Representing externally positive systems through minimal eventually positive realizations

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Preamble

Preamble (some “alternative facts”)

1970

Preamble (some “alternative facts”)

1970

1990

Preamble (some "alternative facts")

1970

1990

Now

Anders

Preamble (some “alternative facts”)

1970

1990

Now

Anders

Happy birthday “comrade” Anders !!

Representing externally positive systems through minimal eventually positive realizations

Outline:

- Externally vs internally positive linear systems
- Eventually positive matrices
- Eventually positive minimal realizations are externally positive
- Viceversa: constructing an eventually positive minimal realization for an externally positive system
- Downsampling of eventually positive realizations
- Continuous-time minimal eventually positive realizations: a "dual" to Nyquist-Shannon sampling theorem

Externally vs Internally positive systems

- Discrete-Time SISO linear system

$$H(z) = \frac{P(z)}{Q(z)} = \sum_{i=1}^{\infty} h(i) z^{-i}$$

- Externally positive system

$$u(k) \geq 0 \;\; \forall k \quad \Longrightarrow \quad y(k) \geq 0 \;\; \forall k$$

- Equivalent conditions:
  - impulse response is non-negative
  - Markov parameters $h(i) \geq 0 \;\; \forall\, i = 0, 1, \ldots$

Externally vs Internally positive systems

- Discrete-Time SISO linear system

$$\begin{aligned} x(k+1) &= A\,x(k) + b\,u(k) \qquad k = 0,\, 1, \ldots \\ y(k) &= c\,x(k) \end{aligned}$$

- (Internally) positive system

$$u(k) \geq 0 \;\; \forall k \quad \Longrightarrow \quad \begin{array}{l} x(k) \geq 0 \;\; \forall k \\ y(k) \geq 0 \;\; \forall k \end{array}$$

- Equivalent conditions:

$$A \geq 0 \qquad b \geq 0 \qquad c \geq 0$$

- External positivity $\begin{array}{c} \Longleftarrow \\ \not\Longrightarrow \end{array}$ Internal positivity

(Non)-minimal positive realization

Consider $H(z)$ externally positive

Assumption: $H(z)$ has a simple, strictly dominating pole.

Theorem:
$H(z)$ is externally positive $\iff$ $H(z)$ has a (non-minimal) positive realization

Problem: Given $H(z)$ externally positive, a *minimal* positive realization $\{A,\, b,\, c\}$ may not exist!

Our task: Study the "gap" between external and (minimal) internal positivity in the case of simple, strictly dominating pole

Constructing (non-minimal) positive realizations

Theorem: (Ohta, Maeda & Kodama, SIAM J. Con. Opt. 1984)

$H(z)$ has a (non-minimal) positive realization $\iff$ For any minimal realization $\{A, b, c\}$ $\exists$ a polyhedral cone $\mathcal{K}$ such that

$$A\mathcal{K} \subseteq \mathcal{K}$$
$$b \in \mathcal{K}$$
$$c \in \mathcal{K}^*$$

- If $\mathcal{K} \subseteq \mathbb{R}^n_+ \implies \exists$ minimal positive realization
- Condition above is a Perron-Frobenius condition

Theorem: (Valcher & Farina, SIAM J. Mat. An. App. 2000)

$\exists$ polyhedral cone $\mathcal{K}$ s.t. $A\mathcal{K} \subset \mathcal{K}$ $\iff$ P.F. holds: $\rho(A) \in \mathrm{sp}(A)$ with $\rho(A)$ simple, positive and s.t. $\rho(A) > |\lambda| \; \forall \, \lambda \in \mathrm{sp}(A)$

Gap between externally and internally positive

To describe the "gap" between externally positive and internally positive systems:

Approach:

1. relax the positivity of A
2. construct a minimal realization $A \gtrless 0$, $b \geq 0$, $c \geq 0$ s.t. the state $x(k)$ lacks positivity only transiently:

$$\forall\, x(0) \geq 0 \;\; \exists\, \eta_o \in \mathbb{N} \;\; \text{s. t.} \;\; x(k) \geq 0 \quad \forall\, k \geq \eta_o$$

Definition: A realization $\{A_e, b_e, c_e\}$ is said eventually positive if $x(k) \geq 0 \;\; \forall\, k \geq \eta_o$ and $\forall\, x(0) \geq 0$

