SF3810 Convexity and optimization in linear spaces, 2023.

Home assignments, collection number 1.

Due date: February 22, 2023.

Note: You may discuss the problems with other students, but you should write your own solutions, "in your own words".

1. Let S be a given set in a linear space X. S may or may not be a convex set.
A point $x \in X$ is said to be a *convex combination* of elements in S if $x = \sum_{i=1}^{k} \alpha_i s_i$, for some integer $k \geq 1$, some elements $s_1, \ldots, s_k$ in S, and some real numbers $\alpha_1, \ldots, \alpha_k$ which satisfy $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, \ldots, k$.
Let convcomb(S) denote the set of all convex combinations of elements in S.

Show that convcomb(S) is the *smallest* of all *convex* sets which contain S.

Hint: The following steps are recommended:
Show that $S \subseteq$ convcomb(S).
Show that convcomb(S) is a convex set.
Show that if C is a convex set then convcomb$(C) = C$.
Show that if C is a convex set such that $S \subseteq C$, then convcomb$(S) \subseteq C$.

2. Let X be the normed linear space consisting of real $n \times n$ matrices, with the natural definitions of addition and scalar multiplication, and with the norm defined by $\|A\|^2 = \text{trace}(A^{\mathsf{T}}A)$. (The trace of an $n \times n$ matrix is the sum of the n diagonal elements.) Let M be the subset of X consisting of symmetric matrices, and let C be the subset of M consisting of positive semidefinite (psd) matrices.

Show that M is a *closed subspace* of X, and that C is a *closed convex cone* in X.

Do not use any "advanced" properties of psd matrices (like non-negative eigenvalues), just use the basic definition ($y^{\mathsf{T}}A\,y \geq 0$ for all y).

3. Let $X = l_\infty$, and consider the following two subsets M_1 and M_2 of X.
Let M_1 be the set of sequences in l_∞ having only a finite number of non-zero terms.
If $x \in M_1$ then $x = (\xi_1, \xi_2, \dots) \in l_\infty$ with only a finite number of $\xi_j \neq 0$.
Let M_2 be the set of sequences in l_∞ which converge to zero.
If $x \in M_2$ then $x = (\xi_1, \xi_2, \dots) \in l_\infty$ with $\xi_j \to 0$ as $j \to \infty$.

Show that M_1 is a subspace of l_∞, but not a closed subspace.
Show that M_2 is a closed subspace of l_∞.
Is the closure of M_1 equal to M_2? Proof or counterexample!

4. Let $S = \{x \in l_2 \mid \|x\| \leq 1\}$ = the unit sphere in l_2,
and let $f\colon l_2 \to R$ be defined by $f(x) = \sum_j a_j \xi_j^2$, where $a_j = j/(j+1)$.
Show that S is not compact, by considering a certain (very simple) sequence $\{x_n\}$ with $x_n \in l_2$ and $\|x_n\| = 1$.
Show that f is continuous (Hint: Hölders inequality), but that there is no $\hat{x} \in S$ such that $f(\hat{x}) \geq f(x)$ for all $x \in S$.

Good luck!