In Celebration of Anders Lindqusit's 75th Birthday

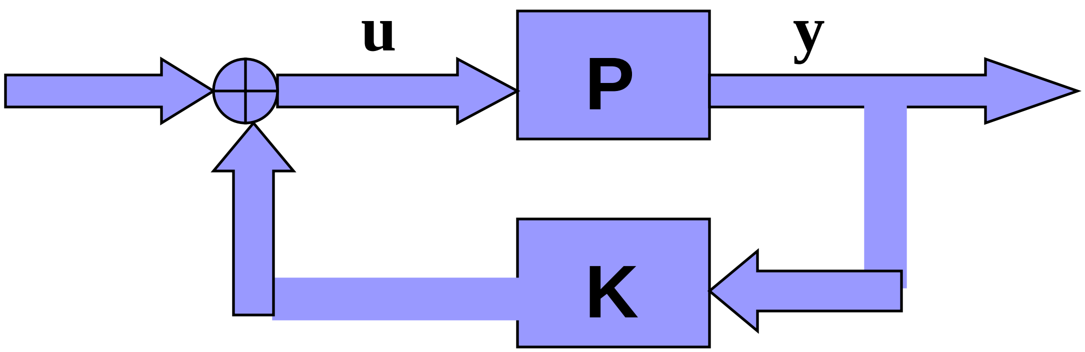
Happy Birthday, 老林，or 小林！

Delay Margin: A "Trivial" Application of Moment Theory to a Nontrivial Problem

Jie Chen
Department of Electronic Engineering
City University of Hong Kong
Hong Kong, China

The Lindquist Symposium in Systems Theory
KTH-Royal Institute of Technology
Stockholm, Sweden
November 23-25, 2017

Fundamental Issue-Feedback Stabilization



- ***Delay Stabilization Margin***: For a delay system $P(s) = e^{-\tau s} P_0(s)$, or in state space form

$$\dot{x}(t) = Ax(t) + Bu(t-\tau)$$
$$y(t) = Cx(t)$$

what is the largest interval $[0, \bar{\tau})$ so that it is possible to find a *single* feedback controller K(s) to stabilize P(s)?

Systems with Time-Varying Delay

- For a delay system with time-varying delay:

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t))$$
$$y(t) = Cx(t)$$

$$0 \le \tau(t) \le \bar{\tau}, |\dot{\tau}(t)| \le \rho < 1$$

what are the range of $(\bar{\tau}, \rho)$ so that it is possible to find a *single* feedback controller K(s) to stabilize the system?

MIMO Systems

$$P_\tau(s) = \begin{bmatrix} e^{-\tau_1 s} & & \\ & \ddots & \\ & & e^{-\tau_l s} \end{bmatrix} P_0(s)$$

- ***Delay Radius***:

$$\|\tau\|_d = \begin{cases} (\sum_i \tau_i^d)^{1/d} & d \in [1, \infty) \\ \max_i \tau_i & d = \infty \end{cases}$$

What is the largest $\|\tau\|_d$ so that it is possible to find a *single* feedback controller K(s) to stabilize the system?

The Delay Margin Problem

- ***Delay Margin***:

$$\tau^* = \sup\{\nu : K(s) \text{ stabilizes } P_\tau(s), \forall \tau \in [0, \nu)\}$$

- ***Previous Work***:

1. First-order system, state feedback (Michiels & Niculescu, 07)

$$\tau^* = 1/p$$

2. First-order system, PID control (Silva, Datta, Bhattacharyya, 02)

$$\tau^* = 2/p$$

3. General plants with one unstable pole p (Middleton & Miller, 07)

$$\tau^* = 2/p$$

The Delay Margin Problem

- ***Delay Margin***:

$$\tau^* = \sup\{\nu : K(s) \text{ stabilizes } P_\tau(s), \forall \tau \in [0, \nu)\}$$

- ***The Difficulty***:

