Exam December 16, 2020 in SF2852 Optimal Control.

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Allowed aids: The formula sheet and mathematics handbook (by Råde and Westergren). (Note that calculator is **not** allowed.)

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Consider the sequential decision problem

$$\max \sum_{k=0}^{N-1} \beta^k u_k^{1-\nu} \quad \text{subj. to} \quad \begin{cases} x_{k+1} = \alpha(x_k - u_k), & 0 \le u_k \le x_k \\ x_0 = W \end{cases}$$

The problem is to maximize the utility of spending u_k amount of capital each time instance given an initial fund $x_0 = W$. We assume that $\beta, \alpha > 0$, and $1 > \nu > 0$ are given.

   (*a*) Do two iterations of the dynamic programming algorithm and determine the optimal control at stage $N-1$ and $N-2$. . (6p)

   (*b*) Determine the optimal value function at stage $N-1$ and $N-2$. *Hint: The derivation of the value function at stage $N-2$ is simplified if you put $\gamma = (\beta\alpha^{1-\nu})^{1/\nu}$.* . . . . (4p)

2. Find the optimal solution (if such exists) and the optimal value of the following optimal control problem

$$\min \int_0^2 (u^2(t) + (1-t)u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = u(t), \\ x(0) = 0, \; x(2) = 1/2, \\ u(t) \ge 0 \end{cases}$$

. . . . (10p)

3. Consider the following nonlinear optimal control problem

$$\min x(1)^2 + \int_0^1 (x(t)u(t))^2 dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = x(t)u(t) \\ x(0) = 1, \end{cases}$$

Solve the problem using dynamic programming.

Hint: Use the ansatz $V(t,x) = p(t)x^2$. . . . . (10p)

4. Consider the following optimal control problem

$$\max x_2(T) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1 = -x_2 + u, & x_1(0) = 0 \\ \dot{x}_2 = x_1, & x_2(0) = 0 \\ \int_0^T u^2(t)dt = 1, \end{cases}$$

where $T > 0$ is given.

(*a*) Reformulate the optimal control problem as a problem on state space form (with constraints as considered in the course). . (1p)

(*b*) Solve the optimal control problem. .... (7p)

(*c*) What happens with $x_2(T)$ when $T \to \infty$. .... (2p)

5. Consider the following infinite horizon optimal control problem

$$\min \int_0^\infty \left(x_1(t)^2 + x_2(t)^2 + u(t)^2\right) dt \quad \text{s.t.} \quad \begin{cases} \dot{x}_1(t) = x_2(t) + u(t), \\ \dot{x}_2(t) = 2x_2(t) - x_1(t) + u(t), \\ x_1(0) = x_{10}, \\ x_2(0) = x_{20}. \end{cases} \tag{1}$$

(*a*) Find the optimal feedback solution (when such exists) and the optimal value of the optimal control problem (1). .... (8p)

(*b*) Is the solution in (*a*) unique? .... (1p)

(*c*) Is the closed loop system stable? .... (1p)

Good luck!

Solutions

1. Dynamic programming gives

$$V_k(x) = \sup_{0\le u\le x} \{\beta^k u^{1-\nu} + V_{n+1}(\alpha(x-u))\}$$
$$V_N(x) = 0$$

   We obtain

$$V_{N-1}(x) = \beta^{N-1}x^{1-\nu} \quad \text{and} \quad u_{N-1}(x)^* = x,$$

   i.e., we should spend all our capital in the last step. For the next prior step we have

$$V_{N-2}(x) = \sup_{0\le u\le x} \left\{\beta^{N-2}u^{1-\nu} + \beta^{N-1}\alpha^{1-\nu}(x-u)^{1-\nu}\right\}$$

   Setting the derivative of the above expression to zero and solving for u gives

$$(1-\nu)u^{-\nu} = \beta\alpha^{1-\nu}(1-\nu)(x-u)^{-\nu}.$$

   From this we get

$$u^*_{N-2} = (1 + (\beta\alpha^{1-\nu})^{1/\nu})^{-1}x$$

   which is in the interval $(0, x)$. We have

$$V_{N-2}(x) = \beta^{N-2}(1 + (\beta\alpha^{(1-\nu)})^{1/\nu})^{\nu}x^{1-\nu}.$$

   Note that if $\nu = 1$ then the control is irrelevant.

2. The Hamiltonian is

$$H(x, u, \lambda) = u^2 + (1-t)u + \lambda u,$$

   and note that λ is constant since

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0.$$

   Minimizing the Hamiltonian with respect to $u \ge 0$ gives

$$u(t) = \max(0, (t-1-\lambda)/2).$$

   We now need to find λ such that this control is feasible, i.e.,

$$1/2 = x(2) = x(0) + \int_0^2 u(t)dt = \int_0^2 \max(0, (t-1-\lambda)/2)dt$$
$$= \int_{1+\lambda}^2 (t-1-(\lambda)/2)dt = (\lambda-1)^2/4.$$

This is satisfied if $\lambda = 1 \pm \sqrt{2}$, and the only reasonable value is $\lambda = 1 - \sqrt{2}$. This gives

$$u(t) = \max(0, (t - 2 + \sqrt{2})/2).$$

Plugging this in gives the objective value $(2\sqrt{(2)} - 3)/6$.

