PMP proof outline

Step 1: Needle perturbation

$$u(t) = \begin{cases} u^*(t), & t \leq \tau - \Delta\tau \\ v, & \tau - \Delta\sigma \leq t \leq \tau \\ u^*(t), & t \geq \tau \end{cases}$$

Resulting change in trajectory:

$$\begin{aligned} \delta\tilde{x}(\tau) = \tilde{x}(\tau) - \tilde{x}^*(\tau) &= \int_{\tau-\Delta\tau}^{\tau} [\tilde{f}(x(t), v) - \tilde{f}(x^*(t), u^*(t))]dt \\ &= (\tilde{f}(x^*(\tau), v) - \tilde{f}(x^*(\tau), u^*(\tau)))\Delta\tau + o(\Delta\tau). \end{aligned}$$

Transported by linearized dynamics

$$\delta\tilde{x}(t_f^*) = \Phi(t_f^*, \tau)\delta\tilde{x}(\tau) + o(\Delta\tau)$$

where Φ is transition matrix $\sim A(t) = \tilde{f}_{\tilde{x}}(x^*(t), u^*(t))$

PMP proof outline

Change in end time

$$u(t) = \begin{cases} u^*(t), & t \in [0, t_f^*] \\ u^*(t_f^*), & t \in [t_f^*, t_f^* + \Delta t]. \end{cases}$$

Change in final state:

$$\begin{aligned} \delta\tilde{x}(t_f^*) = \tilde{x}(t_f^* + \Delta t) - \tilde{x}^*(t_f^*) &= \int_{t_f^*}^{t_f^* + \Delta t} \tilde{f}(x(t), u(t))dt \\ &= \tilde{f}(x^*(t_f^*), u^*(t_f^*))\Delta t + o(\Delta t) \end{aligned}$$

where Δt may be positive or negative

PMP proof outline

Step 2: Combined perturbations:

$$u(t) = \begin{cases} v_k, & t \in I_k, \quad v_k \in U \\ u^*(t), & t \notin I_k \cup [t_f^*, t_f^* + \Delta t] \\ u^*(t_f^*), & t \in [t_f^*, t_f^* + \Delta t] \end{cases}$$

and similarly if $\Delta t < 0$.

Linearized perturbation to final state in cone:

$$\mathcal{K}(t_f^*) = \Big\{ \tilde{f}(x^*(t_f^*), u^*(t_f^*))\Delta t + \sum_{k=1}^{p} \alpha_k \Phi(t_f^*, \tau_k)\delta\tilde{x}_k :$$

$$\Delta t \in \mathbf{R};\ \alpha_k \geq 0;\ \tau_k \in (0, t_f^*), p \text{ is an integer},$$

$$\delta\tilde{x}_k = (\tilde{f}(x^*(\tau_k), v_k) - \tilde{f}(x^*(\tau_k), u^*(\tau_k)))\Delta\tau,\ v_k \in U \Big\}$$

PMP proof outline

Step 3: Separating hyperplane

- By optimality $\mathcal{K}(t_f^*)$ cannot intersect the line A^*B
- There exists separating hyperplane H, specified by its normal $\begin{bmatrix} \lambda_0 \\ a \end{bmatrix}$ and the point $A^* = \tilde{x}^*(t_f^*)$

Cone above Hyperplane H:

$$\begin{bmatrix} \lambda_0 \\ a \end{bmatrix}^T \delta\tilde{x}(t_f^*) \geq 0, \quad \forall\, \delta\tilde{x}(t_f^*) \in \mathcal{K}(t_f^*)$$

The ray A^*B is below H

$$\begin{bmatrix} \lambda_0 \\ a \end{bmatrix}^T (B - A^*) \leq 0 \Leftrightarrow \begin{bmatrix} \lambda_0 \\ a \end{bmatrix}^T \begin{bmatrix} -\beta \\ 0 \end{bmatrix} \leq 0,$$

hence $\lambda_0 \geq 0$.

PMP proof outline

Step 4: Define adjoint equation

$$\lambda(t_f^*) = a \quad \text{and} \quad \begin{bmatrix} \lambda_0(t) \\ \lambda(t) \end{bmatrix} = \Phi(t_f^*, t)^T \begin{bmatrix} \lambda_0 \\ a \end{bmatrix}$$

Then

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \lambda_0(t) \\ \lambda(t) \end{bmatrix} &= A(t)^T \Phi(t_f^*, t)^T \begin{bmatrix} \lambda_0 \\ a \end{bmatrix} \\ &= -\frac{\partial \tilde{f}}{\partial \tilde{x}}(x^*(t), u^*(t)) \begin{bmatrix} \lambda_0(t) \\ \lambda(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\frac{\partial}{\partial x}\underbrace{(\lambda_0 f_0(x, u) + \lambda(t)^T f(x, u))}_{H(x,u,\lambda)} \end{bmatrix} \end{aligned}$$

Then

$$
\begin{aligned}
& H(x^*(\tau), v, \lambda(\tau)) - H(x^*(\tau), u^*(\tau), \lambda(\tau)) \\
&= \begin{bmatrix} \lambda_0 \\ \lambda(t) \end{bmatrix}^T \left(\tilde{f}(x^*(\tau, v)) - \tilde{f}(x^*(\tau, u^*(\tau))\right) \\
&= \begin{bmatrix} \lambda_0 \\ a \end{bmatrix}^T \underbrace{\Phi(t_f^*, t) \left(\tilde{f}(x^*(\tau, v)) - \tilde{f}(x^*(\tau, u^*(\tau))\right)}_{\in \mathcal{K}(t_f^*)} \geq 0 \quad \forall v \in U.
\end{aligned}
$$

Therefore:

$$
H(x^*(\tau), u^*(\tau), \lambda(\tau)) = \min_{v \in U} H(x^*(\tau), v, \lambda(\tau)), \quad \forall \tau \in [0, t_f^*].
$$

PMP proof outline

Prove $H^* = 0$ (for free final time).

$$\tilde{f}(x^*(t_f^*), u^*(t_f))\Delta t \in \mathcal{K}(t_f^*) \quad \text{for } \Delta t \in \mathbb{R}$$

Therefore

$$\begin{bmatrix} \lambda_0 \\ a \end{bmatrix}^T \tilde{f}(x^*(t_f^*), u^*(t_f))\Delta t \geq 0 \quad \text{for } \Delta t \in \mathbb{R}$$

$$\Leftrightarrow H(x^*(t_f^*), u^*(t_f^*)) = \begin{bmatrix} \lambda_0 \\ a \end{bmatrix}^T \tilde{f}(x^*(t_f^*), u^*(t_f)) = 0$$

H^* constant.
Similar to previous proof for time invariant systems (under certain conditions).

Summary

- Dynamic programming
- MPC
- PMP
- Numerical optimization methods (homeworks)

Main emphasis of exam:
Dynamic programming, MPC, PMP
Solving small problems analytically + concepts

Dynamic Programming

- Discrete/continuous time
- Finite/infinite time horizon
- Feedback control
- Numerical optimization suffer from curse of dimensionality (unless LQ)

PMP

- Continuous time (discrete time can also be formulated)
- Fixed/free final time
- Control to a manifold
- Open loop control
- Iterative methods can handle large dimensional problems