* $a(s) = \alpha(s)/\tau(s)$, $b(s) = \beta(s)/\tau(s)$, *and*

$$\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \tau(s)\tau(-s). \tag{4.22}$$

In particular,

$$2\mathrm{Re}\{f(\mathrm{i}\omega)\} = \frac{2s_0\Pi(s_0)P^{-1}\Pi(s_0)^*}{\Pi(s_0)P^{-1}\Pi^*(\mathrm{i}\omega)\Pi(\mathrm{i}\omega)P^{-1}\Pi(s_0)^*}, \tag{4.23}$$

where $\Pi^*(s) = \overline{\Pi(-\bar{s})}^{\mathsf{T}}$ *for functions*, *and where* $M^* = \overline{M}^{\mathsf{T}}$; *i.e.*, *the Hermitian transpose*, *for any matrix* M.

Proof. From (4.9) and (4.11) we obtain

$$\begin{aligned}\mathbb{J}_1(Q) :=& \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\mathrm{i}\omega)^*[w(\mathrm{i}\omega) + w(\mathrm{i}\omega)^*]a(\mathrm{i}\omega)\frac{\mathrm{d}\omega}{\omega^2+s_0^2} \\ &-\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\log|a(\mathrm{i}\omega)|\frac{\mathrm{d}\omega}{\omega^2+s_0^2}.\end{aligned} \tag{4.24}$$

Since $a(s) := \Pi(s)a$, the first term in (4.24) can written a^*Pa, where

$$P := \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi(\mathrm{i}\omega)^*[w(\mathrm{i}\omega) + w(\mathrm{i}\omega)^*]\Pi(\mathrm{i}\omega)\frac{\mathrm{d}\omega}{\omega^2+s_0^2}, \tag{4.25}$$

which is actually the Pick matrix (4.5). To see this, first note that $P = P_w + P_w^*$, where P_w is the matrix obtained with only the term $w(\mathrm{i}\omega)$ within the square brackets in (4.25). Clearly, by Cauchy's Theorem,

$$(P_w)_{j\ell} = \frac{1}{2\pi\mathrm{i}}\oint \frac{1}{s-s_j}\frac{w(s)}{s+\bar{s}_\ell}\mathrm{d}s,$$

where we integrate counter-clockwise along a closed contour consisting of the interval $(-\mathrm{i}r, \mathrm{i}r)$ and the half-circle in $\mathbb{C}_+$ with center zero and radius r that encircles the point $s = s_j$. In fact,

$s \mapsto w(s)/(s+\bar{s}_\ell)$ is analytic inside the contour, and the integral along the half-circle tends to zero as $r \to \infty$. However, by Cauchy's integral formula, this equals

$$(P_w)_{j\ell} = \frac{w_j}{s_j + \bar{s}_\ell},$$

which added to P_w^* establishes that (4.25) is the Pick matrix (4.5).

To evaluate the second term in (4.24), we first observe that, since the real, rational function a is outer in $\mathbb{C}_+$, the real, rational function $\hat{a}$, obtained from a via the Möbius transform (4.12), is outer in $\mathbb{D}$. Therefore, by Jensen's formula [1, p. 184],

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\hat{a}(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta = \log|\hat{a}(0)|,$$

which, via the inverse Möbius transform (4.13), can be written

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \log|a(\mathrm{i}\omega)|\frac{2s_0\mathrm{d}\omega}{\omega^2+s_0^2} = \log|a(s_0)|.$$

Consequently, minimizing (4.24) is equivalent to minimizing

$$J_1(a) = a^*Pa - \frac{1}{2s_0}\log a(s_0)^*a(s_0) \tag{4.26}$$

over all $a \in \mathbb{R}^{k+1}$. Since $P > 0$ and $a(s_0) = \Pi(s_0)a$, the minimizer is given by

$$a = \frac{1}{2s_0 a(s_0)} P^{-1}\Pi(s_0)^*.$$

Therefore, $a(s_0) = \sqrt{\Pi(s_0)P^{-1}\Pi(s_0)^*/2s_0}$, and hence (4.20) follows. By Theorem 4, the numerator $b(s)$ has the form $\Pi(s)b$. Moreover, since $a(s)$ is a Hurwitz polynomial, (4.21) is a linear systems of equations with unique solution. Finally, (4.23) is obtained by inserting (4.20) in $2\mathrm{Re}\{f(\mathrm{i}\omega)\} = 1/|a(\mathrm{i}\omega)|^2$. □

Theorem 4 can be generalized to tangential Nevanlinna–Pick interpolation, as established in [30]. Hence, in view of Proposition 2, the theory of analytic interpolation with degree constraint could also be applied in the multivariable case ($m > 1$). However, for clarity of presentation, in the sequel we will restrict our attention to the singe-input/single-output case ($m = 1$).

5. The Antoulas–Sorensen method as the maximum entropy solution

In the Antoulas–Sorensen approach, one interpolates not only at the unstable spectral zeros $s_1, s_2, \ldots, s_k$ but also at $s_0 := \infty$. More specifically,

$$\hat{G}(\infty) = D = G(\infty). \tag{5.1}$$

However, $s_0 := \infty$ lies on the boundary of the analyticity region $\mathbb{C}_+$ – a situation to which Corollary 5 and Theorem 4 do not immediately apply. Therefore, next we demonstrate that choosing the interpolation point s_0 to be a positive number, sufficiently large for the Pick matrix (4.5) to be positive definite, determining the corresponding central solution, and then taking the limit as $s_0 \to \infty$, results precisely in the Antoulas–Sorensen solution.

