

Multi-marginal optimal transport for state estimation and information fusion



Johan Karlsson

Optimization and Systems Theory
KTH Royal Institute of Technology

Based on joint work with:

**Yongxin Chen, Filip Elvander, Isabel Haasler
Andreas Jakobsson, Axel Ringh**

Acknowledgements:

Swedish Science Council, SSF, CIAM, ACCESS

Conference on Decision and Control, Nice, December 10, 2019

- 1 Optimal transport for state-estimation of multi-agent systems
 - Optimal transport with underlying dynamics
 - Partial information from multi-agent systems and measures
 - Optimal mass transport formulations for estimation and information fusion
 - State estimation examples

- 2 Regularized multi-marginal optimal transport for sensor fusion
 - Multimarginal optimal transport and Sinkhorn iterations
 - Efficient computations of projections and generalized Sinkhorn iterations
 - Localization with partially calibrated sensors

Optimal mass transport with dynamics

Compare sets of agent:

$$\{x_k^0\}_{k=1}^n \sim \mu_0 = \sum_{k=1}^n \delta_{x_k^0}$$

$$\{x_k^1\}_{k=1}^n \sim \mu_1 = \sum_{k=1}^n \delta_{x_k^1}$$

Define transport cost

$$T(\mu_0, \mu_1) = \min_{\phi \in \text{perm}(\{1, \dots, n\})} \sum_{k=1}^n c(x_k^0, x_{\phi(k)}^1)$$

$$= \min_{M \geq 0} \langle C, M \rangle$$

$$\text{s.t. } M\mathbf{1} = \mathbf{1}, M^T\mathbf{1} = \mathbf{1},$$

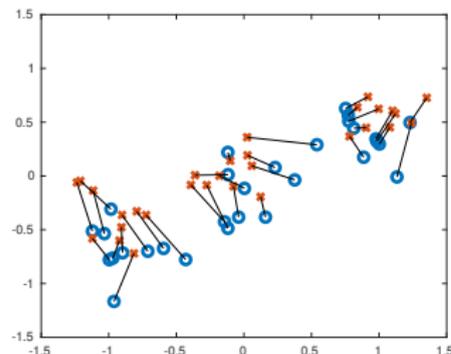


Figure: The black lines illustrate optimal optimal transport plan for cost $c(x_0, x_1) = \|x_0 - x_1\|^2$.

Optimal mass transport with dynamics

Compare sets of agent:

$$\{x_k^0\}_{k=1}^n \sim \mu_0 = \sum_{k=1}^n \delta_{x_k^0}$$

$$\{x_k^1\}_{k=1}^n \sim \mu_1 = \sum_{k=1}^n \delta_{x_k^1}$$

Define transport cost

$$T(\mu_0, \mu_1) = \min_{\phi \in \text{perm}(\{1, \dots, n\})} \sum_{k=1}^n c(x_k^0, x_{\phi(k)}^1)$$

$$= \min_{M \geq 0} \langle C, M \rangle$$

$$\text{s.t. } M\mathbf{1} = \mathbf{1}, M^T\mathbf{1} = \mathbf{1},$$

For quadratic cost function $C_{k,\ell} = c(x_k^0, x_\ell^1) = \|x_k^0 - x_\ell^1\|_2^2$.

Interpretation: each agent cost correspond to the optimal control problem

$$c(x_0, x_1) = \min_u \int_0^1 \|u(t)\|^2 dt$$

subject to $\dot{x}(t) = u(t), x(0) = x_0$ and $x(1) = x_1$

where x_0 is initial position and x_1 is the final position.

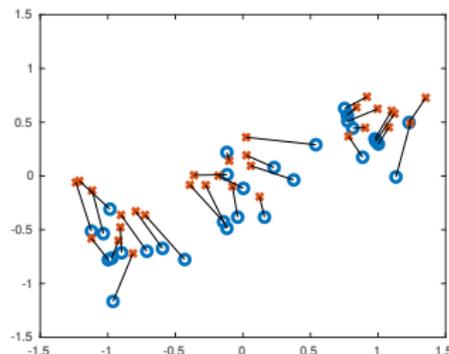


Figure: The black lines illustrate optimal optimal transport plan for cost $c(x_0, x_1) = \|x_0 - x_1\|_2^2$.

