The Lindquist Symposium in Systems Theory:

Flows in Wasserstein space

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Univ. of California, Irvine

Cambridge, July 2017

dedicated to Anders
on the occasion of his 75th birthday

Anders' birthday puzzle

Anders' birthday puzzle

Anders' birthday puzzle

Anders' birthday puzzle

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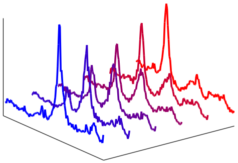
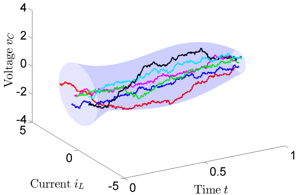
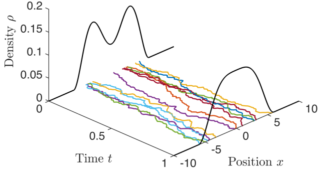
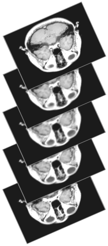
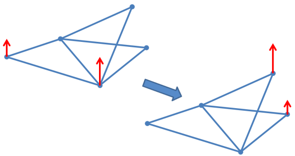
Flows in Wasserstein space









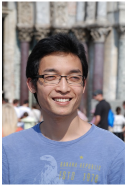


Flows in Wasserstein space

Mass Transport $\Leftrightarrow$ Schrödinger bridges $\Leftrightarrow$ Stochastic control

with connections to:
LQG, Riccati, Discrete-spaces & Networks, Matrix/Quantum flows, etc.

Collaboration with:



Yongxin Chen Michele Pavon Allen Tannenbaum

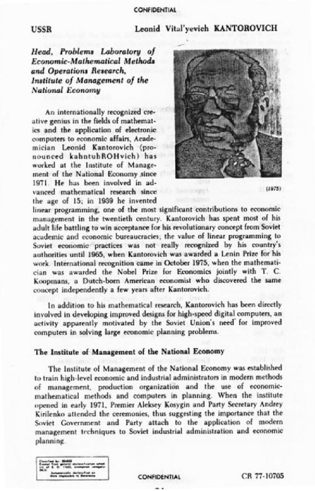
also W. Gangbo, H. Farooq, K. Yamamoto, E. Haber

Optimal Mass Transport (OMT)

Gaspard Monge 1781

Leonid Kantorovich 1976

Work in early 1940's, Nobel 1975



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USSR — Leonid Vital'yevich KANTOROVICH

Head, Problems Laboratory of Economic-Mathematical Methods and Operations Research, Institute of Management of the National Economy

(1975)

An internationally recognized creative genius in the fields of mathematics and the application of electronic computers to economic affairs, Academician Leonid Kantorovich (pronounced kahntuhROHvich) has worked at the Institute of Management of the National Economy since 1971. He has been involved in advanced mathematical research since the age of 15; in 1939 he invented linear programming, one of the most significant contributions to economic management in the twentieth century. Kantorovich has spent most of his adult life battling to win acceptance for his revolutionary concept from Soviet academic and economic bureaucracies; the value of linear programming to Soviet economic practices was not really recognized by his country's authorities until 1965, when Kantorovich was awarded a Lenin Prize for his work. International recognition came in October 1975, when the mathematician was awarded the Nobel Prize for Economics jointly with T. C. Koopmans, a Dutch-born American economist who discovered the same concept independently a few years after Kantorovich.

In addition to his mathematical research, Kantorovich has been directly involved in developing improved designs for high-speed digital computers, an activity apparently motivated by the Soviet Union's need for improved computers in solving large economic planning problems.

The Institute of Management of the National Economy

The Institute of Management of the National Economy was established to train high-level economic and industrial administrators in modern methods of management, production organization and the use of economic-mathematical methods and computers in planning. When the institute opened in early 1971, Premier Aleksey Kosygin and Party Secretary Andrey Kirilenko attended the ceremonies, thus suggesting the importance that the Soviet Government and Party attach to the application of modern management techniques to Soviet industrial administration and economic planning.

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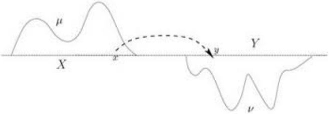
CIA file on Kantorovich (wikipedia)