Theorem 6. *Let G be a scalar, positive real function with spectral zeros in $\{s_j\}_{j=1}^k$ and let $\hat{G}$ be the reduced order interpolant constructed by the method of Antoulas–Sorensen. Set $w_j := G(s_j)$*

and $w_0 := D = G(\infty)$. *For* $s_0 > 0$ *sufficiently large, let* f_{s_0} *be the central solution corresponding to the interpolation conditions*

$$f_{s_0}(s_0) = w_0,$$
$$f_{s_0}(s_j) = w_j, \quad j = 1, \dots, k;$$

i.e., the positive real function maximizing the entropy functional (4.19) *subject to the interpolation constraints. Then, as* $s_0 \to \infty$, $f_{s_0} \to \hat{G}$ *pointwise except in the poles of* $\hat{G}$. *Moreover, setting*

$$\Lambda := \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_k \end{bmatrix}, \qquad \widetilde{P} = \left[\frac{w_j + \bar{w}_\ell}{s_j + \bar{s}_\ell}\right]_{j,\ell=1}^k,$$

$w := (w_1 - w_0, w_2 - w_0, \dots, w_k - w_0)^\mathsf{T}$ *and* $h := (1, 1, \dots, 1)^\mathsf{T} \in \mathbb{R}^k$, *we have*

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D \tag{5.2}$$

with $(\hat{A}, \hat{B}, \hat{C})$ *given by*

$$\hat{A} = -\Lambda + h\hat{C}, \tag{5.3}$$
$$\hat{B} = 2w_0(Q\hat{C}^* + h), \tag{5.4}$$
$$\hat{C} = (\widetilde{P}^{-1}w)^\mathsf{T}, \tag{5.5}$$

where $\hat{A}$ *has all it eigenvalues in the open left half plane, and where* Q *is the unique solution of the Lyapunov equation*

$$\hat{A}Q + Q\hat{A}^* + hh^* = 0. \tag{5.6}$$

Proof. By Corollary 5,

$$\Phi_{s_0}(s) := f_{s_0}(s) + f_{s_0}(-s) = \frac{1}{a_{s_0}(s)a_{s_0}(-s)} = \frac{\tau(s)\tau(-s)}{\alpha_{s_0}(s)\alpha_{s_0}(-s)}, \tag{5.7}$$

where $f_{s_0}(s)$, $a_{s_0}(s)$ and $\alpha_{s_0}(s)$ are as defined in Corollary 5 with an subscript added to denote the dependence on s_0.

We begin by showing that $a_{s_0}(s)$ tends to a limit $a(s)$ as $s_0 \to \infty$ pointwise, except at the poles of $a(s)$, and that consequently $\alpha_{s_0}(s)$ tends to a limit $\alpha(s)$. To this end, we take the inverse of the Pick matrix (4.5) to obtain

$$P^{-1} = \frac{\operatorname{adj} P}{\det P} = \frac{\operatorname{diag}(\det \widetilde{P}, 0, \dots, 0) + \mathrm{O}(s_0^{-1})}{\frac{w_0}{s_0}\det \widetilde{P} + \mathrm{O}(s_0^{-2})} = \frac{s_0}{w_0}\operatorname{diag}(1, 0, \dots, 0) + \mathrm{O}(1),$$

from which it follows that

$$\lim_{s_0 \to \infty} \frac{\Pi(s_0)P^{-1}\Pi(s_0)^*}{2s_0} = \frac{1}{2w_0}.$$

Moreover, since

$$\Pi(s) = \begin{bmatrix} 1 & h^\mathsf{T} \end{bmatrix} \begin{bmatrix} \frac{s+s_0}{s+s_0} & & & \\ & \frac{s+s_0}{s+s_1} & & \\ & & \ddots & \\ & & & \frac{s+s_0}{s+s_k} \end{bmatrix},$$

we have

$$\frac{\Pi(s)P^{-1}\Pi(s_0)^*}{2s_0} = [1 \quad h^{\mathsf{T}}]\left(\left[\frac{(s+s_\ell)(w_j+\bar{w}_\ell)(s_0+\bar{s}_j)}{(s+s_0)(s_j+\bar{s}_\ell)}\right]^k_{j,\ell=0}\right)^{-1}\begin{bmatrix}1\\h\end{bmatrix}.$$

Therefore taking the limit, we obtain, with $\tilde{w} := (w_1+w_0, w_2+w_0, \ldots, w_k+w_0)^{\mathsf{T}}$,

$$\begin{aligned}\lim_{s_0\to\infty}\frac{\Pi(s)P^{-1}\Pi(s_0)^*}{2s_0} &= [1 \quad h^{\mathsf{T}}]\begin{bmatrix}2w_0 & 0\\ \tilde{w} & \widetilde{P}(sI+\Lambda)\end{bmatrix}^{-1}\begin{bmatrix}1\\h\end{bmatrix}\\ &= [1 \quad h^{\mathsf{T}}]\begin{bmatrix}(2w_0)^{-1} & 0\\ -(2w_0)^{-1}(sI+\Lambda)^{-1}\widetilde{P}^{-1}\tilde{w} & (sI+\Lambda)^{-1}\widetilde{P}^{-1}\end{bmatrix}\begin{bmatrix}1\\h\end{bmatrix}\\ &= \frac{1}{2w_0}[1-h^{\mathsf{T}}(sI+\Lambda)^{-1}\widetilde{P}^{-1}w].\end{aligned}$$

Transposing this, it follows from Corollary 5 that

$$a(s) = \lim_{s_0\to\infty} a_{s_0}(s) = \frac{1}{\sqrt{2w_0}}[1-\hat{C}(sI+\Lambda)^{-1}h] = \frac{\alpha(s)}{\tau(s)}, \tag{5.8}$$

where $\hat{C}$ is given by (5.5) and $\alpha(s) = \lim_{s_0\to\infty}\alpha_{s_0}(s)$.