- Generalization to linear dynamics: $\dot{x}(t) = Ax(t) + Bu(t)$.
- Optimal control problem for each agent:

$$\begin{aligned} \min_u \quad & \int_0^1 \|u(t)\|^2 dt \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t), \\ & x(0) = x_0 \text{ and } x(1) = x_1. \end{aligned}$$

- The cost is then given by

$$\begin{aligned} c_{A,B}(x_0, x_1) &= (x_1 - \Phi x_0)^T M_{10}^{-1} (x_1 - \Phi x_0), \text{ where} \\ \Phi &= e^A, M_{10} = \int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)} d\tau. \end{aligned}$$

Y. Chen, T.T. Georgiou, and M.Pavon. Optimal Transport Over a Linear Dynamical System. IEEE TAC, 2017.

Optimal mass transport with dynamics

Insight: Sufficient to change the cost to account for agent dynamics

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Corresponding optimal transport formulations:

$$\begin{aligned} T_{A,B}(\mu_0, \mu_1) &= \min_{u_k, \phi \in \text{perm}(\{1, \dots, n\})} \sum_{k=1}^n \int_0^1 \|u_k(t)\|^2 dt \\ &\text{subject to } \dot{x}_k(t) = Ax_k(t) + Bu_k(t), \\ &x_k(0) = x_k \text{ and } x_k(1) = y_{\phi(k)}, \end{aligned}$$

$$\begin{aligned} &= \min_{M \in \mathcal{M}_+(X^2)} \int_{X^2} c_{A,B}(x_0, x_1) M(x_0, x_1) dx_0 dx_1 \\ &\text{subject to } \int_{x_1 \in X} M(x_0, x_1) dx_0 = \mu_0(x_0) \\ &\int_{x_0 \in X} M(x_0, x_1) dx_1 = \mu_1(x_1) \end{aligned}$$

where the cost is then given by

$$c_{A,B}(x_0, x_1) = (x_1 - \Phi x_0)^T M_{10}^{-1} (x_1 - \Phi x_0), \text{ where}$$

$$\Phi = e^A, M_{10} = \int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)} d\tau.$$

Partial information

Observation models

- Standard observations in state space: $C : \mathcal{X} \rightarrow \mathcal{Y}$

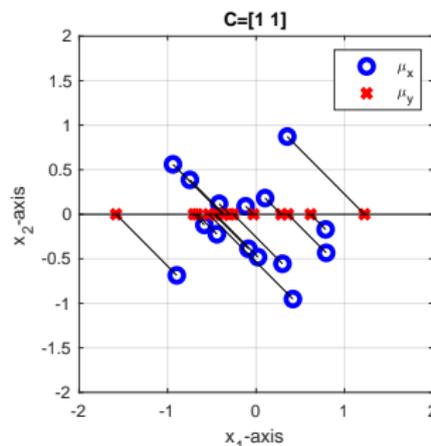
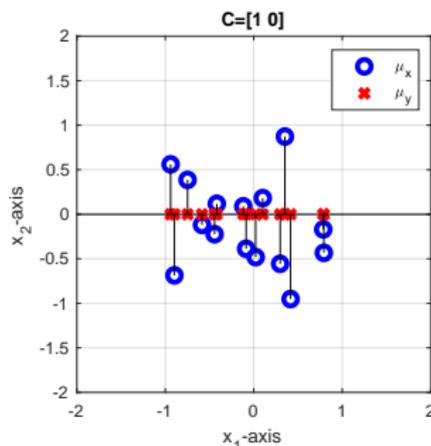
$$y = Cx$$

hold for each agent.

- Corresponding observations for measures $\mu_y = C_{\#}\mu_x$:

$$\mu_y(U) = \mu_x(C^{-1}(U)) \quad \text{for all subsets } U \in \mathcal{Y}.$$

- Projection according to $C = [1 \ 0] : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $C = [1 \ 1] : \mathbb{R}^2 \rightarrow \mathbb{R}$:



Information obtained from (other) integral operator:

- Fourier operator (DOA, Spectral estimation)

$$\Gamma(\mu) := \int_{-\pi}^{\pi} \begin{pmatrix} 1 \\ e^{-i\theta} \\ \vdots \\ e^{-in\theta} \end{pmatrix} d\mu(\theta)$$

- Radar localization

$$\Gamma(\mu) = \int_X a(x)\mu(x)a(x)^H dx.$$

where a is steering vector.

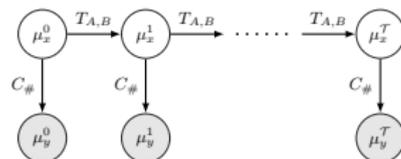
- Radon transform (CT-imaging, radar imaging), blurring, etc.