Monge's problem

Le mémoire sur les déblais et les remblais
Gaspard Monge 1781



$$\inf_{T} \int \|x - \underbrace{T(x)}_{y}\|^2 d\mu(x) \qquad =: W_2(\mu, \nu)$$

where $T\#\mu = \nu$

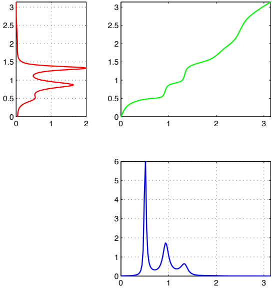
Kantorovich's formulation

$$\inf_{\pi \in \Pi(\rho_0,\rho_1)} \iint \|x - y\|^2 \, d\pi(x,y)$$

where $\Pi(\mu,\nu)$ are "couplings":

$$\int_y \pi(dx,dy) = \rho_0(x)dx = d\mu(x)$$

$$\int_x \pi(dx,dy) = \rho_1(y)dy = d\nu(y).$$



B&B's fluid dynamic formulation

Benamou and Brenier (2000):

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|\mathrm{v}(x,t)\|^2 \rho(x,t)\,dt\,dx$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathrm{v}\rho) = 0$$
$$\rho(x,0) = \rho_0(x), \quad \rho(y,1) = \rho_1(y)$$

McCann, Gangbo, Otto, Villani, ...

Stochastic control formulation

$$\inf_{\mathrm{v}} \mathbb{E}_\rho \left\{ \int_0^1 \|\mathrm{v}(x,t)\|^2 dt \right\}$$

$$\dot{x}(t) = \mathrm{v}(x,t)$$

$$\begin{aligned} x(0) &\sim \rho_0(x)dx \\ x(1) &\sim \rho_1(y)dy \end{aligned}$$

OMT as a control problem – derivation

$$\|x-y\|^2 = \inf_{\mathrm{x}\in\mathcal{X}_{xy}} \int_0^1 \|\dot{\mathrm{x}}\|^2 dt,$$

$$\mathcal{X}_{xy} = \{\mathrm{x} \in C^1 \mid \mathrm{x}(0) = x, \mathrm{x}(1) = y\}.$$

Inf attained at constant speed geodesic $\mathrm{x}^*(t) = (1-t)x + ty$

OMT as a control problem – derivation

Dirac marginals:

Also, Inf = any probabilistic average in $\mathcal{X}_{xy}$

$$\|x-y\|^2 = \inf_{P_{xy}\in\mathbb{D}(\delta_x,\delta_y)} \mathbb{E}_{P_{xy}}\left\{\int_0^1 \|\dot{\mathrm{x}}(t)\|^2 dt\right\},$$

General marginals:

$$\inf_{\pi\in\Pi(\rho_0,\rho_1)}\int_{\mathbb{R}^n\times\mathbb{R}^n}\|x-y\|^2 d\pi(x,y) = \inf_{P\in\mathbb{D}(\rho_0,\rho_1)}\mathbb{E}_P\left\{\int_0^1 \|\dot{\mathrm{x}}(t)\|^2 dt\right\}.$$

Stochastic control:

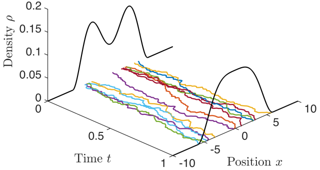
$$\inf_{\mathrm{v}} \mathbb{E}\left\{\int_0^1 \|\mathrm{v}\|^2 dt\right\}$$

$$\dot{x}(t) = \mathrm{v}(x(t),t), \quad \text{a.s.}, \quad x(0)\sim\rho_0 dx, \quad x(1)\sim\rho_1 dy.$$

Schrödinger's Bridges

Erwin Schrödinger
Work in 1926, Nobel 1935
Bridges 1931/32

$$\rho = \Psi \bar{\Psi}$$



Schrödinger's Bridge Problem (SBP)

- Cloud of N independent Brownian particles (N large)
- empirical distr. $\rho_0(x)dx$ and $\rho_1(y)dy$ at $t = 0$ and $t = 1$, resp.
- ρ_0 and ρ_1 not compatible with transition mechanism

$$\rho_1(y) \neq \int_0^1 p(t_0, x, t_1, y)\rho_0(x)dx,$$

where

$$p(s, y, t, x) = [2\pi(t - s)]^{-\frac{n}{2}} \exp\left[-\frac{|x - y|^2}{2(t - s)}\right], \quad s < t$$

Particles have been transported in an unlikely way

Schrödinger (1931): Of the many unlikely ways in which this could have happened, which one is the most likely?

Large deviations formulation of SBP

$$\text{Minimize} \quad H(Q,W) = E_Q\left[\log \frac{dQ}{dW}\right]$$

over $Q \in \mathbb{D}(\rho_0, \rho_1)$ distributions on paths with marginals ρ's

$H(\cdot,\cdot)$: relative entropy

Föllmer 1988: This is a problem of *large deviations of the empirical distribution* on path space connected through Sanov's theorem to a *maximum entropy problem*.

