Optimal Control Theory SF 2852

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Fall 2024

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1. Course information and course logistics
2. Course overview and application examples

Course Information and Course Logistics

- This is a course on optimal control
  - Optimization problems involving difference or differential equations
  - Many engineering problems are naturally posed as optimal control problems
    - Economics and logistics
    - Aerospace systems
    - Automotive industry
    - Autonomous systems and robotics
    - Bio-engineering
    - Process control
    - Power systems

Course Contents

- Optimal control is an important branch of mathematics
    - Roots in the calculus of variations
    - Dynamic programming
    - Pontryagin minimum principle (PMP)
    - Linear quadratic control
    - Model predictive control (MPC)

Teachers

- Johan Karlsson (Email: johan.karlsson@math.kth.se)
- Michele Mascherpa (Email: micmas@kth.se),
    - Office hours: One hour a week. Time/place will be on homepage.
- Lectures: Results, how to use results, proofs
- Exercises: Examples (some background)
- Exam: Focus on using the results to solve problems.

Course Material

- Optimal Control: Lecture notes
- Exercise notes on optimal control theory
- Some complementary material will be handed out. It will also be posted on the course homepage.

Homework

Four homework sets

HW0 Some basics on linear systems and get started on convex optimization solvers. (No bonus points).

HW1 Discrete dynamic programming and model predictive control.

HW2 Computational methods (Project):

HW3 PMP and related topics.

- Each homework has 3-5 problems.
- Must get at least one point on each of the homework sets 1-3.
- Each homework set handed in in time may also give maximum 2 bonus credits for the exam.

Final Written Exam

The final exam takes place on October 24, 2024 at 8.00-13.00.

You must register for the exam during the time period Sept 17-Oct 1. Use "My Pages".

For questions about registrations, etc. contact students affairs office: `studentoffice@math.kth.se`.

Grade	A	B	C	D	E	FX
Total credit (points)	45-56	39-44	33-38	28-32	25-27	24

- Total credit = exam score + homework score.
- The maximum exam score = 50. Maximum bonus from the homework sets = 6.
- You may use Mathematics Handbook and a formula sheet found on the course homepage.

Project

Requirement for PhD version (SF3852, 7.5hp.)

- Project related to development in the world (use data from Gapminder or John Hopkins)
- Presentation ($\sim$ 15 min) and report
- At least B on exam.
- The requirement of B on exam is quite tough.

1. Discrete time optimal control theory
2. Continuous time optimal control theory

Discrete Time Optimal Control

$$\min \phi(x_N) + \sum_{k=0}^{N-1} f_0(k, x_k, u_k) \quad \text{subj. to} \quad \begin{cases} x_{k+1} = f(k, x_k, u_k) \\ x_0 \text{ given}, \ x_k \in X_k \\ u_k \in U(k, x_k) \end{cases}$$

The variables $x_1, \cdots, x_N$ (states) and $u_0, \cdots, u_{N-1}$ (controls) are the ones we optimize over.

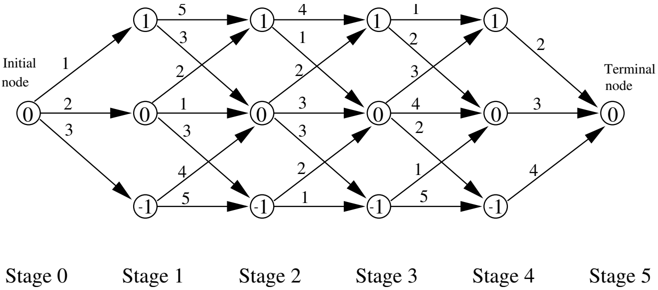
The functions ϕ and f_0 determine the terminal and running costs.

The function f and sets X_k and U determine feasible states and controls.

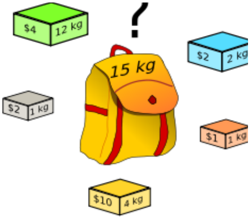
Sequential decision problem which contain as special cases

- Graph search problems
- Combinatorial optimization
- Discrete time optimal control

Shortest Path problem



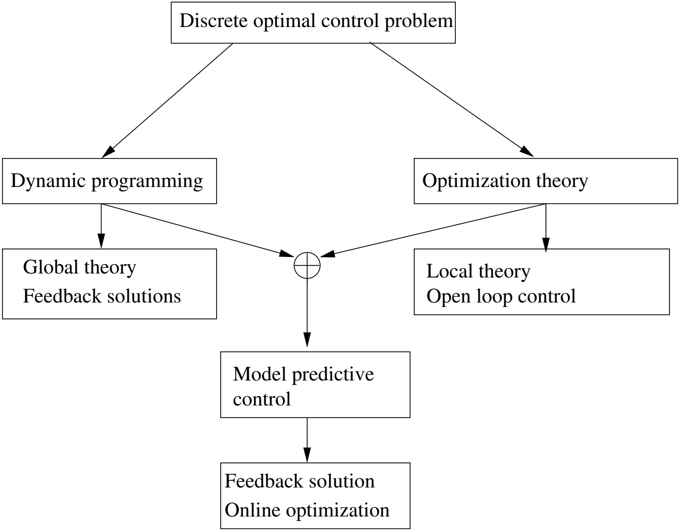
Find the shortest path from the initial node at stage 0 to the terminal node at stage 5



$$\max \sum_{j=1}^{n} p_j x_j$$

$$\text{subject to} \sum_{j=1}^{n} w_j x_j \leq c, \quad x_j = 0 \text{ or } 1, \quad j = 1, \ldots, n.$$

Discrete Optimal Control Theory



Continuous Time Optimal Control

$$\min \Phi(t_1, x(t_1)) + \int_{t_0}^{t_1} f_0(t, x(t), u(t))dt \text{ subj. to } \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_0) \in S_0, \\ x(t_1) \in S_1 \\ u \in U(x) \end{cases}$$

The variables are the functions $x(t)$ (state trajectory) and $u(t)$ (control signal) that we optimize over.