Eventually positive matrices

Definition: A matrix A_e is called eventually positive if $\exists\ \eta_o$ such that $(A_e)^\eta > 0\ \forall\, \eta \geq \eta_o$

- notation for eventually positive: $A_e \overset{\vee}{>} 0$
- meaning: the negative entries of A_e disappear for higher powers
  $\Longrightarrow$ disregarding the transient, the matrix is "positive"
- equivalent characterization: Perron-Frobenius property

Theorem: (Noutsos & Tsatsomeros, SIAM J. Mat. An. App. 2008)

$A_e \overset{\vee}{>} 0 \quad \Longleftrightarrow \quad$ P.F. holds: $\rho(A_e) \in \mathrm{sp}(A_e)$ with $\rho(A_e)$ simple, positive and s.t. $\rho(A_e) > |\lambda|$ $\forall\, \lambda \in \mathrm{sp}(A_e)$, with positive right and left P.F. eigenvectors: $v > 0$ and $w > 0$

Perron-Frobenius property & Eventual Positivity

Theorem: (Altafini & Lini, IEEE Tr. Aut. Con., 2015)

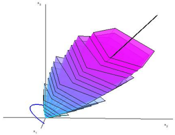
$$A_e \overset{\vee}{>} 0 \qquad \Longleftrightarrow \qquad \exists \text{ cone } \mathcal{K} \text{ s. t. } A_e\mathcal{K} \subset \mathcal{K} \text{ and}$$

$$\forall\, \eta \geq \eta_o \quad \begin{cases} (A_e)^\eta \mathcal{K} \subset \mathbb{R}_+^n \\ (A_e^T)^\eta \mathcal{K}^* \subset \mathbb{R}_+^n \end{cases}$$

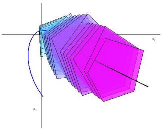
- iterated cone "enters" in $\mathbb{R}_+^n$ (since $v > 0$)

$$A_e\mathcal{K},\ A_e^2\mathcal{K}, \quad \dots, \quad A_e^\eta \mathcal{K} \subset \mathrm{int}(\mathbb{R}_+^n)$$

3D view



top view



Perron-Frobenius property & Eventual Positivity

- Combining $A_e \overset{\vee}{>} 0$ with cone conditions on b_e and c_e:

Theorem:

A minimal realization $\{A_e, b_e, c_e\}$ of $H(z)$ is eventually positive $\iff$ $A_e \overset{\vee}{>} 0$, $b_e \geq 0$, $c_e \geq 0$ and $\exists$ a cone $\mathcal{K}$ such that

$$A_e \mathcal{K} \subseteq \mathcal{K}$$
$$b_e \in \mathcal{K}$$
$$c_e \in \mathcal{K}^*$$

- difference w.r.t. conditions in the literature: the cone $\mathcal{K}$ becomes positive (after η_o iterations), hence the minimal realization $\{A_e, b_e, c_e\}$ itself can be used (no need to construct a "larger" realization based on the rays of $\mathcal{K}$)

Sketch of the proof (practical meaning)

$$x(k) = x_o(k) + x_f(k) = A_e^k x(0) + \sum_{j=0}^{k-1} A_e^{k-j-1} b_e u(j)$$

1. Forced evolution $x_f(k)$
   - since $x_f(k) \in \mathcal{R} = \mathrm{cone}(b_e, A_e b_e, \ldots)$

     $$\text{if } \mathcal{R} \subset \mathbb{R}_+^n \implies x_f(k) \geq 0 \; \forall \, k$$

   - next slides: for a Markov observability realization this is always true

2. Free evolution $x_o(k)$
   - when P.F. holds, then

     $$x_o(k) \to \mathrm{span}(v)$$

     but the sign of

     $$\lim_{k \to \infty} x_o(k) = \frac{v \, w^T x(0)}{w^T \, v}$$

     is not determined;
   - if in addition $v > 0$ and $w > 0$ then

     $$x(0) \geq 0 \implies x_o(k) > 0 \;\; \text{for } k \text{ sufficiently large}$$

Main result (and a conjecture)

Consider $H(z)$ with a simple, strictly dominating pole.

