Exam October 28, 2021 in SF2852 Optimal Control.

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Allowed aids: The formula sheet and mathematics handbook (by Råde and Westergren). (Note that calculator is **not** allowed.)

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Determine the optimal control for the following two problems.

   (*a*) Let t_f be a fixed time and solve:

   $$\min_{u(\cdot)} \frac{1}{2}\int_0^{t_f} (t^3 + u(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = u(t), \ x(0) = x_0 \\ x(t_f) = 0 \end{cases}$$

   Hint: When t_f is fixed the objective function can be simplified. (4p)

   (*b*) Let t_f be a free variable and solve:

   $$\min_{u(\cdot), t_f \geq 0} \frac{1}{2}\int_0^{t_f} (t^3 + u(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = u(t), \ x(0) = x_0 \\ x(t_f) = 0, \ t_f \geq 0 \end{cases}$$

   .... (6p)

2. The following subproblems do not require full solutions. It is enough with an answer and a brief motivation. Remember that the value of a minimization problem is ∞ if the constraint cannot be satisfied.

   (a) Consider the optimal control problem

   $$\min x_1(T) + x_3(T) + \int_0^T f_0(x,u)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = f(x,u), \\ x(0) = x_0, \\ x_4(T) = 10 \end{cases}$$

   The state vector has n-variables ($x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$). What are the boundary conditions on the adjoint vector λ that can be derived from PMP.

   .... (4p)

(b) Determine the optimal value of the time optimal control problem

$$\min T \quad \text{subj.to} \quad \begin{cases} \dot{x}_1 = u, & x_1(0) = 1 \quad x_1(T) = 0 \\ \dot{x}_2 = 2u, & x_2(0) = 1, \quad x_2(T) = 0 \\ |u| \leq 1. \end{cases}$$

.... (2p)

(c) Determine the optimal value of the time optimal control problem

$$\min T \quad \text{subj.to} \quad \begin{cases} \dot{x} = x + u, & x(0) = 2, \qquad x(T) = 0 \\ |u| \leq 1. \end{cases}$$

.... (2p)

(d) Determine the optimal value of the time optimal control problem

$$\min T \quad \text{subj.to} \quad \begin{cases} \dot{x} = -x + u, & x(0) = 2, \qquad x(T) = 0 \\ |u| \leq 1. \end{cases}$$

.... (2p)

3. An investor receives an annual income of amount x_k (each year k). From the x_k received, the investor may reinvest one part $x_k - u_k$ and keep u_k for spending. The reinvestment results in an increase of the capital income as

$$x_{k+1} = x_k + \theta(x_k - u_k)$$

where $\theta > 0$ is given.

The investor wants to maximize his total consumption over N years, i.e., she wants to maximize the utility $\sum_{k=0}^{N-1} u_k$. The resulting optimization problem is

$$\max \sum_{K=0}^{N-1} u_k \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k + \theta(x_k - u_k) \\ 0 \leq u_k \leq x_k, \ x_0 > 0 \text{ is given} \end{cases}$$

(a) Formulate the dynamic programming recursion that solves this optimization problem. .... (5p)

(b) Solve the problem when $N = 4$, $x_0 = 10$, $\theta = 0.4$. .... (5p)

4. Solve the following infinite horizon control problem

$$\min \int_0^\infty \left(3x(t)^2 + \left(\int_0^t (x(s) + 2u(s))ds \right)^2 + u(t)^2 \right) dt$$
$$\text{subj. to} \quad \dot{x}(t) = u(t), \quad x(0) = x_0.$$

Give an expression for the optimal "feedback" (describe the optimal $u(t)$ in terms of $x(t), x(s)$, and $u(s)$ for $s < t$). .... (10p)

5. Consider the following infinite horizon optimal control problem

$$J^*(x_i) = \min_{x_k, u_k, k=0,1,\ldots} \sum_{0}^{\infty} \left(\|Cx_k\|_2^2 + u_k^T R u_k \right) \tag{1}$$

$$\text{subject to } \begin{cases} x_{k+1} = Ax_k + Bu_k \\ x_0 = x_i, \end{cases}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$, $R \in \mathbf{R}^{m \times m}$, $C \in \mathbf{R}^{n \times n}$. Assume that (A, B) is controllable, C is full rank, and $R > 0$.

(a) Check if all assumptions hold in Theorem 2 in the formula sheet. (2p)

(b) Make the ansatz $V(x) = x^T P x$ and determine the minimizing argument in the Bellman equation, i.e., the optimal feedback u expressed as a function of P and x (and the system matrices). (3p)

(c) Determine the matrix equation that P needs to satisfy in order for the Bellman equation to hold for all $x \in \mathbf{R}^n$, i.e., so that $J^*(x) = x^T P x$ is the minimum cost for (1). .... (5p)

Good luck!