- The results are limited to simple systems. The techniques are case-by-case.
- The general problem remains open.
- Suggested as an unresolved challenge by Davison and Miller (2004) in *Unsolved Problem in Mathematical Systems and Control Theory, eds. Blondel and Megretski.*

Delay Stabilization Problem

$$1 + T(s)(e^{-\tau s} - 1) \neq 0, \forall s \in \mathrm{RHP}$$

$$T(s) = [1 + P_0(s)K(s)]^{-1}$$

Gain Margin

$$1 + T(s)(k - 1) \neq 0, \forall s \in \mathrm{RHP}$$

$$k \in [1, k_0)$$

Phase Margin

$$1 + T(s)(e^{j\theta} - 1) \neq 0, \forall s \in \mathrm{RHP}$$

$$\theta \in [-\theta_0, \theta_0]$$

The delay stabilization problem is considerably more difficult. No viable necessary and sufficient condition exists.

Sufficient Conditions

- ***Bound for guaranteed stabilization***: There exists a controller K(s) that stabilizes $P(s) = e^{-\tau s} P_0(s)$ for all $\tau \in [0, \tau_{\min})$,

$$\tau_{\min} = \sup\{\tau : \inf \| T(s)(e^{-\tau s} - 1) \|_{\infty} < 1\}$$

- **Solving the weighted H_{∞} probem**:

1. Rational approximation: A wide variety of approximations including bilinear transformation are available for $e^{-\tau s}$.

2. Solution via Nevanlinna-Pick interpolation.

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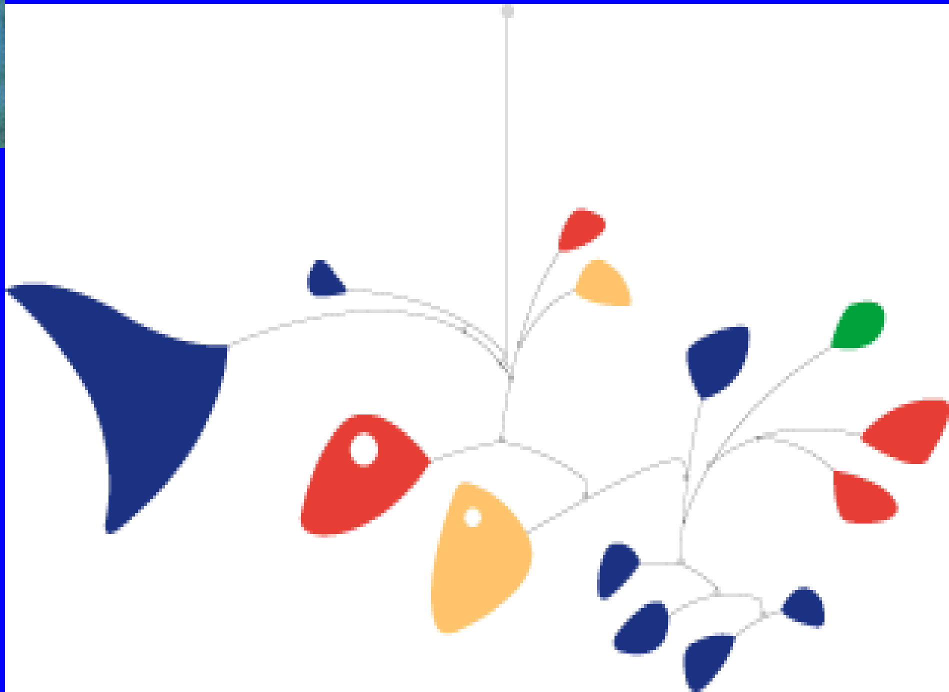

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Rational Function Approximation

- Construct a parameter-dependent rational approximation

$$w_\tau(s) = \frac{b_\tau(s)}{a_\tau(s)} = \frac{b_q(\tau s)^q + \cdots + b_1(\tau s) + b_0}{a_q(\tau s)^q + \cdots + a_1(\tau s) + a_0},$$

such that $g(\omega) \leq |w_\tau(j\omega)|$ for all ω, where

$$g(\omega) = \left|e^{-j\omega\tau} - 1\right| = \begin{cases} 2\sin(\omega\tau/2) & |\omega\tau| \leq \pi \\ 2 & \text{otherwise.} \end{cases}$$