State tracking and information fusion for multi-agent systems

Optimization problems based on optimal transport

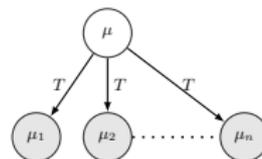
- State estimation:

$$\begin{aligned} \min_{\mu_t^x} \quad & \sum_{t=1}^{\mathcal{T}} T(\mu_{t-1}^x, \mu_t^x) \\ \text{subject to} \quad & \mu_t^y = C_{\#} \mu_t^x, \quad t = 0, 1, \dots, \mathcal{T} \end{aligned}$$



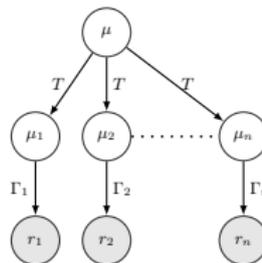
- Barycenter (centroid):

$$\min_{\mu} \sum_{k=1}^n T(\mu, \mu_k)$$



- Barycenter with partial information

$$\begin{aligned} \min_{\mu, \mu_k, k=1, \dots, n} \quad & \sum_{k=1}^n T(\mu, \mu_k) + \lambda \sum_{k=1}^n \|\Delta_t\|^2 \\ \text{subject to} \quad & \Gamma_k(\mu) = r_k + \Delta_k \end{aligned}$$



State tracking of multi-agent systems

Problem formulation

Assume that the underlying dynamics and output measurements corresponds to the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1a)$$

$$y(t) = Cx(t) \quad (1b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$ and (A, B) is a controllable pair.

Problem: track the distribution $\hat{\mu}_t$, where each particle abides by (1a), based on the output distributions of (1b) at the times $k = 0, 1, \dots, T$.

Example:

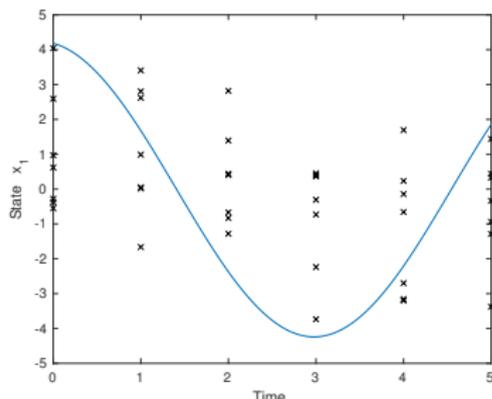
Consider the tracking of N systems with oscillatory dynamics

$$dx(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)dt + \sigma dw(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

where dw is normalized white Gaussian noise. Note that only the distribution of the output is available.

Identifiability of this problem studied in (Zeng, Waldherr, Ebenbauer, Allgöwer, 2016).



State tracking of multi-agent systems

Example

$$\min_{\mu_t^{(x)}} \sum_{t=1}^{\mathcal{T}} T_{A,B}(\mu_{t-1}^{(x)}, \mu_t^{(x)})$$

$$\text{subject to } \mu_t^{(y)} = C_{\#} \mu_t^{(x)}, \quad t = 0, 1, \dots, \mathcal{T}.$$

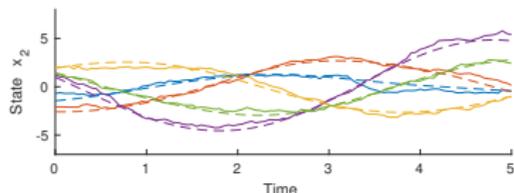
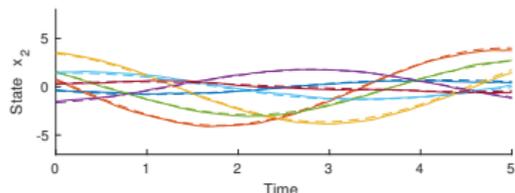
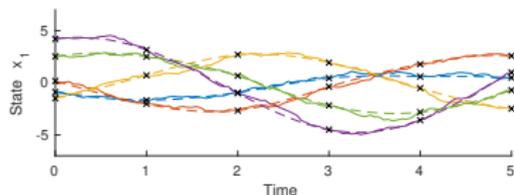
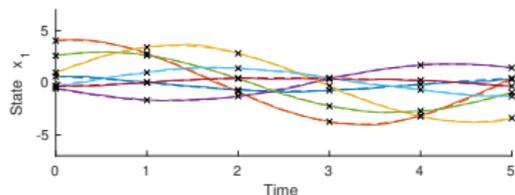


Figure: Example with $N = 7$ systems to be tracked. Noise level: $\sigma = 0.1$.