For each s_0, all the roots of $\alpha_{s_0}(s)$ lie in the open left half plane. We want to show that the same is true for the limit polynomial $\alpha(s)$; i.e., that $\alpha(s)$ has no root on the imaginary axis. To this end, first observe that, in view of (4.22),

$$\alpha_{s_0}(s)\beta_{s_0}(-s)+\alpha_{s_0}(-s)\beta_{s_0}(s) = \tau(s)\tau(-s)$$

has no zeros on the imaginary axis, for $\tau(s)$ has none. Therefore, by continuity, nor does

$$\alpha(s)\beta_{s_0}(-s)+\alpha(-s)\beta_{s_0}(s)$$

for sufficiently large s_0. However, then $\alpha(s)$ cannot have a root on the imaginary axis, because, since $\alpha(s)$ is real, $\alpha(-\mathrm{i}\omega)=0$ whenever $\alpha(\mathrm{i}\omega)=0$. Consequently, the limit polynomial $\alpha(s)$ is a Hurwitz polynomial, as claimed. Consequently, given this $\alpha(s)$, the corresponding equation (4.22) has a unique solution $\beta(s)$, which can be seen to be Hurwitz in the same way. Moreover, f_{s_0} tends to $f(s) := \beta(s)/\alpha(s)$ pointwise as $s_0\to\infty$ except in the poles of f. Since $f_{s_0}(s_j) = w_j$, $j = 1, 2, \ldots, k$, for each s_0, then $f(s_j) = w_j$, $j = 1, 2, \ldots, k$. Moreover, $f(\infty) = D$.

Next we demonstrate that

$$f(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B}+D, \tag{5.9}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ are given by (5.3)–(5.5). To this end, note that f has all its poles and zeros in the open left half plane and

$$f(s)+f(-s) = \frac{1}{a(s)a(-s)}.$$

Hence f is positive real. From (5.8) we have

$$\frac{1}{a(s)} = \sqrt{2w_0}[1+\hat{C}(sI-\hat{A})^{-1}h], \tag{5.10}$$

and consequently (5.9) follows from the Positive-Real-Lemma equations [2,20]. Indeed, since $\alpha(s)$ is a Hurwitz polynomial, $\hat{A}$ has all its eigenvalues in the open left half plane.

It remains to prove that the limit interpolant f is indeed equal to the Antoulas–Sorensen solution $\hat{G}$. The rational functions f and $\hat{G}$ satisfy the same interpolation conditions. More specifically,

$$(\hat{G} - f)(\infty) = 0, \quad (\hat{G} - f)(s_j) = 0, \quad j = 1, 2, \ldots, k; \tag{5.11}$$

i.e., $\hat{G} - f$ has the $k + 1$ zeros $s_1, \ldots, s_k$ and ∞. Moreover, the spectral densities $\hat{G}(s) + \hat{G}(-s)$ and $f(s) + f(-s)$ can have zeros only at the spectral zeros $\{\pm s_1, \pm s_2, \ldots, \pm s_k\}$.

Clearly, $m := \deg f \leqslant k$ and $\hat{m} := \deg \hat{G} \leqslant k$. To show that $f \equiv \hat{G}$, we use the fact that any rational function of degree $\leqslant \ell$ with more than ℓ zeros must be identically equal to zero. Consider two cases. First, suppose that $\deg \hat{G} + \deg f \leqslant k$. Then, in view of (5.11), $\hat{G} - f$ is identically zero. Next, suppose that $\deg \hat{G} + \deg f = \hat{m} + m = k + p > k$. Since $f(s) = -f(-s)$ for m points in $\{s_j\}_{j=1}^k$, and $\hat{G}(s) = -\hat{G}(-s)$ for $\hat{m}$ points in $\{s_j\}_{j=1}^k$, there are p points in $\{s_j\}_{j=1}^k$ for which $f(-s) = \hat{G}(-s)$. Therefore, $f - \hat{G}$ has $k + 1 + p$ zeros. But, since it is only of degree $k + p$, $\hat{G} \equiv f$. $\square$

Example 7. Consider a positive real function

$$G(s) = \frac{1/3s + 1}{(s+1)(s+2)} + 1$$

with stable spectral zeros

$$\lambda_1 = -\sqrt{3}, \quad \lambda_2 = -\sqrt{2}.$$

This is the transfer function of second-order passive system with a minimal realization

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 1/3], \quad D = 1.$$

Applying the Antoulas–Sorensen method to approximate this system by a first-order passive system having a transfer function $\hat{G}$ with spectral zeros $\pm\sqrt{3}$ yields

$$\hat{G}(s) = \frac{2s + 4}{2s + 3}, \tag{5.12}$$

which clearly satisfies the interpolation conditions

$$\hat{G}(\infty) = G(\infty), \quad \hat{G}(\pm\sqrt{3}) = G(\pm\sqrt{3}).$$

On the other hand, the maximum entropy solution f_{ME} with interpolation conditions

$$f_{\mathrm{ME}}(\infty) = G(\infty), \quad f_{\mathrm{ME}}(\sqrt{3}) = G(\sqrt{3})$$

can be determined as in Theorem 6. In fact, $w_0 = 1$, $w = (2\sqrt{3} - 3)/3$, $\widetilde{P} = 2/3$, and $\Lambda = \sqrt{3}$, and hence $\hat{C} = \sqrt{3} - 3/2$ and $\hat{A} = -3/2$. Solving the Lyapunov equation (5.6), we have $Q = 1/3$, and hence $\hat{B} = 2\sqrt{3}/3 + 1$. Inserting this into (5.9), we have

$$f_{\mathrm{ME}}(s) = \frac{2s + 4}{2s + 3}, \tag{5.13}$$

which is the same as (5.12).