- Lower bound of delay margin

$$\underline{\tau} = \sup\left\{\tau > 0 : \inf_{K(s)} \|T_0(s)w_\tau(s)\|_\infty < 1\right\}.$$

Infinite-dimensional optimization $\rightarrow$ Finite-dimensional optimization

Rational Approximation for $g(\omega)$

$$w_{1\tau}(s) = \tau s,$$

$$w_{2\tau}(s) = \frac{\tau s}{1+\tau s/3.465},$$

$$w_{3\tau}(s) = \frac{1.216\tau s}{1+\tau s/2},$$

$$w_{4\tau}(s) = \frac{\tau s(2\times 0.2152^2\tau s+1)}{(0.2152\tau s+1)^2},$$

$$w_{5\tau}(s) = \frac{\tau s}{1+\tau s/2}\times \frac{0.1791(\tau s)^2+0.7093\tau s+1}{0.1791(\tau s)^2+0.5798\tau s+1}.$$

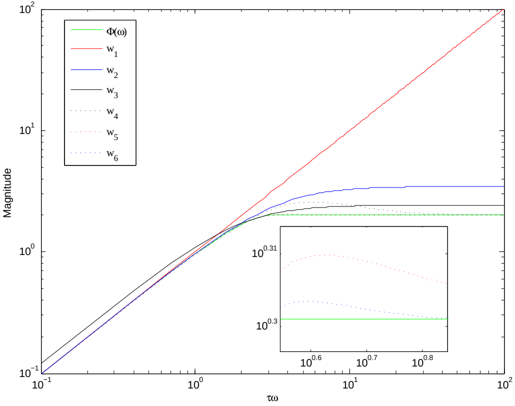


Figure 1: Rational approximation for $g(\omega)$.

$$w_{6\tau}(s) = \frac{\tau s}{1+\tau s/2}\frac{0.03061(\tau s)^4+0.2102(\tau s)^3+0.7087(\tau s)^2+1.203\tau s+1}{0.03061(\tau s)^4+0.1918(\tau s)^3+0.6457(\tau s)^2+1.104\tau s+1}.$$

Nevanlinna-Pick Tangential Interpolation

Consider distinct points $z_i \in \mathbb{C}_+, i = 1, \dots, m$ and $p_i \in \mathbb{C}_+, i = 1, \dots, n$. Assume that $z_i \neq p_j$ for any i and j. Then, there exists a rational matrix function $H(s)$ such that i) $H(s)$ is stable; ii) $\|H(s)\|_\infty \leq 1$; and iii) $H(s)$ satisfies the conditions

$$
\begin{aligned}
x_i^H H(z_i) &= y_i^H, \quad i = 1, \dots, m, \\
H(p_i) u_i &= v_i, \quad i = 1, \dots, n,
\end{aligned}
$$

for some vectors $x_i, y_i, i = 1, \dots, m$ and $u_i, v_i, i = 1, \dots, n$ with compatible dimensions, if and only if

$$
Q = \begin{bmatrix} Q_1 & Q_{12}^H \\ Q_{12} & Q_2 \end{bmatrix} \geq 0,
$$

where

$$
Q_1 = \left[\frac{x_i^H x_j - y_i^H y_j}{z_i + \bar{z}_j} \right], \quad Q_2 = \left[\frac{u_i^H u_j - v_i^H v_j}{\bar{p}_i + p_j} \right], \quad Q_{12} = \left[\frac{y_i^H u_j - x_i^H v_j}{z_i - p_j} \right].
$$

$\mathcal{H}_\infty$ computation $\rightarrow$ Analytical interpolation

Lower Bound on Delay Margin

Suppose that $p_i \in \mathbb{C}_+, i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+, i = 1, \ldots, m$ are the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then for any $w_\tau(s)$,

$$\underline{\tau} = \overline{\lambda}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right).$$

Remarks:

- $\overline{\lambda}(\cdot)$: The largest positive real eigenvalue.
- Φ_k are constructed based on $a_i,\ b_i,\ p_i,\ z_i$.
- A more accurate $w_\tau(s)$ gives a tighter $\underline{\tau}$.
- $\underline{\tau}$ depends on the unstable poles and nonminimum phase zeros.