Figure: Example with $N = 5$ systems to be tracked. Noise level: $\sigma = 0.5$.

Available measurement points (x).

True system states (solid). Estimated system states (dashed).

Complex-valued, zero mean, discrete-time stochastic process, $y(t)$, for $t \in \mathbb{Z}$.

If the signal is wide sense stationary (WSS), we may define covariance

$$r_k \triangleq E(y(t)\overline{y(t-k)})$$

being independent of t .

The measured covariances are the Fourier coefficients of the spectrum:

$$r_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(\theta) e^{-ik\theta} d\theta.$$

A signal is typically only stationary for short time intervals. To handle this, solve the problem¹

$$\begin{aligned} \min_{\mu_t \in \mathcal{M}_+(\mathbb{T})} \quad & \sum_{t=1}^{\mathcal{T}} T(\mu_{t-1}, \mu_t) \\ \text{subject to} \quad & \hat{r}_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu_t(\theta) e^{-ik\theta} d\theta \end{aligned}$$

where $\hat{r}_k(t)$ it the k :th covariance estimated in time interval t .

¹ Includes regularization.

Ground truth:
Frequency tracking. Two frequencies.

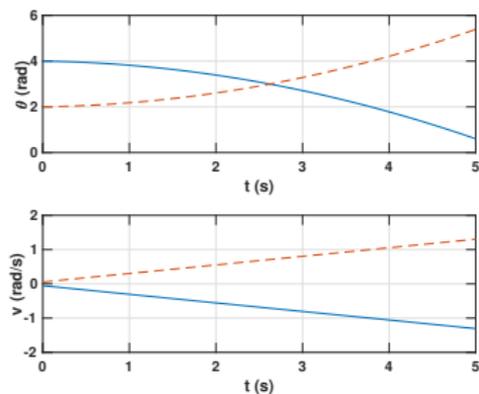


Figure: Top panel: the frequency locations.
Bottom panel: the velocities (chirp rate).

Estimates of the frequencies.

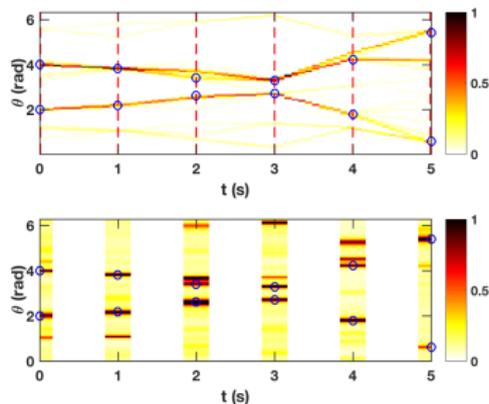


Figure: Top panel: reconstruction using standard OMT formulation. Bottom panel: estimates obtained using the Capon estimator.

Optimal mass transport

Tracking using dynamics

Ground truth:
Frequency tracking. Two frequencies.

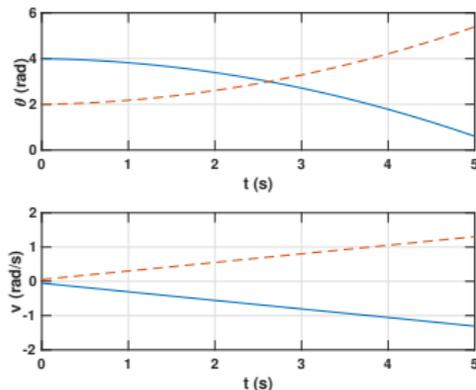


Figure: Top panel: the frequency locations.
Bottom panel: the velocities (chirp rate).

Estimates of the locations:
Dynamics in the OMT formulation.

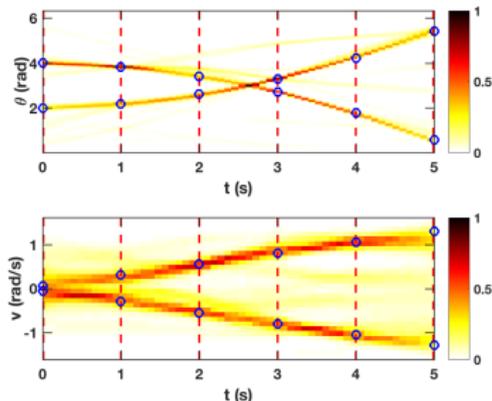


Figure: Top panel: reconstructed location spectrum. Bottom panel: reconstructed velocity spectrum.

$$\text{Dynamics : } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = [0 \quad 1]^T, C = [1 \quad 0],$$