Theorem:

$H(z)$ admits a minimal eventually positive realization $\implies$ $H(z)$ externally positive

$\overset{??}{\Longleftarrow}$

Proof:

"$\implies$" $\exists$ a cone $\mathcal{K}$ such that $A_e\mathcal{K} \subseteq \mathcal{K}$, $b_e \in \mathcal{K}$ and $c_e \in \mathcal{K}^*$.
Then $c_e b_e \geq 0$, $A_e b_e \in \mathcal{K} \implies c_e A_e b_e \geq 0, \ldots$

"$\overset{??}{\Longleftarrow}$" A proof is missing, but a constructive algorithm is available, and always terminate successfully...

Example

$$H(z) = \frac{0.105z^3 + 0.13z^2 - 0.022z - 0.015}{z^5 - 0.96z^4 - 0.058z^3 + 0.035z^2 - 0.01z - 0.003}$$

- externally positive system, without minimal positive realization
- Markov observability form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.003 & 0.01 & -0.035 & 0.058 & 0.96 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0.105 \\ 0.23 \\ 0.21 \\ 0.19 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

- by construction: $\mathcal{R} \subset \mathbb{R}_+^n \implies x_f(k) \geq 0 \, \forall \, k$

Example

- spectral radius: $\rho(A) = 1$
- P.F. eigenvectors:

$$w = \begin{bmatrix} 0.003 \\ 0.014 \\ -0.021 \\ 0.037 \\ 0.999 \end{bmatrix} \qquad v = \mathbb{1} \qquad \Longrightarrow \quad \begin{cases} (A)^\eta \mathcal{K} \subset \mathbb{R}^5_+ \\ (A^T)^\eta \mathcal{K}^* \not\subseteq \mathbb{R}^5_+ \end{cases}$$

- $\Longrightarrow A$ is not eventually positive

$$\lim_{k\to\infty} x_o(k) = \frac{v\, w^T x(0)}{w^T\, v}$$

- $\Longrightarrow x_o(k)$ can have any sign $\forall\ k$

Example

- changing basis with $M = I_5 + [m_{43}]$

$$A_e = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.608 & 1 & 0 \\ 0 & 0 & -0.369 & -0.608 & 1 \\ 0.003 & 0.01 & 0.0003 & 0.058 & 0.96 \end{bmatrix}, \quad b_e = \begin{bmatrix} 0 \\ 0.105 \\ 0.23 \\ 0.067 \\ 0.19 \end{bmatrix} \quad c_e = c$$

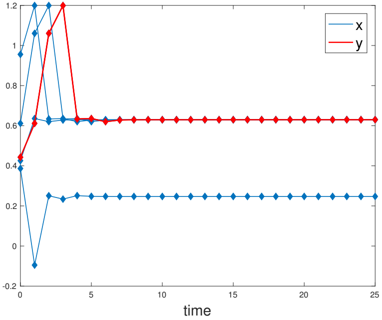
- P.F. eigenvectors:

$$w_e = \begin{bmatrix} 0.003 \\ 0.014 \\ 0.0012 \\ 0.037 \\ 1 \end{bmatrix} \quad v_e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0.39 \\ 1 \end{bmatrix} \implies \begin{cases} (A_e)^\eta \mathcal{K} \subset \mathbb{R}^5_+ \\ (A_e^T)^\eta \mathcal{K}^* \subset \mathbb{R}^5_+ \end{cases} \implies A_e \overset{\vee}{>} 0$$

Example

Practical meaning:

- in the "Markov observability basis":
  - y never violates positivity
  - x may violate positivity (and remain non-positive forever)
- in the "eventually positive basis"
  - y never violates positivity
  - x can transiently violate positivity

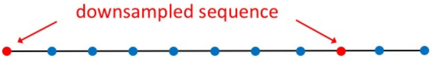


Recovering positivity through dowsampling

Consider $H(z) = \sum_{i=1}^{\infty} h(i) z^{-i}$ with a simple strictly dominating pole.