Special Cases: One Unstable Pole

- One unstable pole: For a general $w_\tau(s)$:

$$\underline{\tau} = \frac{\lambda_{\min}}{p}$$

$$\lambda_{\min} = \min\left\{\lambda > 0\colon \sum_{k=0}^{q}(b_k - a_k)\lambda^k = 0\right\}.$$

- One unstable pole: For a particular $w_\tau(s)$:

$$w_{6\tau}(s) = \frac{\tau s}{1+\tau s/2}\frac{0.03061(\tau s)^4+0.2102(\tau s)^3+0.7087(\tau s)^2+1.203\tau s+1}{0.03061(\tau s)^4+0.1918(\tau s)^3+0.6457(\tau s)^2+1.104\tau s+1}.$$

$$\underline{\tau} \approx 1.828/p. \qquad \boxed{\tau^* = 2/p.}$$

Special Cases: One Pole/Multiple Zeros

- One unstable pole/multiple zeros: For a general $w_\tau(s)$:

$$\underline{\tau} = \frac{\lambda_{\min}}{p}$$

$$\lambda_{\min} = \min\left\{\lambda > 0 : \ \sum_{k=0}^{q} (b_k - M a_k)\lambda^k = 0\right\}, \quad M = \prod_{i=1}^{m} \left|\frac{z_i - p}{\bar{z}_i + p}\right|.$$

- One unstable pole/multiple zeros: For the particular $w_\tau(s) = \tau s$:

$$\underline{\tau} = \frac{1}{p} \prod_{i=1}^{m} \left|\frac{z_i - p}{\bar{z}_i + p}\right|.$$

Delay Radius for MIMO Systems

- Delay radius:

$$r_d = \sup \left\{ \nu : \ K(s) \text{ stabilizes } P_\tau(s) \forall \tau = \begin{bmatrix} \tau_1 & \cdots & \tau_l \end{bmatrix}^T, \ \|\tau\|_d \leq \nu \right\}.$$

- Lower bounds on delay radius:

$$r_d \geq \sup \left\{ \|\tau\|_d : \ \inf_{K(s)} \|W_\tau(s) T_0(s)\|_\infty < 1 \right\},$$

$$W_\tau(s) = \text{diag}\left(w_{\tau_1}(s), ..., w_{\tau_l}(s)\right), \left|e^{-j\omega\tau_i} - 1\right| \leq |w_{\tau_i}(j\omega)|, \ i = 1, ..., l.$$

Comments:

1. The delay radius and its lower bound are difficult to compute. The difficulty can be likened to that of μ-synthesis.
2. Estimates on the delay radius are available by solving LMI feasibility problems.

Special Cases: One Pole

Suppose that $P_0(s)$ has only one unstable pole $p \in \mathbb{C}_+$ with input direction vector $\eta = [\eta_1^H, \ldots, \eta_l^H]^H$. Then $P_\tau(s)$ can be stabilized by some $K(s)$ for any $\tau = [\tau_1 \ \cdots \ \tau_l]^T$ if

$$p^2 \sum_{i=1}^{l} \tau_i^2 |\eta_i|^2 < 1.$$

Furthermore,
(1) $r_\infty > 1/p$;
(2) $r_2 > 1/\left(p \max_i |\eta_i|\right)$;
(3) $r_1 > 1/p$.