State tracking of distributions

Summary

- Approach for state tracking of ensembles, applicable for discrete set agents/systems or general distributions.
- Convex problem, but computationally expensive: discretize state space in M points and solve optimization problem with M^2 variable (curse of dimensionality).
- Works ok for low dimensions: $n - m \lesssim 2$.
- Non-convex approach (similar to K-means) for similar problems (Zeng, Ishii, Allgöwer. 2017).
- What if system has higher dimension?
 - Method can be extended if assumptions are made on the distribution, e.g., gaussian distributions (see [L-CSS: Chen, Karlsson 2018]).
 - Use Sinkhorn and utilize problems structures

Multimarginal optimal mass transport

Discrete formulation

Generalization to optimal transport problem with several margins.

Cost and transport plan are tensors $\mathbf{C}, \mathbf{M} \in \mathbb{R}^{n^{T+1}}$

$$\min_{\mathbf{M} \in \mathbb{R}_+^{n^{T+1}}} \langle \mathbf{C}, \mathbf{M} \rangle$$

$$\text{subject to } P_t(\mathbf{M}) = \mu_t, \quad t = 0, \dots, T,$$

where

$$\langle \mathbf{C}, \mathbf{M} \rangle = \sum_{i_0, \dots, i_T} \mathbf{M}_{i_0, \dots, i_T} \mathbf{C}_{i_0, \dots, i_T},$$

$$P_t(\mathbf{M})_j = \sum_{\substack{i_0, \dots, i_{t-1}, \\ i_{t+1}, \dots, i_T}} \mathbf{M}_{i_0, \dots, i_{t-1}, j, i_{t+1}, \dots, i_T}, \quad \text{for } j = 1, \dots, n.$$

Multimarginal optimal mass transport

Discrete formulation

Generalization to optimal transport problem with several margins.

Cost and transport plan are tensors $\mathbf{C}, \mathbf{M} \in \mathbb{R}^{n^{T+1}}$

$$\min_{\mathbf{M} \in \mathbb{R}_+^{n^{T+1}}} \langle \mathbf{C}, \mathbf{M} \rangle$$

$$\text{subject to } P_t(\mathbf{M}) = \mu_t, \quad t = 0, \dots, T,$$

where

$$\langle \mathbf{C}, \mathbf{M} \rangle = \sum_{i_0, \dots, i_T} \mathbf{M}_{i_0, \dots, i_T} \mathbf{C}_{i_0, \dots, i_T},$$

$$P_t(\mathbf{M})_j = \sum_{\substack{i_0, \dots, i_{t-1}, \\ i_{t+1}, \dots, i_T}} \mathbf{M}_{i_0, \dots, i_{t-1}, j, i_{t+1}, \dots, i_T}, \quad \text{for } j = 1, \dots, n.$$

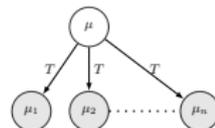
Barycenter problem:

\Leftrightarrow Multimarginal problem:

$$\min_{\mu} \sum_{k=1}^n T(\mu, \mu_k)$$

$$\min_{\mathbf{M} \in \mathcal{M}_+(\mathbf{X})} \left\langle \mathbf{M}, \left[\sum_{k=1}^n \mathbf{C}_{i_0, i_k} \right]_{i_0, \dots, i_T} \right\rangle$$

subject to $P_k(\mathbf{M}) = \mu_k$ for $k = 1, \dots, n$.



Discretized and entropy regularized multi-marginal optimal transport problem

$$\begin{aligned} \min_{\mathbf{M} \in \mathbb{R}_+^{n^{T+1}}} \quad & \langle \mathbf{C}, \mathbf{M} \rangle + \epsilon \mathcal{D}(\mathbf{M}) \\ \text{subject to} \quad & P_t(\mathbf{M}) = \mu_t, \quad t = 0, \dots, T, \end{aligned}$$

where $\epsilon > 0$ is a regularization parameter, with the entropy term being

$$\mathcal{D}(\mathbf{M}) \triangleq \sum_{i_0, \dots, i_T} (\mathbf{M}_{i_0, \dots, i_T} \log(\mathbf{M}_{i_0, \dots, i_T}) - \mathbf{M}_{i_0, \dots, i_T} + 1).$$