Solution

1. Solution 1

   The Hamiltonian is

   $$H(t, x, u, \lambda) = \frac{1}{2}(t^3 + u^2) + \lambda u,$$

   and minimizing with respect to u gives

   $$u = -\lambda.$$

   The dynamics for the adjoint system is

   $$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0,$$

   hence λ is constant and thus u is also constant.

   (a) In order for the control to be feasible the control must be

   $$u = -x_0/t_f.$$

   (b) When t_f is free, this must be determined. Note that

   $$\begin{aligned} 0 =& H(t_f, x^*(t_f), u^*(t_f), \lambda(t_f)) \\ =& \frac{1}{2}(t_f^3 + u^*(t_f)^2) + \lambda(t_f)u^*(t_f) \\ =& \frac{t_f^3}{2} - \left(\frac{x_0}{t_f}\right)^2/2 \end{aligned}$$

   which implies that $t_f = (x_0^2)^{1/5}$.

2. Solution 2

   (a) The solution is

   $$\lambda_k(T) = \begin{cases} 0, & \text{for } k = 2, 5, 6, \ldots, n \\ 1, & \text{for } k = 1, 3 \\ \text{free} & \text{for } k = 4. \end{cases}$$

   (b) No feasible solution.

   (c) No feasible solution.

   (d) Optimal control $u \equiv -1$, which results in $T^* = \log(3)$.

3. Solution 3

(a) The dynamic programming recursion is

$$J(N,x) = 0,$$
$$J(n,x) = \max_{0\le u\le x} u + J(n+1, x+\theta(x-u)), \quad \text{for } n = N-1, N-2, \dots, 0.$$

(b) Applying the recursion for $N = 4, \theta = 0.4$ we get

$$J(4,x) = 0$$
$$J(3,x) = \max_{0\le u\le x} u = x, \text{ optimal } u = x$$
$$J(2,x) = \max_{0\le u\le x} u + x + \theta(x-u) = 2x, \text{ optimal } u = x$$
$$J(1,x) = \max_{0\le u\le x} u + 2(x+\theta(x-u)) = 3x, \text{ optimal } u = x$$
$$J(1,x) = \max_{0\le u\le x} u + 3(x+\theta(x-u)) = 4.2x, \text{ optimal } u = 0.$$

4. Solution 4

Let $y_1(t) = x(t)$, $y_2(t) = \int_0^t (x(s) + 2u(s))ds$. Then the problem can be written as

$$\begin{array}{ll} \min & \int_0^\infty \left(3y_1(t)^2 + y_2(t)^2 + u(t)^2\right) dt \\ \text{subject to} & \dot y_1(t) = u(t), \qquad y_1(0) = x_0. \\ & \dot y_2(t) = x(t) + 2u(t), \quad y_2(0) = 0. \end{array}$$

This is a standard infinite horizon LQ problem with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, Q = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, R = 1.$$

Let $P \in \mathbf{R}^{2\times 2}$ be the matrix satisfying the Riccati equation

$$A^T P + PA + Q = PBR^{-1}B^T P.$$

By solving this, we get $P = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$, and thus the optimal cost is $y(0)^T P y(0) = 3x_0^2$, and the optimal control is

$$u(t) = -R^{-1}B^T P y(t) = \begin{pmatrix} -1 & -1 \end{pmatrix} y(t) = -x(t) - \int_0^t (x(s) + 2u(s))ds.$$

5. Solution 5

(a) Assumption 1 is trivial. Assumption 2 can be verified by noting that

$$\|Cx\|_2^2 + u^T R u = f_0(x,u) \ge \epsilon(\|x\|_2 + \|u\|_2)$$

whenever $0 < \epsilon < \min(\lambda_{\min}(R), \lambda_{\min}(C^T C))$, where $\lambda_{\min}(R)$ denotes the smallest eigenvalue of R. Both $\lambda_{\min}(R)$, $\lambda_{\min}(C^T C)$ are positive since $R > 0$ and C is full rank.

(b) Let $V(x) = x^T P x$ in the Bellman equation, which gives

$$x^T P x = \min_u x^T C^T C x + u^T R u + (Ax + Bu)^T P (Ax + Bu).$$

Note that the objective of the right hand side is a non-negative quadratic from (whenever $P > 0$), thus the minimum is attained when the gradient of the objective function is zero, i.e.,

$$0 = 2Ru + 2B^T P(Ax + Bu) \Leftrightarrow u = -(R + B^T PB)^{-1} B^T Ax.$$

(c) By plugging in the expression for the optimal controller in the objective we arrive at

$$x^T P x = x^T (C^T C + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA)x.$$

For this to hold for all x we need that

$$P = C^T C + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA,$$

which is the discrete time Algebraic Riccati equation.