Special Cases: One Pole/One Zero

Suppose that $P_0(s)$ has one unstable pole $p \in \mathbb{C}_+$ with input direction vector $\eta = [\eta_1^H, \ldots, \eta_l^H]^H$ and one nonminimum phase zero $z \in \mathbb{C}_+$ with output direction vector $\xi = [\xi_1^H, \ldots, \xi_l^H]^H$. Then $P_\tau(s)$ can be stabilized by some $K(s)$ for any $\tau = [\tau_1 \ \cdots \ \tau_l]$ if

$$p^2 \left[\sum_{i=1}^{l} \tau_i^2 |\eta_i|^2 + \frac{\cos^2 \angle(\eta, \xi)}{\sum_{i=1}^{l} \tau_i^{-2} |\xi_i|^2} \left(\left| \frac{z+p}{z-p} \right|^2 - 1 \right) \right] < 1.$$

where $\cos^2 \angle(\eta, \xi) = |\eta^H \xi|^2$. Furthermore,

$$r_\infty > \frac{1}{p} \frac{1}{\sqrt{\cos^2 \angle(\eta, \xi) \left| \frac{z+p}{z-p} \right|^2 + \sin^2 \angle(\eta, \xi)}}.$$

Time-Varying Delays

Consider the system with input time-varying delay

$$\begin{aligned} \dot{x} =& Ax + Bu(t - \tau(t)), & \quad 0 \le \tau(t) \le \bar{\tau}, \\ y =& Cx, & \quad |\dot{\tau}(t)| \le \delta < 1. \end{aligned}$$

with output feedback

$$u(s) = K(s)y(s).$$

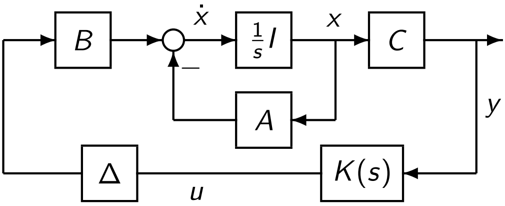
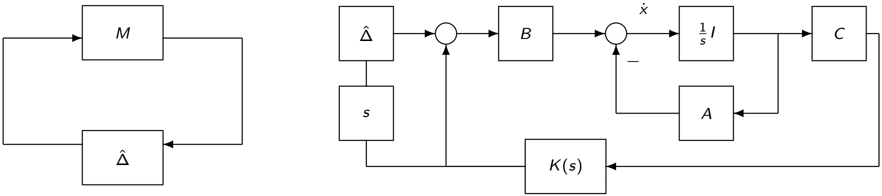


Figure 2: Feedback system with time-varying input delay.

Small Gain Reformulation

- Remodel the system:

$$M(s) = sT(s), \quad \hat{\Delta}x = -\int_{t-\tau(t)}^{t} x(\sigma)\mathrm{d}\sigma.$$



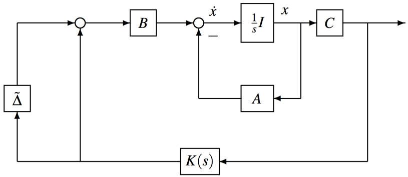
- Estimate the induced norm of the uncertainty: $\|\hat{\Delta}\|_{2,2} \leq \bar{\tau}$.
- Small-gain condition: $K(s)$ stabilizes the system if it stabilizes $P_0(s)$ and

$$\boxed{\|\bar{\tau}sT(s)\|_\infty < 1.}$$

Another Small Gain Reformulation

- Remodel the system:

$$M(s) = T(s), \quad \tilde{\Delta} = \Delta - I.$$



- Small-gain condition (Kao & Rantzer, 2007): For any rational approximant $|w_\tau(j\omega)| \geq g(\omega) + \epsilon$, $K(s)$ stabilizes the system if it stabilizes $P_0(s)$ and

$$\inf_{K(s)} \|T(s)w_\tau(s)\|_\infty < \sqrt{\frac{2-\delta}{2}}.$$

The Bounding Function

$$g(\omega) = \begin{cases} 2\sin(\omega\bar{\tau}/2) & |\omega| \leq \pi/\bar{\tau}, \\ 2 & \text{otherwise.} \end{cases}$$

Comments:

- $$g(\omega) = \sup_{\tau \in [0,\ \bar{\tau}]} |e^{-j\omega\tau} - 1|.$$
- All $w_\tau(s)$ can be used with an additional constant factor. In particular, when $\delta = 0$, i.e., when the delay is constant, the condition reduces completely to that of the constant delay case.