Note that the purpose of this example is not to provide an alternative algorithm for the Antoulas–Sorensen method but to illustrate Theorem 6.

6. Tuning both interpolation points and spectral zeros

We have thus established that the reduced-order model computed by the Antoulas–Sorensen method coincides (in the limit) with the central (maximum-entropy) Nevanlinna-Pick interpolant with interpolation points in the mirror-image of the selected spectral zeros. However, the central solution is quite special, and an important question is whether better approximants can be obtained by using the full power of the theory of analytic interpolation with degree constraints.

Let us first stress that we do not need to strictly enforce the interpolation condition $f(\infty) = D$. For all practical purposes we may exchange this condition with $f(s_0) = D$, where s_0 is a very large real number. For the bounded real case and for strictly proper interpolants, we have an alternative strategy to avoid interpolation at infinity, as we shall discuss in detail in Section 7.

In the two examples presented below, we have used the solver proposed in [6]. Although the code can be rewritten for the continuous-time case studied in this paper, the formulation in [6] is for the discrete-time case, and hence the appropriate linear fractional transformation has been applied.

A key point in applying this algorithm for tuning is the appropriate choice of spectral zeros and interpolation points. In the examples below we have used the following criteria. The spectral zeros are placed at frequencies where the original spectrum has valleys, the deeper the valley the closer to the imaginary axis. The interpolation points are selected close to the imaginary axis in regions where god fit is required. The closer interpolation points are to the imaginary axis, the more accurate is the fit, but it is also more localized to a smaller part of the spectrum.

All of this requires manual tuning of the type characteristic for engineering practice. An alternative way would be to use the procedure introduced in [29]. Instead of contractive functions, considered there, we will here describe briefly the analogous procedure for positive real functions. Given a (rational) function f, the idea is to find a function Ψ such that f is precisely the unique minimizer of

$$\mathbb{I}_\Psi(g) \quad \text{subject to } g(s_j) = w_j, \quad j = 0, 1, \ldots, k,$$

where $w_j = f(s_j)$ are values of the given function. We refer to this as the inverse problem of analytic interpolation. Such a function Ψ exists, even if $\deg f > n$, but in general it is not of the form

$$\left\{ \Psi \,\middle|\, \Psi(\mathrm{i}\omega) = \left| \frac{\sigma(\mathrm{i}\omega)}{\tau(\mathrm{i}\omega)} \right|^2, \quad \text{where } \deg \sigma = k, \quad \tau(s) = \prod_{j=1}^{k} (s + s_j) \right\}. \tag{6.1}$$

Instead, Ψ is in general a rational function of higher order. Given such a Ψ, which can be computed using the theory developed in [26], the spectral zeros of the reduced order system can be selected as the zeros of $\hat{\Psi}$, where $\hat{\Psi}$ is the function in the class (6.1) closest to Ψ. More precisely, $\hat{\Psi}$ is the minimizer in the class (6.1) of

$$\left\| 1 - \frac{\hat{\Psi}(\mathrm{i}\omega)}{\Psi(\mathrm{i}\omega)} \right\|_\infty .$$

This can be written as a convex problem with infinitely many linear constraints and can be solved by method of convex optimization; see, e.g., [7]. The model-reduced interpolant is then taken to be the unique minimizer $\hat{f}$ of

$$\mathbb{I}_{\hat{\Psi}}(g) \quad \text{subject to } g(s_j) = w_j, \quad j = 0, 1, \ldots, k.$$

Even though this is a systematic procedure for selecting spectral zeros, there is presently no quantitative bounds in the positive real case on the error between the original system and the degree reduced system. This is a subject for further research.

Example 8. Consider the benchmark problem to approximate

$$G(s) = \frac{s^5 + 3s^4 + 6s^3 + 9s^2 + 7s + 3}{s^5 + 7s^4 + 14s^3 + 21s^2 + 23s + 7} \tag{6.2}$$

taken from [3, p. 359] and [38]. The stable spectral zeros of G are

$$\lambda_1 = -1.8355, \quad \lambda_2 = -1.3018, \quad \lambda_3 = -0.7943, \quad \lambda_{4,5} = -1.833 \pm 1.5430\mathrm{i}.$$

Choosing the interpolation points $s_1 = -\lambda_1$, $s_2 = -\lambda_2$ and $s_3 = -\lambda_3$, the Antoulas–Sorensen method yields a third-order passive system with transfer function

$$\hat{G}(s) = \frac{s^3 + 2.553s^2 + 2.906s + 1.173}{s^3 + 6.681s^2 + 8.459s + 3.07}, \tag{6.3}$$

which is positive real and has spectral zeros also in $\lambda_1, \lambda_2, \lambda_3$.