Discretized and entropy regularized multi-marginal optimal transport problem

$$\begin{aligned} \min_{\mathbf{M} \in \mathbb{R}_+^{n^{T+1}}} \quad & \langle \mathbf{C}, \mathbf{M} \rangle + \epsilon \mathcal{D}(\mathbf{M}) \\ \text{subject to} \quad & P_t(\mathbf{M}) = \mu_t, \quad t = 0, \dots, T, \end{aligned}$$

where $\epsilon > 0$ is a regularization parameter, with the entropy term being

$$\mathcal{D}(\mathbf{M}) \triangleq \sum_{i_0, \dots, i_T} (\mathbf{M}_{i_0, \dots, i_T} \log(\mathbf{M}_{i_0, \dots, i_T}) - \mathbf{M}_{i_0, \dots, i_T} + 1).$$

Two marginal

- $K = \exp(-C/\epsilon)$
- Optimal solution
 $M = \text{diag}(u_0)K\text{diag}(u_1) = K \odot (u_0 u_1^T)$
- Sinkhorn iterations
 $u_0 = \mu_0 ./ (K u_1)$
 $u_1 = \mu_1 ./ (K^T u_0).$

Multimarginal optimal mass transport

Sinkhorn iterations

Discretized and entropy regularized multi-marginal optimal transport problem

$$\begin{aligned} \min_{\mathbf{M} \in \mathbb{R}_+^{n^{T+1}}} \quad & \langle \mathbf{C}, \mathbf{M} \rangle + \epsilon \mathcal{D}(\mathbf{M}) \\ \text{subject to} \quad & P_t(\mathbf{M}) = \mu_t, \quad t = 0, \dots, T, \end{aligned}$$

where $\epsilon > 0$ is a regularization parameter, with the entropy term being

$$\mathcal{D}(\mathbf{M}) \triangleq \sum_{i_0, \dots, i_T} (\mathbf{M}_{i_0, \dots, i_T} \log(\mathbf{M}_{i_0, \dots, i_T}) - \mathbf{M}_{i_0, \dots, i_T} + 1).$$

Two marginal

- $K = \exp(-C/\epsilon)$
- Optimal solution
 $M = \text{diag}(u_0)K\text{diag}(u_1) = K \odot (u_0 u_1^T)$
- Sinkhorn iterations
 $u_0 = \mu_0 ./ (K u_1)$
 $u_1 = \mu_1 ./ (K^T u_0).$

Multimarginal

- $\mathbf{K} = \exp(-\mathbf{C}/\epsilon)$
- Optimal solution
 $\mathbf{M} = \mathbf{K} \odot \mathbf{U}$, where $\mathbf{U} = (u_0 \otimes u_1 \otimes \dots \otimes u_T)$
- Sinkhorn iterations
 $u_t \leftarrow u_t \odot \mu_t / P_t(\mathbf{K} \odot \mathbf{U}), \quad t = 0, 1, \dots, T.$

Benamou, Carlier, Cuturi, Nenna, and Peyré, "Iterative Bregman Projections for Regularized Transportation Problems," SIAM J. Sci. Comput., 2015.

Multimarginal optimal mass transport

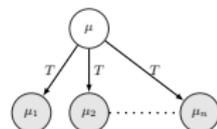
Entropy regularization

Barycenter problem:

$$\min_{\mu} \sum_{k=1}^n T(\mu, \mu_k)$$

\Leftrightarrow Multimarginal problem:

$$\begin{aligned} \min_{\mathbf{M} \in \mathcal{M}_+(\mathbf{X})} \left\langle \mathbf{M}, \left[\sum_{k=1}^n C_{i_0, i_k} \right]_{i_0, \dots, i_T} \right\rangle \\ \text{subject to } P_k(\mathbf{M}) = \mu_k \text{ for } k = 1, \dots, n. \end{aligned}$$



Pairwise regularized barycenter problem

or Multimarginal regularized problem

$$\begin{aligned} \min_{\mu, M_k, k=1, \dots, n} \sum_{k=1}^n \left(\text{trace}(C^T M_k) + \epsilon \mathcal{D}(M_k) \right) \\ \text{subject to } M_k \mathbf{1} = \mu, \\ M_k^T \mathbf{1} = \mu_k, \text{ for } k = 0, \dots, n. \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{M} \in \mathcal{M}_+(\mathbf{X})} \left\langle \mathbf{M}, \left[\sum_{k=1}^n C_{i_0, i_k} \right]_{i_0, \dots, i_T} \right\rangle + \epsilon \mathcal{D}(\mathbf{M}) \\ \text{subject to } P_k(\mathbf{M}) = \mu_k \text{ for } k = 1, \dots, n. \end{aligned}$$

Which regularization to select?

