SF3810 Convexity and optimization in linear spaces, 2023.

Home assignments, collection number 2.

Due date: March 8, 2023.

Note: You may discuss the problems with other students, but you should write your own solutions, "in your own words".

1. Let K be a closed convex cone in a Hilbert space H, let $h \in H$ be a given vector, and let $K^{\ominus} = \{y \in H \; ; \; (y \,|\, x) \leq 0 \text{ for all } x \in K\}$ = "negative conjugate cone" of K.

(a) Prove that $K^{\ominus}$ is a closed convex cone in H.

(b) Consider the problem minimize $\|h - k\|$ subject to $k \in K$.
Prove (by using Theorem 1 in section 3.12) that k_0 is the unique optimal solution if and only if: $k_0 \in K$, $h - k_0 \in K^{\ominus}$ and $(h - k_0 \,|\, k_0) = 0$.

(c) Prove that each $h \in H$ can be written in a unique way as $h = k + g$, where $k \in K$, $g \in K^{\ominus}$ and $(g \,|\, k) = 0$.

(d) Prove that $(K^{\ominus})^{\ominus} = K$.

(e) Consider the problem minimize $\|z\|$ subject to $h - z \in K^{\ominus}$.
Prove that $z = k_0$, with k_0 from (b) above, is the unique optimal solution.

2. Let H be the Hilbert space consisting of $n \times n$ matrices, with the inner product defined by $(A \,|\, B) = \text{trace}(A^{\mathsf{T}} B)$, and let M be the closed subspace in H consisting of symmetric matrices.

(a) What is $M^{\perp}$ in this case?

(b) Given a nonsymmetric $n \times n$ matrix A, what is the optimal solution S_0 to the problem minimize $\|A - S\|$ subject to $S \in M$?

3. Let M be the Hilbert space consisting of symmetric $n \times n$ matrices, with the inner product defined as above, and let K be the closed convex cone in M consisting of positive semidefinite symmetric matrices.

(a) What is $K^{\ominus}$ in this case?

(b) Given an indefinite symmetric $n \times n$ matrix S, what is the optimal solution P_0 to the problem minimize $\|S - P\|$ subject to $P \in K$?

4. Let H be as in problem 2 above, and let K be the closed convex cone in H consisting of positive semidefinite symmetric matrices.
Given a nonsymmetric $n \times n$ matrix A, what is the optimal solution P_0 to the problem minimize $\|A - P\|$ subject to $P \in K$?

5. Prove Lemma 1 in section 3.8:
An orthonormal sequence $\{e_i\}$ in a Hilbert space H is complete *if and only if* the only vector $y \in H$ which satisfies $(y \,|\, e_i) = 0$ for all i is the null vector.

Good luck!