Next, using the freedom in the choice of the interpolation points and spectral zeros, offered by the theory of analytic interpolation with degree constraint, a system of degree three is computed. We specify interpolation points at[1]

$$s_0 \approx \infty, \quad s_{1,2} = 0.2038 \pm 0.9029i, \quad s_3 = 0.1010,$$

and the corresponding interpolation values at

$$w_j = G(s_j), \quad j = 0, 1, 2, 3.$$

Then, choosing three stable spectral zeros at

$$\lambda_{1,2} = -0.4150 \pm 0.4596\mathrm{i}, \quad \lambda_3 = -3,$$

we obtain the positive real function

$$f(s) = \frac{1.002s^3 + 2.84s^2 + 1.927s + 0.8978}{s^3 + 7.298s^2 + 6.084s + 2.099}. \tag{6.4}$$

In Fig. 6.1, the singular value of the original system (6.2) of degree five is plotted together with the third-order systems with transfer functions (6.3) and (6.4), respectively. For comparison, we also include the corresponding third-order model obtained by stochastically balanced truncation. One can see that the reduced-order model computed by the Antoulas–Sorensen method is close to the original one at high frequencies. Instead, by an appropriate choice of the interpolation points and the spectral zeros, the corresponding system (6.4) matches the original one quite well at both low and high frequencies. The DC gain matching is due to the interpolation point s_3 which is close to the origin, while the matching at high frequencies is guaranteed by the interpolation condition close to infinity (s_0). The modeling of the "valley" has been done by placing the spectral zeros $\lambda_{1,2}$ appropriately. Stochastically balanced truncation gives a good approximation for high frequencies but not for low frequencies.

7. A large-scale numerical example: CD player model

Next, using a high-order model of a portable CD player taken from [39], we will compare reduced-order models computed by the Sorensen algorithm and analytic interpolation with degree constraint. The full-order model, the transfer function of which is denoted by F, is a single-input

[1] The point s_0 is approximated by a large number to avoid a boundary interpolation condition.

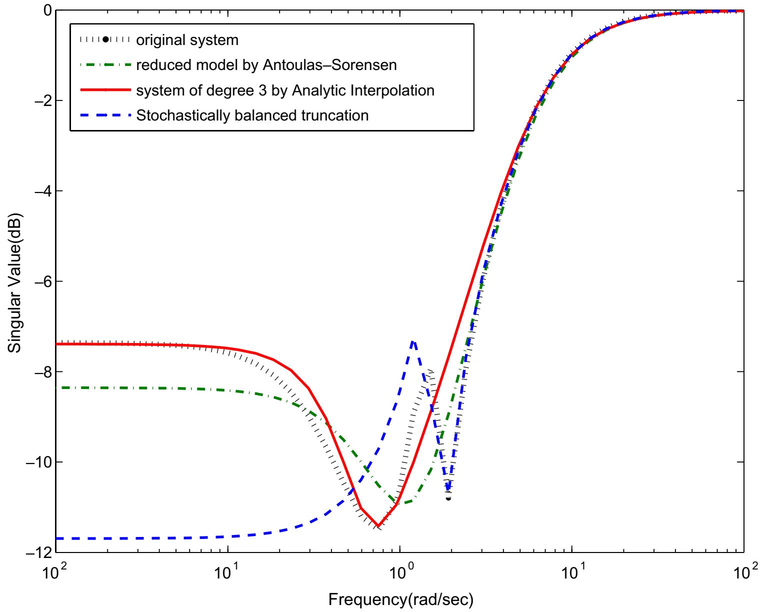

Fig. 6.1. Singular values of the frequency response of the original system of degree five, together with the third-order systems computed with the Antoulas–Sorensen method, analytic interpolation with degree constraint, and stochastically balanced truncation.

single-output, continuous-time, linear time-invariant, and stable model of order 120 and of relative degree two. We aim at reducing the model order to $k = 12$, as well as maintaining the stability and the uniform upper bound $\rho = 80$ ($\approx$ 38dB) of the gain of F.

The model F can be regarded as a function which is analytic in the right half-plane $\mathbb{C}_+$ and maps $\mathbb{C}_+$ into $\rho\mathbb{D}$. In other words, F is a bounded-real function in continuous-time. To be consistent with the problem setting in this paper, we will introduce a positive real function G in continuous-time by a bilinear transformation:

$$G(s) := \frac{\rho - F(s)}{\rho + F(s)}.$$

After computing the positive real reduced-order model $\hat{G}$ of G, we will obtain the bounded real reduced-order model $\hat{F}$ of F by the inverse transformation

$$\hat{F}(s) := \rho \frac{1 - \hat{G}(s)}{1 + \hat{G}(s)}.$$

Model reduction by the Antoulas–Sorensen method

We use the algorithm suggested by Sorensen in [38]. This algorithm is based on the Implicitly Restart Arnoldi (IRA) method [37], and computes automatically the reduced-order system $\hat{G}$ without an explicit computation of all the spectral zeros of G. Following Sorensen, we select $k := 12$ spectral zeros of the system G by determining k eigenvalues of the matrix

$$\mathscr{C}_\mu := (\mu\mathscr{E} - \mathscr{A})^{-1}(\mu\mathscr{E} + \mathscr{A}), \tag{7.1}$$

where $\mu \geqslant 0$ has to be chosen properly, and $\mathscr{A}$ and $\mathscr{E}$ are defined by (2.1) in terms of a minimal realization of the 120th degree positive-real function G.

These eigenvalues can be computed with the Matlab command *eigs*, which implements the IRA method, and allows us to select k eigenvalues in various ways; for example, with

- largest or smallest magnitude,
- largest or smallest real part,
- largest or smallest complex part.

