Exam October 27, 2022 in SF2852/FSF3852 Optimal Control.

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Allowed aids: The formula sheet and mathematics handbook (by Råde and Westergren). (Note that calculator is **not** allowed.)

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Consider the discrete optimal control problem

$$\min \sum_{k=0}^{2} (|x_k| + 5|u_k|) \quad \text{s.t} \quad \begin{cases} x_{k+1} = 0.5x_k + u_k \\ x_k \in X_k \\ u_k \in \{-1, -0.5, 0, 0.5, 1\} \end{cases}$$

where the state space is defined by

$$X_0 = \{2\},\ X_1 = \{0,1,2\},\ X_2 = \{0,1,2\},\ X_3 = \{1\}$$

Solve the problem using dynamic programming.
Hint: It may be useful to introduce the control constraint sets $U(k,x)$ that specify the feasible control values for each $x_k \in X_k$. .... (10p)

2. Solve the following time-variant optimal control problem

$$\min \int_0^T (u^2 + t)dt$$

subject to

$$\begin{aligned} \dot{x} &= x + u, x \in R, u \in R \\ x(0) &= x_0 \\ x(T) &= 0 \\ T &> 0, \end{aligned}$$

where the terminal time T is also a variable. .... (10p)

3. Consider the scalar linear quadratic optimal control problem

$$\min \int_0^\infty (3x^2 + u^2)dt \quad \text{subject to} \quad \dot{x} = -x + u,\ x(0) = 1 \qquad (1)$$

(a) Compute the optimal stabilizing feedback control and the corresponding optimal cost. .... (3p)

(b) Compute the closed loop poles. .... (2p)

Now consider the finite truncation of (1)

$$\min \int_0^T (3x^2 + u^2)dt \quad \text{subject to} \quad \dot{x} = -x + u, \ x(0) = 1 \qquad (2)$$

(c) Use the Hamilton-Jacobi-Bellman equation to compute the optimal feedback control and the corresponding optimal cost. .... (3p)

(d) Let $p(t, T)$ be the Riccati solution corresponding to (2), where the final time is made explicit as an argument. Compute $\lim_{T\to\infty} p(t, T)$ and compare with the solution to the ARE corresponding to (1). .... (2p)

4. Consider multiplication of N matrices

$$M_1 M_2 \ldots M_k M_{k+1} \ldots M_N$$

where each M_k has dimension $n_k \times n_{k+1}$. The order in which the mutiplications are carried out is generally crucial. As an example, if $n_1 = 1$, $n_2 = 10$, $n_3 = 1$ and $n_4 = 10$ then the calculation $((M_1 M_2) M_3)$ requires 20 scalar multiplications while $(M_1(M_2 M_3))$ requires 200 scalar multiplications.

(a) Determine a dynamic programming recursion for finding the multiplication order which gives the smalest number of scalar multiplications.
Hint 1: Multiplying an $n_{k-1} \times n_k$ matrix by an $n_k \times n_{k+1}$ requires $n_{k-1} n_k n_{k+1}$ scalar multiplications.
Hint 2: As a state space vector you can use the set $x_k \subset \{1, 2, \ldots, N+1\}$ with $N+1-k$ elements, which represents the indices corresponding to the dimensions of the matrices resulting from multiplications done so far. The initial state is $x_0 = \{1, 2, \ldots, N+1\}$ and the terminal state is $x_{N-1} = \{1, N+1\}$. The control u_k corresponds to the element that is removed from $x_k \setminus \{1, N+1\}$ to define the next multiplication of the neighboring matrices with column and row dimension n_{u_k}, respectively. For example, if $x_0 = \{1, 2, 3\}$ then the only possible control is $u_0 = 2$ and the cost becomes $n_1 n_2 n_3$. .... (6p)

(b) Solve the problem when $N = 3$ and $n_1 = 3$, $n_2 = 10$, $n_3 = 5$, and $n_4 = 2$. .... (4p)

5. Compute the optimal control $u(t)$ for the following problem:

$$\min x(1) + \frac{1}{2}\int_0^1 u(t)^2 dt \quad \text{subject to} \quad \begin{cases} \dot{x}(t) = x(t) + u(t), \\ x(0) = 0, \\ \int_0^1 x(t)dt = 1. \end{cases}$$

. . . . (10p)

Good luck!

Solutions

1. Let

$$U(0,2) = \{-1, 0, 1\},$$

$$U(1,x) = \begin{cases} \{0,1\}, & x = 0 \\ \{-0.5, 0.5\}, & x = 1 \\ \{-1, 0, 1\}, & x = 2, \end{cases}$$

$$U(2,x) = \begin{cases} \{1\}, & x = 0 \\ \{0.5\}, & x = 1 \\ \{0\}, & x = 2. \end{cases}$$

These control values ensure that the state constraint $x_k \in X_k$ remains satisfied.