Relative entropy w.r.t. Wiener measure

$$dX = \mathrm{v}\,dt + dB$$

Girsanov:

$$E_Q\left[\log \frac{dQ}{dW}\right] = E_Q\left[\frac{1}{2}\int_0^t \|\mathrm{v}\|^2 ds\right]$$

the relative entropy is a quadratic cost!!!

SBP as a stochastic control problem

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(x,t)\|^2 \rho(x,t)\,dt\,dx,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = \frac{1}{2}\Delta\rho$$

$$\rho(x,0) = \rho_0(x), \quad \rho(y,1) = \rho_1(y).$$

Blaquière, Dai Pra, ...

Fluid-dynamic formulation of SBP

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \left[\|v(x,t)\|^2 + \|\frac{1}{2}\nabla \log \rho(x,t)\|^2 \right] \rho(x,t) dt dx,$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0,$$
$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y).$$

$\|\frac{1}{2}\nabla \log \rho(x,t)\|^2$: *Fisher information*, Nelson's osmotic power

Chen-Georgiou-Pavon, On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint, 2015

Mikami 2004, Mikami-Thieullen 2006,2008, Léonard 2012

Conforti 2017: SBP $\sim$ gradient flow in Wasserstein

LQG - covariance control

$$\min_{u} \mathbb{E}\left\{\int_0^T \|\mathrm{v}(t)\|^2\, dt\right\},$$

s.t.

$$dX = AXdt + B\mathrm{v}dt + B_1 dW$$

$$X(0) \sim \mathcal{N}(0, \Sigma_0), \quad X(T) \sim \mathcal{N}(0, \Sigma_1)$$

Chen-Georgiou-Pavon (TAC 2016)

Beghi (1996), Grigoriadis- Skelton (1997)
Brockett (2007, 2012), Vladimirov-Petersen (2010, 2015)

SBP Riccati's

– nonlinearly coupled Riccati equations $\equiv$ Schrödinger system

$$
\begin{aligned}
\dot{\Pi} &= -A'\Pi - \Pi A + \Pi BB'\Pi \\
\dot{\mathrm{H}} &= -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} \\
&\qquad\qquad + (\Pi + \mathrm{H})\left(BB' - B_1B_1'\right)(\Pi + \mathrm{H})\,. \\
\Sigma_0^{-1} &= \Pi(0) + \mathrm{H}(0) \\
\Sigma_T^{-1} &= \Pi(T) + \mathrm{H}(T).
\end{aligned}
$$

Chen-Georgiou-Pavon, Optimal steering of a linear stochastic system to a final probability distribution, *IEEE Trans. Aut. Control*, May 2016

Application: Cooling

Efficient steering from initial condition $\boldsymbol{\rho_0}$ to $\boldsymbol{\rho_1}$ at finite time

– Efficient stationary state of stochastic oscillators to desired $\boldsymbol{\rho_1}$

– thermodynamic systems, controlling collective response

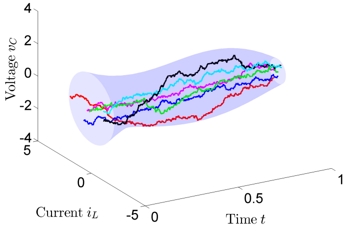
– magnetization distribution in NMR spectroscopy,..

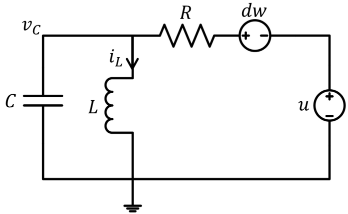
Chen-Georgiou-Pavon, Fast cooling for a system of stochastic oscillators, *J. Math. Phys.* Nov. 2015.

Cooling

Nyquist-Johnson noise driven oscillator

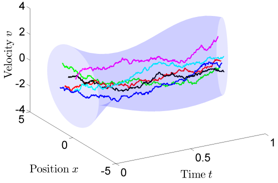
$$\begin{aligned} L di_L(t) &= v_C(t)dt \\ RC dv_C(t) &= -v_C(t)dt - Ri_L(t)dt + u(t)dt + dw(t) \end{aligned}$$

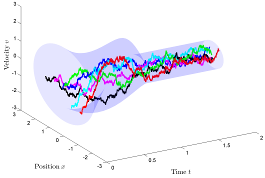


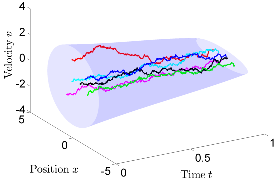


Cool it & keeping it cool

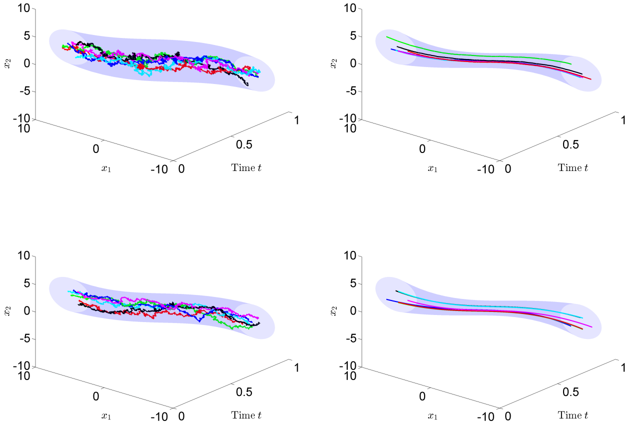
Inertial particles with stochastic excitation