Using this algorithm, we have freedom in choosing μ and the criteria of selecting the eigenvalues of (7.1). In this example, we have tried four different scenarios:

(1) $\mu = 260$ and 12 eigenvalues of largest magnitude; see Fig. 7.1.
(2) $\mu = 20$ and 12 eigenvalues of largest magnitude; see Fig. 7.2.
(3) $\mu = 260$ and 6 eigenvalues of largest magnitude and 6 of smallest magnitude; see Fig. 7.3.
(4) $\mu = 20$ and 6 eigenvalues of largest magnitude and 6 of smallest magnitude; see Fig. 7.4.

In Figs. 7.1–7.4, the frequency response of the original bounded real system F is compared with that of the reduced-order systems $\hat{F}$ corresponding to the four scenarios. As these figures show, the frequency responses of the reduced-order models match the original model only in some frequency bands. This is due to the restriction that interpolation points may only be placed at the mirror images of the spectral zeros. Moreover, the solution depends crucially on the choice of μ and on the eigenvalues selection criteria, choices that are highly nontrival as pointed out in [38].

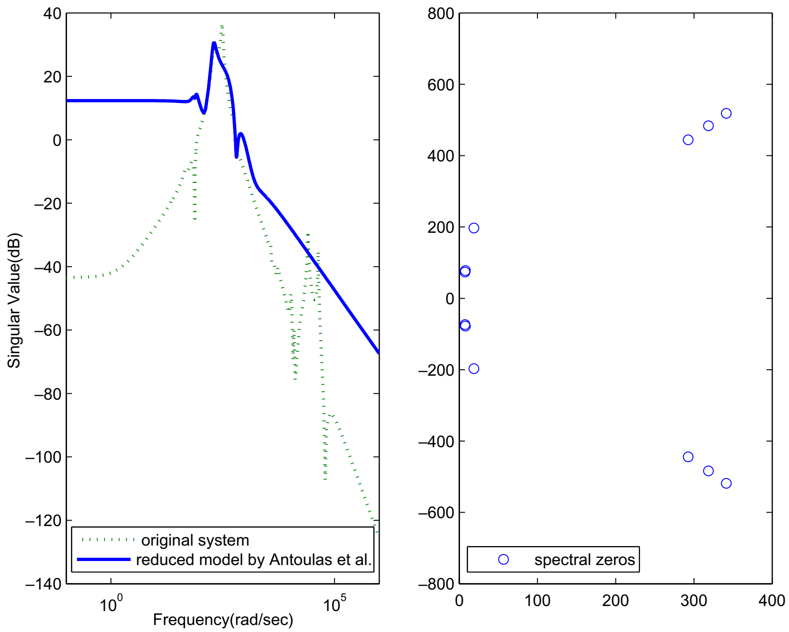

Fig. 7.1. Scenario 1.

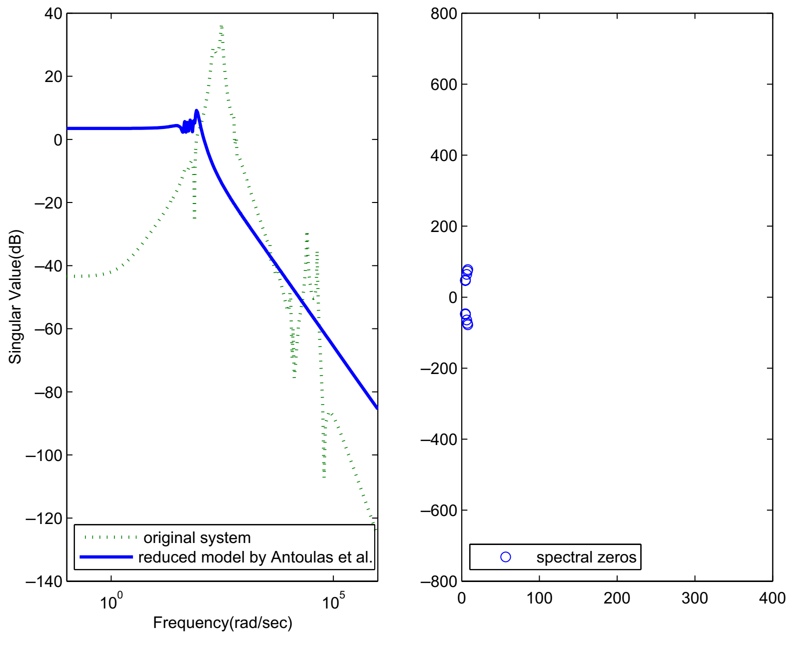

Fig. 7.2. Scenario 2.

Model reduction by analytic interpolation with degree constraint

Applying the theory of Section 4, we may choose the twelve spectral zeros and interpolation points arbitrarily. For comparison with the Antoulas–Sorensen method, which requires an interpolation condition $F(\infty) = 0$, we impose the same interpolation condition on the reduced-order system $\hat{F}$. Since F has relative degree two, it can be factored as $F = F_1 F_2$, where F_2 is of relative degree two and F_1 is of degree 118. In the present example,

$$F_2 = \frac{1}{(s-p)(s-\bar{p})}, \quad p = -12.2708 + 306.5398i,$$

where p is a pole of F close to the frequency peak $\omega = 300$ rad/s. Hence we can restate the problem of reducing the order of F to $k = 12$ as the problem of determining a 10th order approximant $\hat{F}_1$ of F_1 and setting $\hat{F} := \hat{F}_1 F_2$.