The dynamic programming iteration can be formulated as

$$J_3(x) = \begin{cases} 0, & x \in X_3 = \{1\} \\ \infty, & \text{otherwise} \end{cases}$$
$$J_k(x) = \min_{u \in U(k,x)} \{|x| + 5|u| + J_{k+1}(0.5x + u)\}$$

In the stage $k = 2$ we get

$$J_2(x) = \min_{u \in U(k,x)} \underbrace{\{x + 5|u| + J_3(0.5x + u)\}}_{J_2(x,u)}$$

The cost $J_2(x_2, u_2)$ for all feasible pairs (u_2, x_2) are given in the following table.

$x_2 \setminus u_2$	-1	$-1/2$	0	$1/2$	1
0	-	-	-	-	5
1	-	-	-	3.5	-
2	-	-	2	-	-

which implies the follow optimal solution

x_2	$J_2(x_2)$	$\mu(2, x_2)$
0	5	1
1	3.5	$1/2$
2	2	0

In the stage $k = 1$ we get

$$J_1(x) = \min_{u \in U(k,x)} \underbrace{\{x + 5|u| + J_2(0.5x + u)\}}_{J_1(x,u)}$$

The cost $J_1(x_1, u_1)$ for all feasible pairs (u_1, x_1) are given in the following table.

$x_1 \setminus u_1$	-1	$-1/2$	0	$1/2$	1
0	-	-	5	-	8.5
1	-	8.5	-	7	-
2	12	-	5.5	-	9

which implies the follow optimal solution

x_1	$J_1(x_1)$	$\mu(1, x_1)$
0	5	0
1	7	$1/2$
2	5.5	0

In the initial stage $k = 0$ we get

$$J_0(x) = \min_{u \in U(k,x)} \underbrace{\{x + 5|u| + J_1(0.5x + u)\}}_{J_0(x,u)}$$

The cost $J_0(x_0, u_0)$ for all feasible pairs (u_0, x_0) are given in the following table.

$x_0 \setminus u_0$	-1	$-1/2$	0	$1/2$	1
2	12	-	9	-	12.5

which implies the follow optimal solution

x_0	$J_0(x_0)$	$\mu(0, x_0)$
2	9	0

The optimal control is thus

$$u_0^* = 0,\ u_1^* = 1/2,\ u_2^* = 1/2$$

and the corresponding optimal state trajectory is

$$x_0^* = 2,\ x_1^* = 1,\ x_2^* = 1,\ x_3^* = 1.$$

2. Use PMP and one can get that $u^*(t) = -\frac{2x_0 e^{-t}}{1-e^{-2T^*}}$, where the optimal time T^* is to be determined. Plug the optimal control to the objective function

$$\begin{aligned}\int_0^{T^*} (u^*(t)^2 + t)dt &= \left(\frac{2x_0^2}{(1-e^{-2T^*})^2}\right)(1-e^{-2T^*}) + (T^*)^2/2 \\ &= \frac{2x_0^2}{1-e^{-2T^*}} + (T^*)^2/2.\end{aligned}$$

Taking the derivative, we note that the optimal T^* is the unique solution of the equation $T^* = 4x_0^2/(e^{T^*} - e^{-T^*})^2$. This can also be determined by noting that the Hamiltonian is zero at T^*

3. (*a*) The ARE becomes $-2p + 3 = p^2$, which gives $p = -1 \pm 2$. The positive definite solution $p = 1$ corresponds to the stabilizing solution. We get
   i. The optimal stabilizing feedback control $u = -x$.
   ii. The optimal cost $J(x(0)) = x(0)^2 p = 1$.

   (*b*) The closed loop system becomes

$$\dot{x} = -x - x = -2x$$

   Hence, the closed loop pole is at $s = -2$.

   (*c*) HJBE gives rise to the Riccati equation

$$\dot{p} - 2p + 3 - p^2 = 0$$

   Separation of variables gives

$$\begin{aligned} & \frac{dp}{(p+3)(p-1)} = dt \\ \Leftrightarrow\quad & \ln\left(\frac{1-p(t)}{p(t)+3}\right) = 4t + c_1 \\ \Leftrightarrow\quad & \frac{1-p(t)}{p(t)+3} = ce^{4t}\end{aligned}$$

   where $c = e^{c_1}$. The boundary condition $p(T) = 0$ gives $c = e^{-4T}/3$. Hence,

$$p(t,T) = 3\frac{1-e^{4(t-T)}}{3+e^{4(t-T)}}$$

   The optimal feedback solution is $u(t) = -p(t,T)x(t)$ and the optimal cost-to-go is $J(t,x) = p(t,T)x^2$.

   (*d*) We have

$$\lim_{T\to\infty} p(t,T) = 1$$

   which is the same as the stabilizing solution to the ARE in problem (*a*).