Proposition (Elvander, Haasler, Jakobsson, K., 2019)

Let the elements of the cost tensor $\mathbf{C} \in \mathbb{R}^{n^{\mathcal{T}+1}}$ be of the form

$$\mathbf{C}_{i_0, \dots, i_{\mathcal{T}}} = \sum_{\ell=1}^J C_{i_0, i_{\ell}},$$

for a cost matrix $C \in \mathbb{R}^{n \times n}$, and let $\mathbf{K} = \exp(-\mathbf{C}/\epsilon)$, and $\mathbf{U} = (u_0 \otimes u_1 \otimes \dots \otimes u_J)$. Let $K = \exp(-C/\epsilon)$, then

$$P_0(\mathbf{K} \odot \mathbf{U}) = u_0 \odot \bigcirc_{\ell=1}^J (Ku_{\ell}),$$

$$P_j(\mathbf{K} \odot \mathbf{U}) = u_j \odot K^T \left(u_0 \odot \bigcirc_{\ell=1, \ell \neq j}^J (Ku_{\ell}) \right) \quad \text{for } j = 1, \dots, J.$$

The computation of $P_t(\mathbf{K} \odot \mathbf{U})$ requires only \mathcal{T} matrix-vector multiplications of the form Ku_{ℓ} . Similar expressions can be computed for other estimation problem, e.g.,

$$\min \sum_{t=1}^{\mathcal{T}} T(\mu_{t-1}, \mu_t) \Leftrightarrow \mathbf{C}_{i_0, \dots, i_{\mathcal{T}}} = \sum_{t=1}^{\mathcal{T}} C_{i_{t-1}, i_t}.$$

Multimarginal optimal mass transport

Generalized Sinkhorn iterations

Multimarginal optimal mass transport with partial information:

$$\min_{\mathbf{M}, \Delta_t} \langle \mathbf{C}, \mathbf{M} \rangle + \epsilon \mathcal{D}(\mathbf{M}) + \sum_t \gamma_t \|\Delta_t\|_2^2 \quad (2)$$

$$\text{subject to } \Gamma_t P_t(\mathbf{M}) = r_t + \Delta_t \quad \text{for } t = 0, \dots, T,$$

where $\Gamma_t \in \mathbb{R}^{m_t \times n}$ is a mapping from the full state information to the partial information $r_t \in \mathbb{R}^{m_t}$.

Multimarginal optimal mass transport

Generalized Sinkhorn iterations

Multimarginal optimal mass transport with partial information:

$$\min_{\mathbf{M}, \Delta_t} \langle \mathbf{C}, \mathbf{M} \rangle + \epsilon \mathcal{D}(\mathbf{M}) + \sum_t \gamma_t \|\Delta_t\|_2^2 \quad (2)$$

$$\text{subject to } \Gamma_t P_t(\mathbf{M}) = r_t + \Delta_t \quad \text{for } t = 0, \dots, \mathcal{T},$$

where $\Gamma_t \in \mathbb{R}^{m_t \times n}$ is a mapping from the full state information to the partial information $r_t \in \mathbb{R}^{m_t}$.

Theorem (Elvander, Haasler, Jakobsson, K., 2019)

Given an initial set of vectors $\lambda_0, \dots, \lambda_{\mathcal{T}}$, iterate the following steps repeatedly for $t \in \{0, \dots, \mathcal{T}\}$:

- Let

$$v_t = P_t(\mathbf{K} \odot \mathbf{U}) ./ u_t$$

where $\mathbf{U} = u_0 \otimes u_1 \otimes \dots \otimes u_{\mathcal{T}}$ for the vectors $u_t = \exp(\Gamma_t^T \lambda_t / \epsilon)$.

- Update the vector λ_t as the solution to

$$\Gamma_t \left(v_t \odot \exp \left(\frac{\Gamma_t^T \lambda_t}{\epsilon} \right) \right) + \frac{\lambda_t}{2\gamma_t} - r_t = 0.$$

Then, the vectors $\lambda_0, \dots, \lambda_{\mathcal{T}}$ converge q -linearly to the unique optimal solution of the dual problem (2). Furthermore, in the limit point of the iteration, the marginals of \mathbf{M} are given directly as $\mu_t = u_t \odot v_t$, for $t = 0, 1, \dots, \mathcal{T}$.

Multimarginal optimal mass transport

Sensor localization and information fusion

Localization of targets from a sensor array

Covariance of sensor measurements:

$$R = \Gamma(\mu) = \int_X a(x)\mu(x)a(x)^H dx.$$

Steering vector a :

$$a(x) = \left(\frac{1}{\|x_k - x\|_2^{(d-1)/2}} e^{-2\pi j \frac{\|x_k - x\|_2}{\lambda}} \right)_{k=1}^p,$$

target location x , sensor locations $\{x_k\}_k$, wavelength λ .