To reduce the order of F_1 with the methods of Section 4, we need 11 interpolation conditions

$$\hat{F}_1(s_j) = \frac{F(s_j)}{F_2(s_j)} \quad j = 0, \ldots, 10,$$

which we choose at the points

$$s_0 = 199, \quad s_{1,2} = 10^{-5} \pm 0.1\mathrm{i}, \quad s_{3,4} = 0.2 \pm \mathrm{i}, \quad s_{5,6} = 0.01 \pm 74\mathrm{i},$$
$$s_{7,8} = 0.01 \pm 1.3250 \cdot 10^4 i, \quad s_{9,10} = 0.005 \pm 9.9900 \cdot 10^4 \mathrm{i};$$

and a uniform upper bound on the gain, which we take to be $\rho_1 = 3.9811 \cdot 10^7 (\approx 140$ dB). In the family of all such interpolants $\hat{F}_1$, we select the one that has spectral zeros at

$$\lambda_{1,2} = -0.0612 \pm 2.3749i, \quad \lambda_3 = -175.0542, \quad \lambda_4 = -351.5899,$$

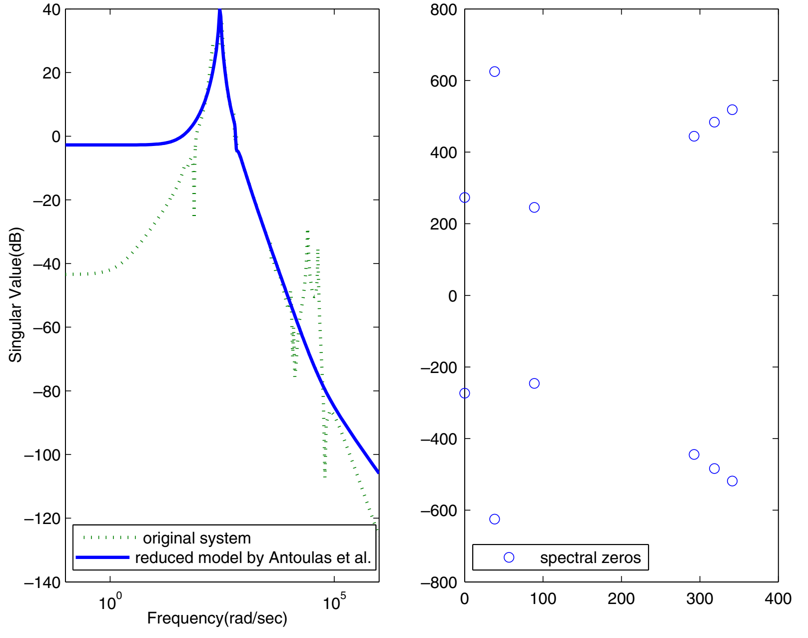

Fig. 7.3. Scenario 3.

$$\lambda_{5,6} = -7.4080 \pm 70.4606i, \quad \lambda_{7,8} = -570.1525 \pm 2.5448 \cdot 10^4 i,$$
$$\lambda_{9,10} = -1.6436 \cdot 10^4 \pm 7.5527 \cdot 10^3 i.$$

In Fig. 7.5, the interpolation points and the spectral zeros of $\hat{F}$ are plotted together with the reduced-order system $\hat{F}$ and the original system F. One can see that $\hat{F}$ matches the peak around $\omega = 300$ rad/s and that it also matches the original system quite well at high and low frequencies. Moreover, it matches the ripple around $\omega = 10^4$ rad/s.

Remark 9. The factorization of the bounded real system F (positive real system G) and the model reduction of the bounded real system F_1 (positive real system G_1) is related to another entropy functional, introduced in [27] in the discrete-time setting, namely

$$\int_{-\infty}^{\infty} \Psi(\mathrm{i}\omega) \log \left(1 - \left| \frac{\hat{F}(\mathrm{i}\omega)}{F_2(\mathrm{i}\omega)} \right|^2 \right) \frac{\mathrm{d}\omega}{s_0^2 + \omega^2},$$

which forces the interpolant to satisfy $|\hat{F}| \leqslant |F_2|$ on $\mathrm{i}\mathbb{R}$ and is useful for imposing bounds on the interpolant. Here we have used it for placing zeros on the unit circle.

Combining the methods

Clearly, for reduced-order models of relatively large order, the appropriate choice of the interpolation points and spectral zeros becomes more intricate. To take advantage of the strengths of the different methods, a possible avenue could be a two-step procedure by which the large-order system is first reduced by the numerical reliable Sorensen algorithm, described in Section 3, and