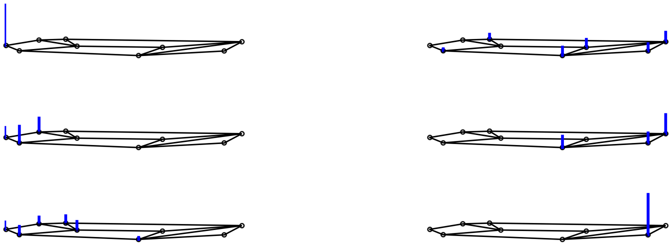




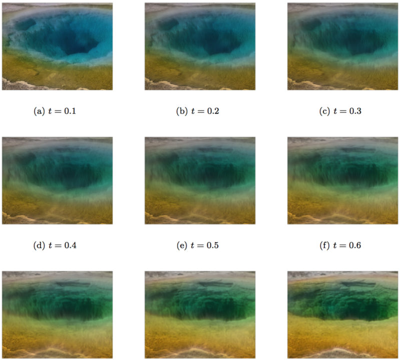
Wasserstein flows and SBP's with dynamics



Flows on discrete spaces



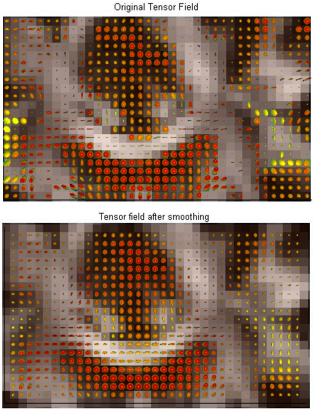
Wasserstein flows for vector & matrix fields



(a) $t = 0.1$ (b) $t = 0.2$ (c) $t = 0.3$

(d) $t = 0.4$ (e) $t = 0.5$ (f) $t = 0.6$

(g) $t = 0.7$ (h) $t = 0.8$ (i) $t = 0.9$



Applications:
color image processing, multivariable spectral analysis, DTI and medical imaging, gradient flows in Quantum mechanics, etc.

… almost the end

Thank you for your attention

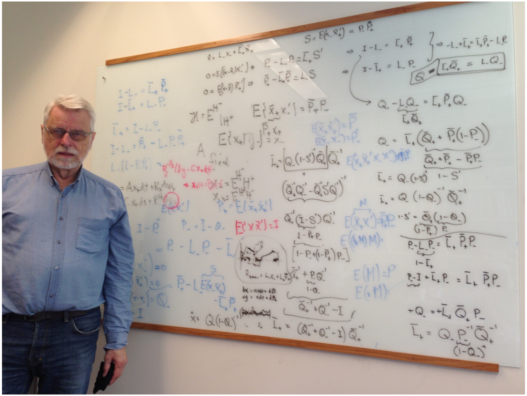
and now

Anders' birthday puzzle

Anders' birthday puzzle

Anders' birthday puzzle

They all preach to the converted... mathematicians & followers
But only Anders has an actual proof...

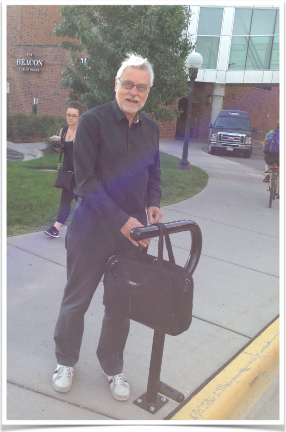


Many happy occasions!

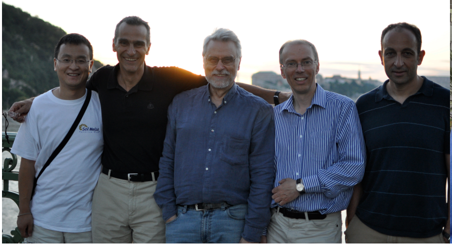
Here come the penguins

Many happy occasions!

Puzzle solving in Minneapolis



Many happy occasions!



Happy 75th!

Athens 2005
two young men enjoying
their $\tau\sigma\iota\pi o\upsilon\rho\alpha$ and $\rho\epsilon\tau\sigma\iota\nu\alpha$