4. (a) With the notation from the second hint we obtain the following optimal control problem for the minimization problem

$$\min \sum_{k=0}^{N-1} f_0(x_k, u_k) \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k \setminus u_k, & u_k \in x_k \setminus \{1, N+1\} \\ x_0 = \{1, 2, \ldots, N+1\} \end{cases}$$

where $f_0(x_k, u_k) = n_l n_{u_k} n_r$ and l is the largest element in x_k which is less than u_k, and r is the smallest element in x_k, which is larger than u_k, i.e.

$$x_k = \{\ldots, l, u_k, r, \ldots\}$$

if the set is ordered as a sequence of increasing numbers. The dynamic programming recursion becomes

$$\begin{aligned} J_{N-1}(x_{N-1}) &= 0 \\ J_k(x_k) &= \min_{u_k \in x_k \setminus \{1, N+1\}} \{f_0(x_k, u_k) + J_{k+1}(x_k \setminus u_k)\} \end{aligned}$$

(b) We have

$$\begin{aligned} J_2(x_2) &= 0 \\ J_1(x_1) &= \min_{u_1 \in x_1 \setminus \{1,4\}} \{f_0(x_1, u_1)\} = \begin{cases} n_1 n_2 n_4, & x_1 = \{1, 2, 4\} \\ n_1 n_3 n_4, & x_1 = \{1, 3, 4\} \end{cases} \\ J_0(x_0) &= \min_{u_0 \in x_0 \setminus \{1,4\}} \{f_0(x_0, u_0) + J_1(x_0 \setminus u_0)\} \\ &= \min(n_1 n_2 n_3 + n_1 n_3 n_4, n_2 n_3 n_4 + n_1 n_2 n_4) \\ &= \min(100 + 60, 150 + 30) = 160 \end{aligned}$$

Hence $M_1(M_2 M_3)$ gives the minimum number of multiplications 160.

5. Let $y(t) = \int_0^t x(t)dt$, then the problem is of the form of a standard optimal control problem

$$\begin{aligned} \min \quad & x(1) + \frac{1}{2} \int_0^1 u(t)^2 dt \\ \text{subject to } & \dot{x}(t) = x(t) + u(t), \quad x(0) = 0, \\ & \dot{y}(t) = x(t), \quad y(0) = 0, \quad y(1) = 1. \end{aligned}$$

The Hamiltonian corresponding to this problem is

$$H((x, y), u, \lambda) = \frac{1}{2} u^2 + \lambda_1 (x + u) + \lambda_2 x.$$

The minimizing argument u of the Hamiltonian is given by

$$\mu((x, y), \lambda) = \arg \min_u H((x, y), u, \lambda) = -\lambda_1$$

The dynamics for the adjoint variables λ are given by

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial H}{\partial x} = -\lambda_1 - \lambda_2 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial y} = 0.\end{aligned}$$

The first adjoint variable satisfies $\lambda_1(1) = \frac{\partial \Phi(x)}{\partial x} = 1$ since the final cost is equal to $\Phi(x(1)) = x(1)$. There is no condition on the second adjoint variable at the end (i.e., $\lambda_2(1)$ is free) since $y(1)$ is fixed. The two point boundary value problems is thus

$$\begin{aligned}
&\dot{x}(t) = x(t) - \lambda_1(t), && x(0) = 0, \\
&\dot{y}(t) = x(t), && y(0) = 0, \quad y(1) = 1, \\
&\dot{\lambda}_1(t) = -\lambda_1(t) - \lambda_2(t), && \lambda_1(1) = 1, \\
&\dot{\lambda}_2(t) = 0.
\end{aligned}$$

Clearly, λ_2 is constant, thus $\lambda_1(t) = (1 + \lambda_2)e^{1-t} - \lambda_2$. This gives $u(t) = \lambda_2 - (1 + \lambda_2)e^{1-t}$, but where we need to determine λ_2.

Next, $x(t)$ is given by

$$\begin{aligned}
x(t) &= \int_0^t e^{t-s}(-\lambda_1(s))ds = \int_0^t e^{t-s}(\lambda_2 - (1 + \lambda_2)e^{1-s})ds \\
&= \left(-e^{t-s}\lambda_2 + \frac{1+\lambda_2}{2} e^{t+1-2s} \right) \Bigg|_0^t \\
&= \lambda_2(e^t - 1) + \frac{1+\lambda_2}{2}(e^{1-t} - e^{1+t}),
\end{aligned}$$

and thus the condition in $y(1)$ becomes

$$\begin{aligned}
1 = y(1) &= \int_0^1 x(t)dt = \int_0^1 \left(\lambda_2(e^t - 1) + \frac{1+\lambda_2}{2}(e^{1-t} - e^{1+t}) \right) dt \\
&= \left(\lambda_2(e^t - t) - \frac{1+\lambda_2}{2}(e^{1-t} + e^{1+t}) \right) \Bigg|_0^1 \\
&= \lambda_2(e - 2) + \frac{1+\lambda_2}{2}(2e - 1 - e^2) \\
&= \lambda_2(4e - 5 - e^2)/2 + (2e - 1 - e^2)/2,
\end{aligned}$$

and gives

$$\lambda_2 = \frac{e^2 + 3 - 2e}{4e - 5 - e^2}.$$

Thus the optimal control is given by

$$u(t) = -e^{1-t} + \frac{e^2 + 3 - 2e}{4e - 5 - e^2}(1 - e^{1-t}).$$