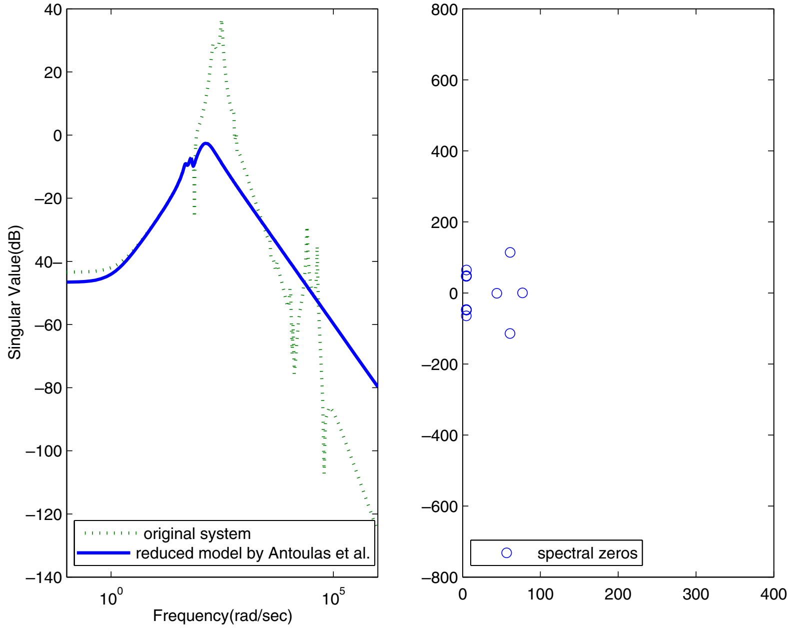

Fig. 7.4. Scenario 4.

then fine-tuned by the more flexible method of Section 4. In the last step, interpolation points $\mathbf{s} := \{s_1, s_2, \ldots, s_k\}$ and spectral zeros $\boldsymbol{\lambda} := \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ could be chosen so as to minimize

$$\min_{\boldsymbol{\lambda},\mathbf{s}} \|G - f(\boldsymbol{\lambda}, \mathbf{s})\|, \tag{7.2}$$

in line with [19,35], or

$$\min_{\boldsymbol{\lambda},\mathbf{s}} \|G + G^* - f(\boldsymbol{\lambda}, \mathbf{s}) - f^*(\boldsymbol{\lambda}, \mathbf{s})\|, \tag{7.3}$$

where $G^*(s) := G(-s)$, and $\| \cdot \|$ denotes some suitable norm. Such minimization problems are in general nonconvex. However, local minimizers may be found numerically starting at the interpolation points and the spectral zeros obtained in the first step.

We believe that this procedure can give reasonable results in most examples and that it provides insight into the question posed by Sorensen in [38], namely what is the best choice of interpolation points. This is the topic of a future study.

8. Conclusions

Over the last decades a quite complete and comprehensive theory of analytic interpolation with degree constraint has been developed. Given a set of $n + 1$ interpolation points and $n + 1$ interpolation values, it provides a complete smooth parameterization of all positive-real interpolants of degree at most n in terms of spectral zeros, as well as a pair of dual convex optimization problems for determining any such interpolant. In particular, if the spectral zeros are chosen in the mirror image of the interpolation points, the problem is linear. The corresponding solution is the central (maximum-entropy) solution.

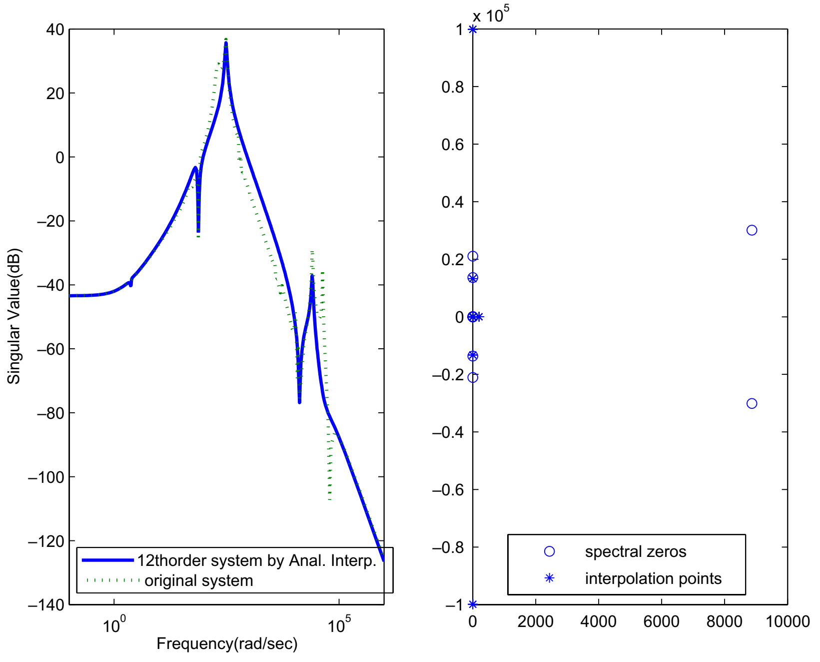

Fig. 7.5. Left: singular values of the frequency response of the original system F and the reduced-order system $\hat{F}$. Right: spectral zeros and the inter-polation points.

We have demonstrated that the passivity-preserving model reduction method proposed by Antoulas and Sorensen can be identified with the central solution. By applying the theory of analytic interpolation with degree constraint, we have demonstrated that it is possible to obtain better approximants by choosing interpolation points that are placed more strategically; i.e., not restricted to the mirror image of the spectral zeros.

It should be noted, however, that the implementation of the central solution provided by Sorensen's algorithm, in which spectral zeros do not have to be determined explicitly, is numerically very efficient, and therefore, for very large problems, preferable even to stochastically balanced truncation, for which there are H^∞ bounds.

Except for the central solution, determining positive-real interpolants with degree constant is a nonlinear problem, and numerically quite a bit more demanding. Therefore, we propose that the high-order model first be reduced by the Sorensen algorithm, and then fine-tuned by moving the interpolation points and spectral zeros to improve the approximation.

In the multi-variable case, the Sorensen algorithm interpolates tangentially in the mirror images of the selected spectral zeros. A generalization of the theory of analytic interpolation with degree constraint to this case is presented in [30] and could be applied to this situation.

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