Sometimes we let also t_1 be a variable.

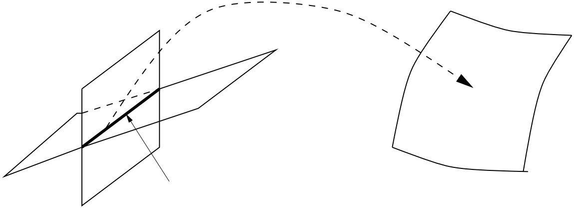
The functions ϕ and f_0 determine the terminal and running costs.

The function f and sets S_0, S_1 and $U(x)$ determine feasible states and controls.

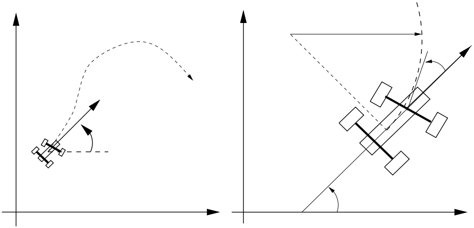
The feasible solutions should satisfy

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_0) \in S_0, \quad x(t_1) \in S_1 \\ u \in U(x) \end{cases}$$

The trajectories $x(t)$ should start in S_0 and end in S_1, *i.e.*,



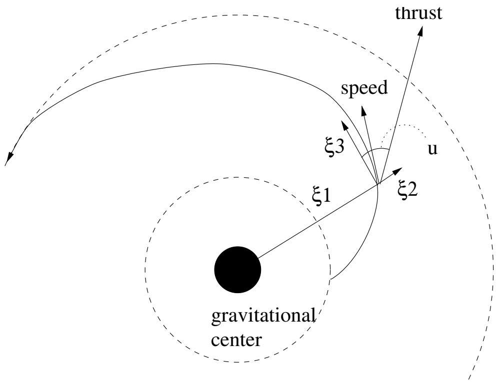
S_0 and S_1 are manifolds (intersection of surfaces).



Problem: Shortest time. (= shortest path when constant speed v)

$$\min_{\omega, T} T \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = v\cos(\theta(t)), \ x(0) = 0, \ x(T) = \bar{x} \\ \dot{y}(t) = v\sin(\theta(t)), \ y(0), \ y(T) = \bar{y} \\ \dot{\theta}(t) = \omega(t), \ \theta(0) = 0, \ \theta(T) = \bar{\theta} \\ |\omega(t)| \leq v/R, \ T \geq 0 \end{cases}$$

Problem: Orbit transfer



Problem:

$$\max_u x_1(T) \qquad \text{s.t.} \quad \begin{cases} \dot{x}_1 &= c_3^2 x_2 \\ \dot{x}_2 &= \frac{x_3^2}{x_1} - \frac{1}{x_1^2} + \frac{\sin u}{1/c_1 - c_2 c_3 t} \\ \dot{x}_3 &= -\frac{c_3^2 x_2 x_3}{x_1} + \frac{c_3 \cos u}{1/c_1 - c_2 c_3 t} \\ x(0) &= (1\ 0\ 1)^T \\ S_f &= \{x \in \mathbf{R}^3 | x_2 = 0,\ x_3 - \frac{1}{\sqrt{x_1}} = 0\} \end{cases}$$

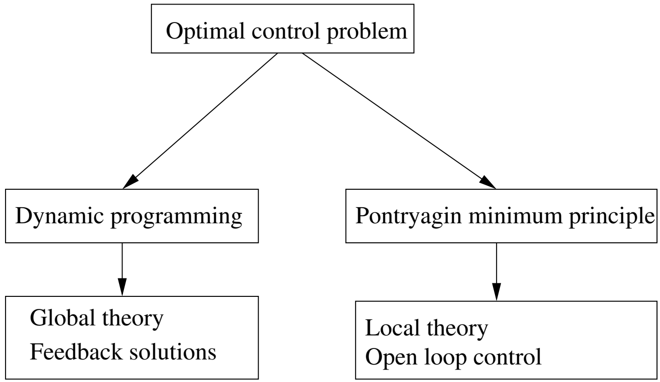
Resource Allocation

$$\max_{u(\cdot)} \int_0^T (1-u(t))x(t)dt + x(T)$$

$$\text{subj. to} \begin{cases} \dot{x}(t) &= \alpha u(t)x(t),\ (0<\alpha<1) \\ x(0) &= x_0 > 0 \\ u(t) &\in [0,1], \forall t \end{cases}$$

- A portion $u(t)$ of the production rate $x(t)$ is invested in the factory (and increases the production capacity)
- The rest $(1-u(t))x(t)$ is stored in a warehouse.
- Maximimize the sum of the total amount of goods stored in the warehouse and the final capacity of the factory.

Optimal Control Theory



- Feedback control $u(t) = \psi(t, x(t))$ (depends on state)
- Open loop control $u(t) = \psi(t)$