Stabilizability Region

Suppose that $p_i \in \mathbb{C}_+, i = 1, \ldots, n$ and $z_i \in \mathbb{C}_+, i = 1, \ldots, m$ are the distinct unstable poles and nonminimum phase zeros of $P_0(s)$, respectively. Assume that $P_0(s)$ has neither zero nor pole on the imaginary axis. Then $K(s)$ can stabilize the system for all $\tau(t) \in [0,\ \bar{\tau})$, $|\dot{\tau}(t)| \leq \delta$ if

$$\bar{\tau} = \overline{\lambda}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right).$$

Remarks:

- The result is almost exactly the same as that for the constant delay case, with the only exception that Φ_0 depends on δ.
- Similar explicit expressions can be obtained for special cases, e.g., when $P_0(s)$ has only one unstable pole and multiple zeros.
- The results can be extended to MIMO systems.

Example: $\tau(t) = \alpha\left(1 - \sin\left(\beta t\right)\right)$

$$\dot{x}(t) = \begin{bmatrix} 0.39 & -0.038 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t - \tau(t)),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t).$$

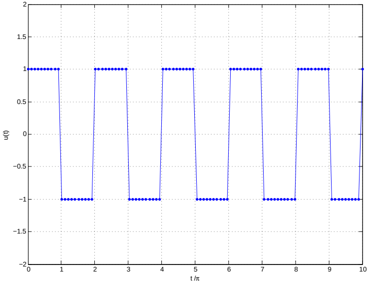
The transfer function of the delay-free plant is

$$P_0(s) = \frac{1}{(s - 0.2)(s - 0.19)}.$$

$\bar{\tau} = 2\alpha, \ \delta = \alpha\beta.$

Input

$$u(t) = \mathrm{sgn}\left(\sin(t)\right).$$



Stabilizability Region

Specifically, set $\beta = 0.1,\ \alpha = 1.4$; that is,

$$\tau(t) = 1.4(1 - \sin(0.1t)).$$

$$K(s) = (s + 0.1)/(s + 1).$$

Stabilized

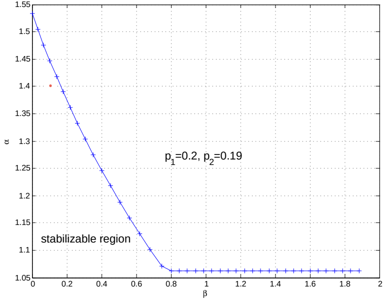


Figure 3: Stabilizability region

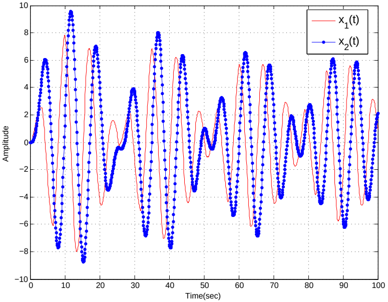


Figure 4: System states (with $K(s) = (s + 0.1)/(s + 1)$).

Conclusion

- Main results:
  1. A general computational formula providing a lower bound on the delay margin, thus a range of delay over which a single LTI controller exists to stabilize the entire family of the delay plants.
  2. Analytical results showing the dependence of the bound on the plant's unstable poles and nonminimum phase zeros in special cases.
  3. The bound can be rather close to the exact delay margin for the special instance when the plant has only one unstable pole.
  4. The results can be extended in a unified manner to MIMO systems and to systems with time-varying delays, with varying degrees of pessimism.
- Possible improvements:
  1. Systematic improvements by higher-order rational approximations.
  2. Potential improvements of the bounding function for time-varying delay.
  3. Possible development of a companion upper bound via rational approximation and analytic interpolation.