Consider two (uncalibrated) sensor arrays, receiving signals from a set of sources

$$\min_{\mu, \mu_0, \mu_1} T(\mu_0, \mu) + T(\mu, \mu_1) + \gamma(\|\Delta_0\|_2^2 + \|\Delta_1\|_2^2)$$

subject to $\Gamma_0(\mu_0) = R_0$, $\Gamma_1(\mu_1) = R_1$.

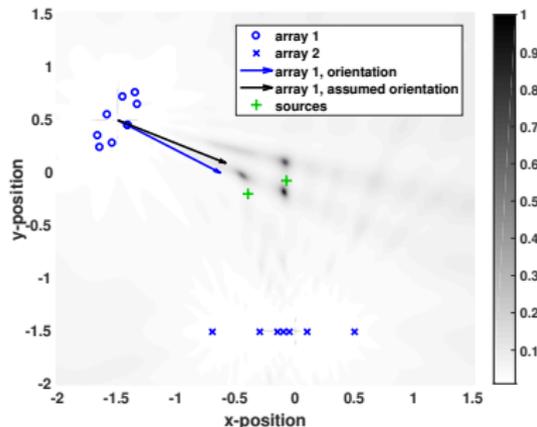
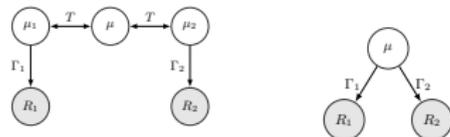


Figure: Spectral estimate with direct use of measurements. The alignment error is 6.7 degrees.



D. H. Johnson and D. E. Dudgeon. Array Signal Processing: Concepts and Techniques. Prentice Hall, Englewood Cliffs, N.J., 1993.

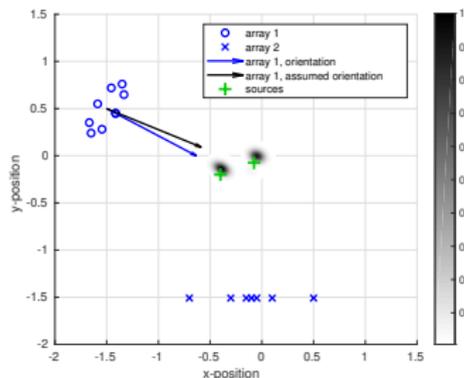
T. T. Georgiou. Solution of the general moment problem via a one-parameter imbedding. IEEE TAC, 2005.

Multimarginal optimal mass transport

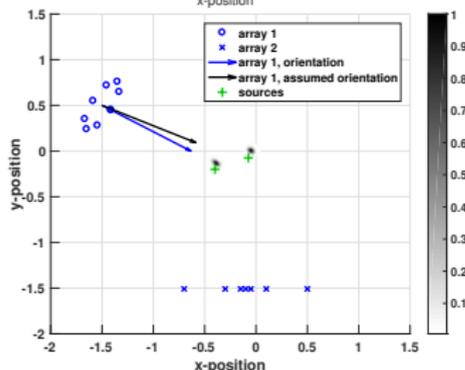
Sensor localization and information fusion

Multimarginal Sinkhorn

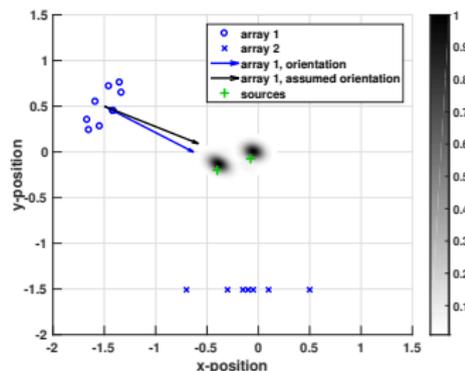
$$\epsilon = 5 \cdot 10^{-3}$$



$$\epsilon = 10^{-3}$$



Pairwise Sinkhorn



Did not converge!

Summary

- Optimal mass transport, a viable framework for estimation, control, and information fusion
- Fast projections utilizing structure in transport cost
- Generalized Sinkhorn iteration for solving multimarginal optimal mass transport problems
- Applications to
 - State-estimation of multiagent systems
 - Tracking of non-stationary spectra
 - Sensor localization via information fusion

Thank you for your attention!

Questions?