3. The HJBE is

$$\begin{aligned}
-\frac{\partial V}{\partial x} &= \min_u \left\{ (xu)^2 + \frac{\partial V}{\partial x} xu \right\} \\
&= \min_u \left\{ x^2 (u + \frac{1}{2x}\frac{\partial V}{\partial x})^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 \right\} \\
V(1,x) &= x^2
\end{aligned}$$

whith the minimizing control $u^* = -\frac{1}{2x}\frac{\partial V}{\partial x}$. With the ansatz $V(t,x) = p(t)x^2$ the HJBE reduces to

$$\begin{aligned}
-\dot{p}(t)x^2 &= -p(t)^2 x^2 \\
p(1)x^2 &= x^2
\end{aligned}$$

which should hold identically for all $(t,x) \in [0,1] \times R$. We get the following ODE for p:

$$\dot{p}(t) = p(t)^2, \quad p(1) = 1,$$

which has the solution $p(t) = 1/(2-t)$. The resulting optimal control is

$$u^*(t) = -\frac{1}{2-t}, \quad 0 \le t \le 1.$$

Note that the solution to the state equation is

$$\dot{x}(t) = -\frac{1}{2-t}x(t), \quad x(0) = 1$$

is $x(t) = 1 - 0.5t$ which is nozero on $0 \le t \le 1$.

4. (*a*) If we let $x_3(t) = \int_0^t u^2(s)ds$ then the optimal control problem can be reformulated as

$$\min -x_2(T) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1 = -x_2 + u, & x_1(0) = 0 \\ \dot{x}_2 = x_1, & x_2(0) = 0 \\ \dot{x}_3 = u^2, & x_3(0) = 0,\ x_3(T) = 1 \end{cases}$$

(b) Let us proceed as usual and introduce the Hamiltonian $H(x, u, \lambda) = \lambda_1(-x_2 + u) + \lambda_2 x_1 + \lambda_3 u^2$. Pointwise minimization gives

$$\arg\min_u H(x, u, \lambda) = \arg\min_u \lambda_1 u + \lambda_3 u^2 = \begin{cases} -\frac{\lambda_1}{2\lambda_3}, & \lambda_3 > 0 \\ \infty, & \lambda_3 \leq 0 \end{cases}$$

The adjoint system is

$$\begin{cases} \dot{\lambda}_1 &=& -\lambda_2, & \lambda_1(T) &=& 0 \\ \dot{\lambda}_2 &=& \lambda_1, & \lambda_2(T) &=& -1 \\ \dot{\lambda}_3 &=& 0, & \lambda_3(T) &=& ? \end{cases}$$

From the last equation we see that λ_3 must be a constant. It is also clear that $\lambda_3 = k > 0$ since otherwise $u^* = \infty$, which is unreasonable.

Note that the transition matrix for both the primal system and the adjoint system is

$$\Phi(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Solving the adjoint equation gives $\lambda_1(t) = \sin(T - t)$ and thus

$$u^*(t) = -\frac{1}{2k} \sin(T - t)$$

where

$$k = \frac{1}{2} \sqrt{\int_0^T \sin(T - t)^2 dt}$$

which follows since $x_3(T) = \int_0^T u^2(t)dt = 1$.

(c) $k \to \infty$ as $T \to \infty$, which implies $u^* \to 0$ as $T \to \infty$. For the state we have

$$x_2(T) = -\frac{1}{2k} \int_0^T \sin^2(T - t)dt = -\sqrt{\int_0^T \sin^2(T - t)dt} \to -\infty$$

as $T \to \infty$.

5. a) The problem is a LQ problem on standard form with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q = I, R = 1.$$

However, note that it is not controllable since

$$[B, AB] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

thus $z = (1, -1)x$ is uncontrollabe and further note that

$$\dot{z} = (1, -1)\dot{x} = z,$$

hence z is unstable. Consider the equivalent formulation in x_1 and $z = x_1 - x_2$, i.e., with x_2 replaced by $x_1 - z$:

$$\min \int_0^\infty \left(x_1(t)^2 + (x_1(t) - z(t))^2 + u(t)^2\right) dt \quad \text{s.t.} \quad \begin{cases} \dot{x}_1(t) = x_1(t) - z(t) + u(t) \\ \dot{z}(t) = z(t) \\ x_1(0) = x_{10}, \\ z(0) = x_{10} - x_{20}. \end{cases}$$

Note that it is impossible to achieve a finite cost unless $z(0) = x_{10} - x_{20} = 0$, which gives $z(t) \equiv 0$ and $x_1(t) = x_2(t)$ for all t. In this case the problem becomes

$$\min \int_0^\infty \left(2x_1(t)^2 + u(t)^2\right) dt \quad \text{s.t.} \quad \begin{cases} \dot{x}_1(t) = x_1(t) + u(t) \\ x_1(0) = x_{10}. \end{cases}$$

This is a standard LQ problem with infinite time horizon, with $a = 1, b = 1, r = 1, q = 2$, thus the Ricatti equation is

$$0 = 2pa + q - b^2p^2r^{-1} = 2p + 1 - p^2 = -(p-1)^2 + 2,$$

thus the solution is the positive root $p = 1 + \sqrt{2}$. The optimal feedback is thus

$$u = -r^{-1}bpx_1 = -(1 + \sqrt{2})x_1$$

and the optimal cost is

$$V(x) = \begin{cases} x_{10}^2(1 + \sqrt{2}) & x_{10} = x_{20} \\ \infty & \text{otherwise.} \end{cases}.$$

b) The optimum is unique if it exists, i.e., if $x_{10} = x_{20}$.

c) The closed loop is unstable since there is one unstable uncontrollable mode ($z = x_1 - x_2$).