Theorem:

$H(z)$ admits a minimal eventually positive realization $\{A_e, b_e, c_e\}$ $\Longrightarrow$ $\exists\, \eta \in \mathbb{N}$ s.t. $\{A_e^{\eta}, b_e, c_e\}$ is a minimal positive realization of the decimated subsequence of Markov parameters $\{h_\eta(k) = h((k-1)\eta + 1)\}_{k=1}^{\infty}$



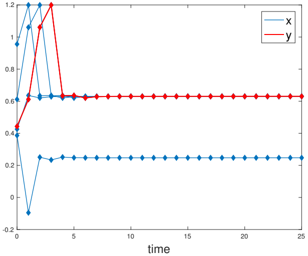
- Meaning: downsampling an eventually positive realization one gets a minimal positive realization

Conjecture: Every externally positive system has subsequences of Markov parameters which admit minimal positive realizations

Example

Original

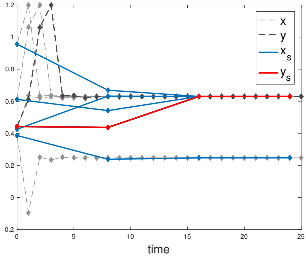
$\{A_e, b_e, c_e\}$ minimal eventually positive



$\Longrightarrow$

Downsampled $(\eta = 8)$

$\{(A_e)^8, b_e, c_e\}$ minimal (internally) positive



Continuous-time eventually positive realizations

- CT SISO linear system

$$\dot{x} = Ax + bu$$
$$y = cx$$

- ZOH discretization

$$x(k+1) = A_\delta \, x(k) + b_\delta \, u(k)$$
$$y(k) = c_\delta \, x(k)$$

where

$$A_\delta = e^{AT} \qquad b_\delta = \int_0^T e^{A\tau} b d\tau \qquad c_\delta = c$$

Continuous-time eventually positive realizations

Consider $H(s)$ with a simple, strictly dominating pole.

Theorem:

$H(s)$ admits a minimal eventually positive realization $\quad \overset{\Longrightarrow}{\overset{??}{\Longleftarrow}} \quad H(s)$ externally positive

Proof:

"$\Longrightarrow$" Same as D.T. case

"$\overset{??}{\Longleftarrow}$" constructive algorithm ...

A "dual" to Nyquist-Shannon sampling theorem

Theorem:

$H(s)$ admits a minimal eventually positive realization and $\mathcal{R} \subset \mathbb{R}^n_+$ $\Longrightarrow$ $\exists$ sample time T_o s.t. $\forall T \geq T_o$ the realization $\{A_\delta, b_\delta, c_\delta\}$ is a minimal positive realization of the ZOH system

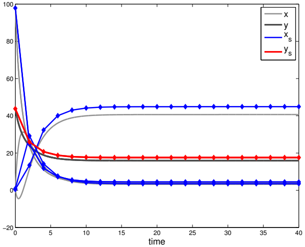
- Meaning: sampling with a sufficiently long sample time, an eventually positive realization leads to a minimal positive realization for the ZOH system
- dual to Nyquist-Shannon sampling theorem: when sampling with sufficiently long sample time the non-positive transient is guaranteed to to be avoided

Conjecture: Every externally positive C.T. system has ZOH discretizations which admit minimal positive realizations

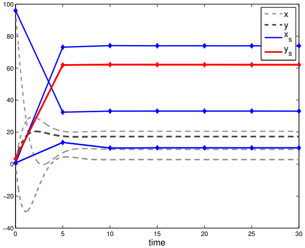
Examples

- ZOH sampling giving a minimal positive realization may or may not lead to a "faithful" DT system

Similar to original



Different from original



Conclusion

- attempt to understand the gap between externally and internally positive linear systems: eventually positive systems
- properties:
  1. A can have negative entries
  2. powers of A become positive
- meaning:
  1. states can become negative even if $x(0) > 0$
  2. after a transient the entire state must becomes positive
- interpretation
  1. the lack of minimal positive realization is due to the transient of the free evolution $x_o(k)$
  2. Perron-Frobenius dictates the asymptotic behavior
- conjecture:
  1. for the case of simple, strictly dominant P.F. eigenvalue, the eventually positive realizations "fill the gap" between externally and internally positive systems
  2. $\exists$ always a basis in which the asymptotic behavior of the state belongs to $\mathbb{R}_+^n$

Thank you!

(other “false positives” when you google “Anders Lindquist”)